An Improved Poisson Distribution and Its Application in Option Pricing

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ABSTRACT

This work, introduces an improve Poisson distribution function. This improved Poisson is equipped with some financial terms, which generate a model for determining the prices of a European call and put option for two period models. Some of its important statistical properties like the mean, variance are given. It was found that the problem of option for non-dividend paying stock can be approached using an improved Poisson distribution function equipped with some financial terms. In comparison it gives exactly the numerical results with the CRR binomial model using the numerical data. An empirical example is given in a concrete setting.

Keywords: Improved Poisson, Generalized Binomial distribution, Option Pricing.

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1. INTRODUCTION

This work focuses a on a particular type of derivative known as option trading for two periods. Option trading gives the holder the right and not obligation to buy or sell an underlying asset at a predetermined price at a specified future time. A good example of underlying asset is a stock.

Stock option is traded on most exchanges around the world. The value of the stock option is dependent on the movement of stock prices. Thus an investor will take a position depending on weather prices of stock move up or down. Investor will only exercise their option if they have a positive payoff value. Determining the value of option at any given point in time is commonly called option pricing.

In recent years, determining an option value or forecasting the price of an option which is to be exercised only at expiration, has become popular in finances, with Black-Scholes model, CRR model and Binomial model, Recently Osu et al [2] develop a model for evaluating the price of call and put an option as

$$C_{(0)} = \frac{1}{R^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\hat{A}^x\hat{B}^{n-x}}{(\hat{A}+\hat{B})^{(n)}}C_T(x) \qquad (1)$$

and

$$P_{(0)} = \frac{1}{R^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\hat{A}^x\hat{B}^{n-x}}{(\hat{A}+\hat{B})^{(n)}}C_T(x), \qquad (2)$$

respectively. Where $C_T(x) = \mathrm{Max}\left[u^x d^{n-x}S_{(0)} - K, 0\right]$, R is the interest rate ,$\frac{\hat{A}}{\hat{A}+\hat{B}}$ and $\frac{\hat{B}}{\hat{A}+\hat{B}}$ are the neutral probabilities,u and $d$are the rates at which the price move up and down respectively and k is the strike price. Egege, et al [9]showed that Polya distribution combined will give;

$$C_{(0)} = \frac{1}{(1+r)^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\left(\check{r}/_{\check{r}+\breve{b}}\right)^x\left(\breve{b}/_{\check{r}+\breve{b}}\right)^{n-x}}{\left(\check{r}/_{\check{r}+\breve{b}}+\frac{\breve{b}}{\check{r}+\breve{b}}\right)^n}C_t(x). \qquad (3)$$

With $C_T(x) = \mathrm{Max}\left[u^x d^{n-x}S_{(0)} - K, 0\right]$, $(1+r)$ the interest rate ,$\frac{\hat{r}}{\hat{r}+\hat{b}}$ and $\frac{b}{\hat{r}+\hat{b}}$ the neutral probabilities. Oduro et al [8] gave binomial model for a two-step binomial as

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}], \qquad (4)$$

with payoff $C_T(x) = [0, S_T - K]$ and neutral probability $p = \frac{e^{rt}-d}{u-d}$. Nyustern[1] gave a Block –Sholes model written as a function of five variablesS, K, T, r and σ^2 as

$$C = SN(d_1) - ke^{-rt}N(d_2), \qquad (5)$$

where $d_1 = \frac{\mathrm{In}\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$,S = current value of the underlying asset ,K= strike price of the option, t = life to expiration of the option , r = riskless interest rate corresponding to the life of the option and σ^2 = variance in the $\mathrm{In(value)}$ of the underlying asset. Chandra et al [4]developed CRR binomial model for the case of two period of the form

$$C_{(0)} = e^{-r\Delta t}[\check{p}^2 C_{uu} + 2\check{p}(1-\check{p})C_{ud} + (1-\check{p})^2 C_{dd}] \qquad (6)$$

and

$$P_{(0)} = e^{-r\Delta t}[\check{p}^2 P_{uu} + 2\check{p}(1-\check{p})P_{ud} + (1-\check{p})^2 P_{dd}], \qquad (7)$$

With neutral probability $\check{p} = \frac{e^{-r\Delta t} - d}{u-d}$.

Instead in this study, a model by equipping an improved Poisson with financial terms is generated of the form;

$$C = \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{N}{x} e^{-\lambda} \frac{\lambda^x}{N^x} e^{\lambda} \left(\frac{B}{A+B}\right)^N fS(N) \qquad (8)$$

and

$$P = \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{N}{x} e^{-\lambda} \frac{\lambda^x}{N^x} e^{\lambda} \left(\frac{B}{A+B}\right)^N fS(N), \qquad (9)$$

for the call and put options with$fS(N)$, the payoff and $\lambda = \frac{NA}{B}$

2. METHOD

The tools for giving the result are generalized binomial distribution and financial terms. The Generalized Binomial distribution in this study was first presented by Dwass (1979). It is a discrete distribution that depends on four parameters A, B, N and α,where A and B are positive , N is a positive integer and α is an arbitrary real number ,satisfying $(N-1) \leq A + B$. Tereapabolan [10] gave Dwass identity of the form $x^{(i)} = x\,(x-\alpha)\,.....(x-(i-1)\alpha)$.

Let X be the generalized Binomial random variable. By Terepabolan[6], its probability function is of the form

$$P_X(x) = \binom{N}{x} \frac{[A(A-\alpha)....(A)-(x-1)\alpha][B(B-\alpha)..(B-(N-x-1)\alpha)]}{(A+B)(A+B-\alpha)....(A+B-(N-1)\alpha)}$$

$$= \binom{N}{x} \frac{A^{(x)} B^{(N-x)}}{(A+B)^N}, x = 0,..N.$$

ASSUMPTIONS FOR THE PROPOSED MODEL

In what follows, the assumes below are made:

1. The initial values of the stock is S_0
2. At the end of the period , the price is either going up or down by a fixed factor $u = e^{\sigma\sqrt{\Delta t}}$ or go down by a factor $d = e^{-\sigma\sqrt{\Delta t}}$.
3. The price of an option is dependent on the following .

a) The strike price K.

b) The expire time T.

c) The risk free rate r.

d) The underlying price S_0.

e) Volatility σ.

4. $e^{\sigma\sqrt{\Delta t}} > e^{r\Delta t} > e^{-\sigma\sqrt{\Delta t}} > 0$.
5. The stock pays no dividends .

6. The continuous compounded interest rate rsuch that $B(0,T) = e^{r\Delta t}$.

7. The length of each period Δt can be positive number.

8. From market data for stock price one can estimate the stock price volatility σ per one time unit (typically one year).

9. Set$N = \frac{T}{\Delta t}$.

OPTION PRICING PARAMETERS

1. The current stock priceS_0 : which is the prevailing market price of the stock at expiration.

2. The strike price K: which is the predetermined price at which the holder will exercised right.

3. The time to expiration T : which is the time duration the holder has to exercise right.

4. The risk-free interest rate r:which is the rate of investment on the stock.

5. The volatility of the stock price σ: which measures the uncertainty of movement in the market.

Change in the above parameters affects the price of the option as discussed in Osu et al [2] and oduro[10]

Lemma 2.1: Let $x \in \mathbb{N} \cup \{0\}$for $N > 0$, then (Dongping Hu et al, [2]),

$$\frac{1}{\prod_{i=0}^{x-1}\left(1-\frac{i}{N}\right)} = \prod_{i=0}^{x-1}\left(1+\frac{i}{N}+\left(\frac{i}{N}\right)^2+\cdots\right) = 1+\frac{x(x-1)}{2N} + 0\left(\frac{1}{N^2}\right). \tag{11}$$

Proof: By mathematical induction; for $x = 1$,

$$\frac{1}{\prod_{i=0}^{1-1}\left(1-\frac{i}{N}\right)} = \prod_{i=0}^{1-1}\left(1+\frac{i}{N}+\left(\frac{i}{N}\right)^2+\cdots\ldots\right) = 1+\frac{1(1-1)}{2N} + 0\left(\frac{1}{N^2}\right) = 1.$$

Let $x = k \in \mathbb{N}$ such that $\frac{1}{\prod_{i=0}^{k-1}\left(1-\frac{1}{N}\right)} = 1+\frac{k(k-1)}{2N} + 0\left(\frac{1}{N^2}\right)$. Thus for $x = k+1$

$$\begin{aligned}
\frac{1}{\prod_{i=0}^{k}\left(1-\frac{1}{N}\right)} &= \left\{1+\frac{k(k-1)}{2N} + 0\left(\frac{1}{N^2}\right)\right\}\left(1+\frac{k}{N}\right) \\
&= 1+\frac{k}{N}+\frac{k(k-1)}{2N} - \frac{k(k-1)}{2N}\frac{k}{N} + O\left(\frac{1}{N^2}\right) - O\left(\frac{1}{N^2}\right).\frac{k}{N} \\
&= 1+\frac{k}{N}+\frac{k(k-1)}{2N} - \frac{k^2(k-1)}{2N^2} + O\left(\frac{1}{N^2}\right) - O\,\frac{k}{N^2} \\
&= 1+\frac{k}{N}+\frac{k(k-1)}{2N} + O\left(\frac{1}{N^2}\right) = 1+\frac{2k+k^2-k}{2N} + O\left(\frac{1}{N^2}\right) \\
&= 1+\frac{k(k+1)}{2N} + O\left(\frac{1}{N^2}\right).
\end{aligned} \tag{12}$$

Therefore for $x = k+1$ we obtain

$$\frac{1}{\prod_{i=0}^{x-1}\left(1-\frac{1}{N}\right)} = 1+\frac{x(x-1)}{2N} + 0\left(\frac{1}{N^2}\right). \tag{13}$$

Lemma 2.2: For $u=\frac{1}{d}=\ e^{\sigma\sqrt{\Delta t}}$, a risk free interest rate e^{2r}=$e^{r\Delta t}$ for two period and $\frac{\breve{A}}{\breve{A}+\breve{B}}=\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}$ holds

If $\sum_{j=1}^{3}\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}=1.$

Proof

Let $\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_1=\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2$, $\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_2=2\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)\left(\frac{e^{\sigma\sqrt{\sigma}}-e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)$ and $\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_3=\left(\frac{e^{\sigma\sqrt{\sigma}}-e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2$

$$\sum_{j=1}^{3}\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}=\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2+2\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)\left(1-\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)+\left(1-\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2$$

$$=\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2+2\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)-2\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2+\left(1-\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2$$

$$=\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2-\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)^2+1=1.\blacksquare$$

Lemma 2.3: If $e^{\sigma\sqrt{\Delta t}}>e^{r\Delta t}>e^{-\sigma\sqrt{\Delta t}}>0$ and no arbitrary principle exist thus the following holds;

1. $E_{\frac{\breve{A}}{\breve{A}+\breve{B}}}(S_2)=e^{2r\Delta t}S_0$.
2. $\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)_j>0$ where $j=1,2,..n.$

Proof (1): For $S(2)$ implies $t=2$ so that

$$\left(\frac{\breve{A}}{\breve{A}+\breve{B}}+1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2=\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2+2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)+\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2.$$

By defining

$$\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_1=\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2,\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_2=2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)\text{and }\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_3=\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2,$$

and

$$E_{\frac{\breve{A}}{\breve{A}+\breve{B}}}S(2)=\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2u^2S_0+2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}udS_0\right)+\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2dS_0\right]. \tag{14}$$

Therefore

$$S_0\left[\frac{\breve{A}}{\breve{A}+\breve{B}}+1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right]^2=S_0\left[\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)u+\left(\frac{e^{\sigma\sqrt{\Delta t}}-e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)d\right]^2=S_0\left[\frac{e^{r\Delta t}u-e^{-\sigma\sqrt{\Delta t}}u+e^{\sigma\sqrt{\Delta t}}d-e^{r\Delta t}d}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right]=S_0e^{2r\Delta t}.$$

Proof (2): Since $e^{\sigma\sqrt{\Delta t}}>e^{r\Delta t}>e^{-\sigma\sqrt{\Delta t}}>0$, it follows that

$\left(\frac{e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}\right)_i>0\forall\ i=1,2..n.$

3. MAIN RESULT

Theorem 3.1: Let $x_0\in\ \mathbb{N}\cup\{0\}$ and $\lambda=\frac{NA}{B}$ then;

$$Gbd(A,B,N,\alpha)\ \cong \widetilde{\wp_\lambda}(x) + O\left(\frac{1}{N^2}\right), \tag{15}$$

and for $\alpha = 0, N, \frac{B}{A+B}$ large, $Gbd(A,B,n)\ \cong\ \widetilde{\wp_\lambda}(x)$ with $\widetilde{\wp_\lambda}(x)\ =\ \wp_\lambda e^\lambda \left(\frac{B}{A+B}\right)^N \frac{1}{1+\frac{x(x-1)}{2N}} + O\left(\frac{1}{N^2}\right)$.

Proof: For $= 0$, $Gbd(A,B,N,\alpha)\ = \left(\frac{B}{A+B}\right)^N\ =\ \widetilde{\wp_\lambda}(0) + O\left(\frac{1}{N^2}\right)$

Thus

$$Gbd(A,B,n,\alpha) = \binom{N}{x}\frac{A(A-\alpha).....A-(x-1)(B(B-\alpha).....B-(N-x-1)\alpha)}{(A+B)(A+B-\alpha)......A+B-(N-1)\alpha)} \tag{16}$$

$$= \binom{N}{x}\frac{A^{(x)}B^{(N-x)}}{(A+B)^{(N)}}$$

$$= \binom{N}{x}\left(\frac{N^{A}/_{A+B}}{N}\right)^x \left(\frac{B}{A+B}\right)^N \left(\frac{B}{A+B}\right)^{-x}$$

$$= \binom{N}{x}\left(\frac{N^{A}/_{A+B}}{N}\right)^x \frac{\left(\frac{B}{A+B}\right)^N}{\left(\frac{B}{A+B}\right)^x}$$

$$= \binom{N}{x}\left(\frac{N^{A}/_{A+B}}{N}\right)^x \frac{\left(\frac{B}{A+B}\right)^N}{\left(\frac{N^{A}/_{A+B}}{\lambda}\right)^x}$$

$$= \binom{N}{x}\left(\frac{\backslash N^x\left(\frac{A}{A+B}\right)^x}{N^x}\right)\left(\frac{B}{A+B}\right)^N \times \frac{\lambda^x}{N^x\left(\frac{A}{A+B}\right)^x}$$

$$= \binom{N}{x}\lambda^x \frac{N^x}{N^x}\left(\frac{\frac{A}{A+B}}{N\frac{A}{A+B}}\right)^x \left(\frac{B}{A+B}\right)^N$$

$$= N_{C_x}\frac{e^{-\lambda}\lambda^x e^\lambda}{N^x}\left(\frac{B}{A+B}\right)^N \tag{17}$$

$$= \frac{N(N-1)(N-2).....(N-x+1)(N-x)..(2)(1)}{x(x-1)(x-2).....(2)(1).(n-x)..(2)(1)}\frac{e^{-\lambda}\lambda^x e^\lambda}{N^x}\left(\frac{B}{A+B}\right)^N$$

$$= \frac{N(N-1)(N-2).......(N-x+1)e^{-\lambda}\lambda^x e^\lambda}{N^x x!}\left(\frac{B}{A+B}\right)^N$$

$$= \frac{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)........\left(1-\frac{x-1}{N}\right)e^{-\lambda}\lambda^x e^\lambda}{x!}\left(\frac{B}{A+B}\right)^N$$

$$= \frac{\lambda^x}{x!}\prod_{i=0}^{x-1}\left(1-\frac{i}{N}\right)\left(\frac{B}{A+B}\right)^N = \frac{e^{-\lambda}\lambda^x e^\lambda}{x!}\left(\frac{B}{A+B}\right)^N \frac{1}{\prod_{i=0}^{x-1}\left(1-\frac{i}{N}\right)}.$$

by lemma 2.1one obtains

$$\cong \frac{\frac{e^{-\lambda}\lambda^x e^\lambda}{x!}\left(\frac{B}{A+B}\right)^N}{1+\frac{x(x-1)}{2N}} + O\left(\frac{1}{N^2}\right). \tag{18}$$

For $\alpha = 0$ and $N, \frac{B}{A+B}$ large $O\left(\frac{1}{N^2}\right) \to 0$, implies

$$Gbd(A,B,N)\ \cong\ \frac{\frac{e^{-\lambda}\lambda^x e^\lambda}{x!}\left(\frac{B}{A+B}\right)^N}{1+\frac{x(x-1)}{2N}}, \tag{19}$$

with

$$E(X) = \frac{\lambda e^{\lambda}\left(\frac{B}{A+B}\right)^{N}}{a},$$

and

$$Var(X) = ae^{\lambda}\left(\frac{B}{A+B}\right)^{N}\frac{\left[(\lambda^2+\lambda)-\lambda^2 e^{2\lambda}\left(\frac{B}{A+B}\right)^{2N}\right]}{a^2},$$

where $a = 1 + \frac{x(x-1)}{2N}$.

Equation (17) can be combine with financial terms to determine the call and put price of an option, provided the following is satisfy

I. $\sum_{x=0}^{n} N_{C_x}\frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x}\left(\frac{B}{A+B}\right)^{N} = 1.$

II. $\left(\frac{B}{A+B}\right)^{N} > 0.$

III. $1 - \frac{B}{A+B} = \frac{A}{A+B}.$

IV. $N > 0.$

V. $\frac{A}{A+B} = \frac{\breve{A}}{\breve{A}+\breve{B}}$ and $1 - \frac{A}{A+B} = 1 - \frac{\breve{A}}{\breve{A}+\breve{B}}.$

Equation (7) applied in finance can be expressed as

$$N_{C_x}\frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x}\left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^{N} \tag{20}$$

Theorem 3.2: let C and P be the value of a European derivation security whose payoff is$f(S(N))$ then if

$$C = e^{-r\Delta t}\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{uu} + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)C_{ud} + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2\right]$$

exist implies that

$$C = \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x}\left(\frac{B}{A+B}\right)^{N} f(S(N))$$

and

$$P = \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x}\left(\frac{B}{A+B}\right)^{N} f(S(N))\,.$$

Where$C$ *and* P the cost of call and put is is option respectively and $f(S(N))$ is the payoff. $\frac{\breve{A}}{\breve{A}+\breve{B}}$ and $\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)$ are the neutral probabilities.

Proof

We show that $C = e^{-r\Delta t}\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{uu} + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)C_{ud} + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2\right]$

For N= 2 we define

$$\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_1 = \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2, \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_2 = 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) \text{ and } \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)_3 = \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2.$$

Then

$$E_{\frac{\breve{A}}{\breve{A}+\breve{B}}} = \left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 f_{uu}S(2) + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) f_{ud}S(2) + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) f_{dd}S(2)\right]$$

$$= \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{uu} + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) C_{ud} + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{dd}$$

$$= C\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 u^2 + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) du + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 d^2\right]$$

$$= C\left[\frac{\breve{A}}{\breve{A}+\breve{B}} u + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) d\right]^2$$

$$=C\left[\frac{e^r-d}{u-d} u + \left(1-\frac{u-e^r}{u-d}\right) d\right]^2 = C\left[\frac{e^r u-du+du-e^r d}{u-d}\right]^2 = Ce^{2r}\left[\frac{u-d}{u-d}\right]^2.$$

By lemma 2.1, one obtains

$$E_{\frac{\breve{A}}{\breve{A}+\breve{B}}} = Ce^{r\Delta t}. \tag{21}$$

Making C the subject we obtained

$$C = e^{-r\Delta t}\left[\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 f_{uu}S(2) + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) f_{ud}S(2) + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) f_{dd}S(2)\right]\right]$$

$$= \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{uu} + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) C_{ud} + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{dd}$$

$$= e^{-r\Delta t}\left[\frac{\breve{A}}{\breve{A}+\breve{B}} u + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right) d\right]^2 f(S(2)).$$

By binomial expansion

$$C = e^{-r\Delta t}\sum_{x=0}^{N}\binom{N}{x}\frac{\breve{A}^x B^{N-x}}{(\breve{A}+\breve{B})^N} f(S(2))$$

$$= \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\binom{N}{x}\frac{\hat{A}^x \hat{B}^{N-x}}{(\hat{A}+\hat{B})^{(x)}(\hat{A}+\hat{B})^{(n-x)}} Max\left[u^x d^{N-x} S_{(0)} - K, 0\right].$$

Where $N \in \mathbb{N}$

$$= \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\left[\binom{N}{x}\left(\frac{N\breve{A}/_{\breve{A}+\breve{B}}}{n}\right)^x \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^N \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^{-x}\right] Max\left[u^x d^{N-x} S_{(0)} - K, 0\right]$$

$$= \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\left[\binom{N}{x}\left(\frac{N\breve{A}/_{\breve{A}+\breve{B}}}{N}\right)^x \frac{\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^N}{\left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^x}\right] Max\left[u^x d^{N-x} S_{(0)} - K, 0\right]$$

$$= \frac{1}{e^{r\Delta t}}\sum_{x=0}^{N}\left[\binom{N}{x}\left(\frac{N\breve{A}/_{\breve{A}+\breve{B}}}{N}\right)^x \frac{\left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N}{\left(\frac{N\breve{A}/_{\breve{A}+\breve{B}}}{\lambda}\right)^x}\right] Max\left[u^x d^{N-x} S_{(0)} - K, 0\right]$$

$$[u^x d^{N-x} S_{(0)} - K, 0]$$

$$= \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \left[\frac{N(N-1)(N-2).....(N-x+1)}{N^x} \frac{e^{-\lambda}\lambda^x}{x!} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N e^{\lambda} \right] Max\, [u^x d^{N-x} S_{(0)} - K, 0]$$

$$= \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{n}{x} \frac{e^{-\lambda}\lambda^x}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N e^{\lambda} \left[Max\, [u^x d^{N-x} S_{(0)} - K, 0]\right]. \qquad (22)$$

Where $\lambda = \frac{nA}{B}$ and $x \in (0,1,..n)$.

Put option follows exactly the same derivation as the call option (10) implies

$$P = \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{n}{x} \frac{e^{-\lambda}\lambda^x}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N e^{\lambda} \left[Max\, [K - u^x d^{N-x} S_{(0)}, 0]\right]. \quad =$$

$$\frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \left[\binom{N}{x} \left(\frac{N^x \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^x}{N^x} \right) \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N \times \frac{\lambda^x}{N^x \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^x} \right] Max\, [u^x d^{N-x} S_{(0)} - K, 0]$$

$$= \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \left[\binom{N}{x} \lambda^x \frac{N^x}{N^x} \left(\frac{\frac{\breve{A}}{\breve{A}+\breve{B}}}{N \frac{\breve{A}}{\breve{A}+\breve{B}}} \right)^x \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N \right] Max\, [u^x d^{N-x} S_{(0)} - K, 0]$$

$$= \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \left[\frac{N(N-1)(N-2).....(N-x+1)(N-x)..(2)(1)}{x(x-1)(x-2).....(2)(1).(n-x)..(2)(1)} \frac{\lambda^x e^{-\lambda}}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N e^{\lambda} \right] Max \quad (23)$$

4 NUMERICAL RESULTS

The following numerical results show to validate the theoretical results in comparison with CRR binomial model [2]

Example 4.1

Suppose a non –dividend pay stock is selling at Rs100 and stock's volatility is 24%. Assume that the continuously compounded risk-free is 5% .A European call and put option is offered on this stock and time of maturity is 4years and strike price Rs125. Calculate the prince of the options

Solution

$$S_0 = Rs100, \qquad K = Rs125\,, T = 4yrs\,, r = 0.05, \sigma = 0.24, \qquad u = \frac{1}{d} = 1.4041\,, d = 0.7122$$

$\frac{\breve{A}}{\breve{A}+\breve{B}} = 0.5680\,, \frac{\breve{B}}{\breve{A}+\breve{B}} = 0.4320$ and $\boldsymbol{\lambda} = 2.6296\,.\, fS(N) = max[u^x d^{N-x} S_0 - K, 0]$

$C_{uu} = 72.1479$, $C_{ud} = 0$, and $C_{dd} = 0$

By (22)

C= $e^{-r\Delta t} \sum_{x=0}^{N} \frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N Max\, [u^x d^{N-x} S_0 - K, 0]$

$$= e^{-0.1} \Big[2C_0 e^{-2.6296} \times \frac{(2.6296)^0}{2^0} \times e^{2.6298} \times (0.4320)^2 \times 0 + 2C_1 \times e^{-2.6296} \times \frac{(2.6296)^1}{2^1} \times e^{2.6296} \times (0.4320)^2$$
$$\times 0 + 2C_2 \times e^{-2.6296} \times \frac{(2.6296)^2}{2^2} \times e^{32.6296} \times (0.4320)^2 \times 96.0664 \Big] = Rs21.056$$

For put price by(23)

$$fS(N) = Max[K - u^x d^{N-x} S_0, 0]$$

$$P_{uu} = 0, P_{ud} = 25 \;, and\; P_{dd} = 74.2771$$

By $= e^{-r\Delta t} \sum_{x=0}^{N} \frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N Max\, [\, K - u^x d^{N-x} S_0, 0]$ we obtained

$$C = Rs\ 23.64$$

By CRR binomial model [4] for call price

$$C = e^{-r\Delta t}[\check{p}^2 C_{uu} + 2\check{p}(1-\check{p})C_{ud} + (1-\check{p})^2 C_{dd}] = Rs21.056$$

For Put price we have $fS(N) = max[K - u^x d^{N-x} S_0]$

$P_{uu} = 0, P_{ud} = 25$ and $P_{dd} = 74.2771$

By CRR binomial model [4] for put price

$$e^{-r\Delta t}[\check{p}^2 C_{uu} + 2\check{p}(1-\check{p})C_{ud} + (1-\check{p})^2 C_{dd}] = Rs23.64$$

Example 4.2 A non-dividend paying stock is currently selling at Rs100 with annual volatility 20% .Assume that the continuously compound risk free interest rate is 5% . Find the price of European call option on this stock with a strike price of Rs 80 and time to expiration 4 years .Using a two period CRR binomial option model and improved Poisson distribution model.

Solution

Given $S_0 = 100, K = 80, T = 4\,, r\ 0.05, \sigma = 0.2\,, \lambda = \frac{nA}{B} = 3.1706$ .Then a fixed up factor and down factor $u = e^{\sigma\sqrt{\Delta t}} = 1.3269,\ d = \frac{1}{u} = 0.7536\,, \frac{\breve{A}}{\breve{A}+\breve{B}} = 0.6132$ and $\frac{\breve{B}}{\breve{A}+\breve{B}} = 0.3868$

Now Payoff values $f(S(N)) = \mathrm{Max}\,[u^x d^{N-x} S_0 - K, 0]$

$C_{uu} = 96.0664$, $C_{ud} = 20$ and $C_{dd} = 0$

By (22) for call price

$$C = e^{-r\Delta t} \sum_{x=0}^{N} \frac{e^{-\lambda}\lambda^x e^{\lambda}}{N^x} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^N \mathrm{Max}\,[u^x d^{N-x} S_0 - K, 0]$$

$$= 2C_0 e^{-3.1706} \times \frac{(3.1706)^0}{2^0} \times e^{3.1706} \times (0.3868)^2 \times 20 + 2C_1 \times e^{-3.1706} \times \frac{(3.1706)^1}{2^1} \times e^{3.1706} \times (0.3868)^2 \times 20$$
$$+ 2C_2 \times e^{-3.1706} \times \frac{(3.1706)^2}{2^2} \times e^{3.1706} \times (0.3868)^2 \times 96.0664$$

$$C = e^{-0.1}[45.6090] = \mathrm{Rs}\ 41.27$$

By (6) Chandra [4] for call price.

$$C_{(0)} = e^{-r\Delta t}[\check{p}^2 C_{uu} + 2\check{p}(1-\check{p})C_{ud} + (1-\check{p})^2 C_{dd}]$$

$$= e^{-0.1}[(0.3760)(96.0664) + (0.4744)(20.00) + 0]$$

=Rs 41.27

To calculate the put price is left as an exercise for the reader.

5. DISSCUSION

From example 4.1-4.2, it was found that CRR binomial model discussed in Chandra[4] for two period model of non-dividend paying stock of a European (call and put) gives exactly the same numerical results with an improved Poisson distribution model, when equipped with financial terms, under the same conditions on its parameters.

6. CONCLUSION

The option pricing model for two period of non- dividend paying has been discussed literature. This work is much interested in a two period model by matching CRR model with a multi-period binomial model. The applicability of the model in literature works in good agreement with the proposed model. Thus problem of option pricing for two period of non-dividend paying stock of call and put option can be evaluated using an improved Poisson distribution with$\lambda = \frac{NA}{B}$ associated with financial terms, which is to be exercised only at expiration date. In comparison it gives the same numerical result faster than the models found in literature.

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