η-EINSTEIN SOLITONS IN $N(k)$-PARACONTACT METRIC MANIFOLDS

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Abstract. The objective of the present paper is to study the η-Einstein solitons on $N(k)$-Paracontact metric manifolds whose potential vector field is the Reeb vector field ξ if and only if the manifold is a para-Sasaki-Einstein. Also, admitting the Ricci Solitons under certain conditions.

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1. Introduction

In recent years the pioneering works of R. S. Hamilton [6] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$. When X is the gradient of a potential $\psi \in C^\infty(M)$, the soliton is called a gradient Ricci soliton [12] and the previous equation (1.1) takes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.2}$$

Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$Ric = \lambda g. \tag{1.3}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X = 0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by* [6]

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively. Also, soliton is called shrinking and it generates an ancient self-similar solution to the Ricci flow with finite extinction time.

It is well known that the quantity $a(g, \psi) := R + \left|\nabla\psi^2\right| - \psi$ must be constant on M and it is often called the *auxiliary constant*. When ψ is constant the gradient Ricci soliton is simply an *Einstein manifold*. Thus Ricci solitons are natural extensions of Einstein metrics, an Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons play an important role in Hamiltonian Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (1.3) is equivalent to ∞-Bakry-Emery Ricci tensor $Ric\psi = 0$. In physics, a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\psi = \lambda$g is called quasi-Einstein manifold. Therefore it is important to study geometry and topology of gradient Ricci solitons and their classifications.

In general one cannot expect potential function ψ to grow or decay linearly along all directions at infinity, because of the product property: the product of any two gradient steady Ricci solitons is also a gradient steady Ricci soliton. Consider for example (R, g, ψ), where g is the standard Euclidean metric, $\psi(x_1, x_2) = x_1$. ψ is constant along x_2 direction, so without additional conditions, ψ may not have linear growth at infinity.

In recent years much effort has been devoted to the classification of self-similar solutions of geometric flows. In 2016, G. Catino and L. Mazzieri introduced the notion of Einstein solitons [5], which generate self-similar solutions to Einstein flow

$$\frac{\partial g}{\partial t} = -2\left(S - \frac{scal}{2} g\right). \tag{1.5}$$

The interest in studying this equation from different points of view arises from concrete physical problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory. In what follows, after characterizing the manifold of constant scalar curvature via the existence of η-Einstein solitons. In the case when the potential vector field ξ is of gradient type i.e., $\xi = grad(f)$, for f a nonconstant smooth function on M and give the Laplacian equation satisfied by f. Under certain assumptions, the existence of an η-Einstein soliton implies that the manifold is quasi-Einstein. Remark that quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations.

In 1925, H. Levy [8] in Theorem: 4, proved that a second order parallel symmetric non-sigular tensor in real space forms is proportional the metric tensor. Later, R. Sharma [13] initiated the study of Ricci solitons in contact Riemannian geometry . In 2009, J. T. Cho and M. Kimura [4] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Recently, in 2018, A. M. Blaga study the notion of η-Einstein solitons [1]. M. D. Siddiqi also studied some properties of η-Einstein solitons in [11] which is closely related to this topic. It is natural and interesting to study η-Einstein solitons in $N(k)$-paracontact metric manifolds.

2. **Preliminaries**

A $(2n+1)$-dimensional smooth manifold M is said to be an almost paracontact manifold if it admits an almost paracontact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ a vector field and its dual 1-form η and for any vector field X on M satisfying [14]

$$\phi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi) = 0, \tag{2.2}$$

the tensor field ϕ induces an almost paracomplex structure on each fibre of $D = ker(\eta)$, that is, the eigen distributions D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact structure is said to be normal [14] if and only if the $(1,2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. If an almost paracontact manifold M equipped with a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on the manifold M, then (M, g) is called an almost paracontact metric manifold. An almost paracontact structure is said to be a paracontact structure if

$$g(X, \phi Y) = d\eta(X, Y) \tag{2.4}$$

where g is the associated metric [14]. For any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis [14], that is, a pseudo orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, ..., E_n, \phi E_1, \phi E_2, ..., \phi E_n\}$, where $\xi, E_1, E_2, ..., E_n$ are space-like vector fields and then, by (2.4) vector fields $\phi E_1, \phi E_2, ..., \phi E_n$ are time-like. In a paracontact metric manifold there exists a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\mathcal{L}_\xi \phi$ satisfying [14]

$$\phi h + h\phi = 0, \qquad h\xi = 0, \tag{2.5}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.6}$$

where ∇ is Levi-Civita connection of the pseudo-Riemannian manifold and for all $X \in \chi(M)$. It is clear that the tensor h satisfies $h = 0$ if and only if ξ is a Killing vector field and then (ϕ, ξ, η, g) is said to be a K-paracontact manifold. An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [14]

$$(\nabla_X \phi)Y = -g(X, Y) + \eta(Y)X \tag{2.7}$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -[\eta(Y)X - \eta(X)Y] \tag{2.8}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.8) does not imply that the paracontact manifold is para-Sasakian manifold. Every para Sasakian manifold is a K-paracontact manifold, but the converse is not always true, as it is shown in three dimensional case. Paracontact metric manifolds have been studied by Cappelletti-Montano et al. ([2], [3]), Martin-Molina ([9], [10]) and many others.

According to Cappelletti-Montano et al [2] we have the following definition.

Definition 2.1. *A paracontact metric manifold is said to be (k,μ)-paracontact manifold if the curvature tensor R satisfies*

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.9}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

In a (k,μ)-paracontact manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n > 1$, the following relations hold [7]:

$$h^2 = (k+1)\phi^2 \tag{2.10}$$

$$(\nabla_X \phi)Y = -g[X - hX, Y]\xi + \eta(Y)[X - hX], \qquad for \quad k \neq -1, \tag{2.11}$$

$$\begin{aligned}(\nabla_X h)Y = &-[(1+k)g(X,\phi Y) + g(X,\phi hY)]\xi \\ &+\eta(Y)\phi h(hX - X) - \mu\eta(X)\phi hY, \qquad for \quad k \neq -1,\end{aligned} \tag{2.12}$$

$$\begin{aligned}QX = &[2(n-1)+\mu]X + [2(n-1)+\mu]hY \\ &+[2(n-1)+n(2k-\mu)]\eta(X)\xi, \quad for \quad k \neq 1,,\end{aligned} \tag{2.13}$$

$$S(X,\xi) = 2nk\eta(X) \tag{2.14}$$

$$Q\xi = 2nk\xi \tag{2.15}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{2.16}$$

for any vector fields $X, Y \in \chi(M)$, where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$. Making use of (2.6) we have

$$(\nabla_X \eta)Y = g(X,\phi Y) + g(\phi hX, Y) \tag{2.17}$$

for all vector fields $X, Y \in \chi(M)$.

In particular, if $\mu = 0$, then the paracontact metric (k,μ)-manifold is called an $N(k)$-paracontact metric manifold. Thus for an $N(k)$-paracontact metric manifold we have

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{2.18}$$

for all $X, Y \in \chi(M)$.

In an $N(k)$-paracontact metric manifold (ϕ, ξ, η, g) the following relation hold (see [2], [3])

$$\begin{aligned}R(X,Y)Z = &\left(\frac{r}{2} - 2k\right)\{g(Y,Z)X - g(X,Z)Y\} \\ &+\left(3k - \frac{r}{2}\right)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\},\end{aligned} \tag{2.19}$$

$$S(X,Y) = \left(\frac{r}{2} - 2k\right) g(X,Y) + \left(3k - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{2.20}$$

$$QX = \left(\frac{r}{2} - 2k\right) X + \left(3k - \frac{r}{2}\right) \eta(X)\xi, \tag{2.21}$$

$$S(X,\xi) = 2k\eta(X) \tag{2.22}$$

where R, S, Q and r are the curvature tensor, Ricci operator, Ricci tensor, and the scalar curvature respectively. From (2.19) it follows that

$$R(\xi,X)Y = k\left\{g(X,Y)\xi - \eta(Y)X\right\}. \tag{2.23}$$

Also using (2.6) we have

$$(\nabla_X \eta)Y = g(X,\phi Y) - g(hX,\phi Y) \tag{2.24}$$

for all $X, Y \in \chi(M)$. Immediately from (2.19) we have the following:

Proposition 2.2. *A 3-dimensional $N(k)$-paracontact metric manifold is a manifold of constant curvature k if and only if the scalar curvature $r = 6k$.*

We recall a result due to Cappelletti-Montano et al (see [2], [3]).

Lemma 2.3. *Any paracontact metric $(k;\mu)$-manifold of dimension three is Einstein if and only if $k = \mu = 0$.*

Though any paracontact metric $(k;\mu)$-manifold of dimension three is Ein- stein if and only if $k = \mu = 0$, it always admits some compatible Einstein metrics [2].

3. η-Einstein solitons on $N(k)$-paracontact manifold

Let (M,ϕ,ξ,η,g) be a $N(k)$-paracontact manifold. Consider the equation [1]

$$\mathcal{L}_\xi g + 2S + (2\lambda - scal) + 2\mu\eta\otimes\eta = 0, \tag{3.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, *scal* is the scalar curvature of the Riemannian metric g and λ and μ are real constants. Writing $\mathcal{L}_\xi$ in terms of the Riemannian connection ∇, we obtain [1]:

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - (2\lambda - scal)g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X, Y \in \chi(M)$.

The data $(g,\xi,\lambda - \frac{scal}{2},\mu)$ which satisfy the equation (4.1) is said to be an η-*Einstein soliton* on M [1]. In particular if $\mu = 0$ then $(g,\xi,\lambda - \frac{scal}{2})$ is called *Ricci soliton* [13] and it is called *shrinking*, *steady* or *expanding*, according as λ is negative, zero or positive respectively [4].

Suppose that a 3-dimensional $N(k)$-paracontact metric manifold admits a η-Einstein soliton whose potential vector field is the Reeb vector field ξ. Then from (3.1) we get

$$g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) + 2S(X,Y) + (2\lambda - scal)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.3}$$

Taking into account of (2.6) the above equation implies

$$g(\phi hX,Y) + S(X,Y) + \left(\lambda - \frac{scal}{2}\right) g(X,Y) + \mu\eta(X)\eta(Y) = 0. \tag{3.4}$$

Replacing Y by ξ in the above equation gives

$$\left[\lambda - \frac{scal}{2} + 2k + \mu\right]\eta(X) = 0. \tag{3.5}$$

Putting $X = \xi$ in (3.5) to get

$$\lambda = -2k + \frac{scal}{2} - \mu. \tag{3.6}$$

Thus from (3.4) and (3.6) together gives

$$S(X,Y) = \left[2k - \lambda - \frac{scal}{2} + \mu\right] g(X,Y) - g(\phi hX, Y). \tag{3.7}$$

Replacing X by ϕX in (3.7) to get

$$S(\phi X,Y) = \left[2k - \lambda - \frac{scal}{2} + \mu\right] g(\phi X,Y) + g(\phi X, Y). \tag{3.8}$$

Also from (2.20) we obtain

$$S(\phi X,Y) = \left(\frac{r}{2} - k\right) g(\phi X,Y). \tag{3.9}$$

Equating the right hand side of (3.8) and (3.9) we get

$$g(hX,Y) = \left(\frac{r}{2} - 3k + \frac{scal}{2} - \mu\right) g(\phi X,Y). \tag{3.10}$$

Replacing X by Y in (3.10) yields

$$g(hY,X) = \left(\frac{r}{2} - 3k + \frac{scal}{2} - \mu\right) g(\phi Y,X). \tag{3.11}$$

Adding (3.10) and (3.11) we have

$$g(hX,Y) = 0, \tag{3.12}$$

which gives

$$h = 0. \tag{3.13}$$

Now $h = 0$ holds if and only if ξ is a Killing vector field and thus $M$is a K-paracontact metric manifold. Then equation (3.1) yields that M is η-Einstein. Also in dimension 3, a K-paracontact metric manifold is a para-Sasakian manifold. Thus M is a para-Sasaki η-Einstein manifold. The converse is trivial. Thus we can state the following:

Theorem 3.1. *A 3-dimensional N(k)-paracontact metric manifold admits a η-Einstein soliton whose potential vector field is the Reeb vector field ξ if and only if the manifold is a para-Sasaki-Einstein.*

Remark 7.2. [[3], Theorem 3.3] is a particular case of Theorem 7.1.

Corollary 3.2. *If a conformally at $N(k)$-paracontact metric manifold admits a η-Einstein soliton, then the manifold is a paraSasaki-Einstein.*

4. Parallel symmetric second order tensors and η-Einstein solitons in $N(k)$-paracontact manifolds

An important geometrical object in studying η-Einstein solitons is well known to be a symmetric $(0,2)$-tensor field which is parallel.

Now, let fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel ∇ that is $\nabla h = 0$. Applying Ricci identity [**?**]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;Z,W) = 0, \tag{4.1}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{4.2}$$

Replacing $Z = W = \xi$ in (4.2) and by using (2.18) and by the symmetry of h follows $h(R(X,Y)\xi,\xi) = 0$ for any $X,Y \in \chi(M)$ and

$$\begin{aligned} &k\eta(Y)h(X,\xi) - k\eta(X)h(Y,\xi) \\ &+k\eta(Y)h(\xi,X) - k\eta(X)h(\xi,Y) = 0. \end{aligned} \tag{4.3}$$

Putting $X = \xi$ in (4.3) we obtain

$$2k[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{4.4}$$

or

$$2k[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{4.5}$$

Since $2k \neq 0$, it results

$$\eta(Y)h(\xi,\xi) - h(Y,\xi) = 0, \tag{4.6}$$

for any $Y \in \chi(M)$, equivalent to

$$g(Y,\xi)h(\xi,\xi) - h(Y,\xi) = 0, \tag{4.7}$$

for any $Y \in \chi(M)$. Differentiating the equation (4.7) covariantly with respect to the vector field $X \in \chi(M)$, we obtain

$$h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) = 2kh(\xi,\xi)[g(\nabla_X Y,\xi) + g(Y,\nabla_X\xi)]. \tag{4.8}$$

Using (2.6) in (4.8), we obtain

$$\eta(Y)h(\xi,\xi) = h(Y,\xi) \tag{4.9}$$

for any $X,Y \in \chi(M)$. The above equation gives the conclusion:

Theorem 4.1. *Let $(M,\phi,\xi,\eta,g,)$ be a $N(k)$-paracontact manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field of type $(0,2)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇, then it is a constant multiple of the metric tensor g.*

On a $N(k)$-paracontact manifoldusing equation (2.6) and $\mathcal{L}_\xi g = 2g(\phi hX,Y)$, the equation (4.2) becomes:

$$\bar{S}(X,Y) = -(2k - \frac{scal}{2} + \mu)g(X,Y) - g(\phi hX,Y). \tag{4.10}$$

In particular, $X = \xi$, we obtain

$$\bar{S}(X,\xi) = -(2k + \lambda - \frac{scal}{2} + \mu)\eta(X). \tag{4.11}$$

In this case, the Ricci operator Q defined by $g(QX,Y) = S(X,Y)$ has the expression

$$\bar{Q}X = -(2k + \lambda - \frac{scal}{2} + \mu)X. \tag{4.12}$$

Remark that on a $N(k)$-paracontact manifold , the existence of an η-Einstein soliton implies that the characteristic vector field ξ is an eigenvector of Ricci operator corresponding to the eigenvalue $-(2k + \lambda + \mu - \frac{scal}{2})$.

Now we shall apply the previous results on η-Einstein solitons.

Theorem 4.2. *Let (M, ϕ, ξ, η, g) be a $N(k)$-paracontact manifold . Assume that the symmetric $(0,2)$-tensor filed $h = \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$ is parallel associated to g. Then $(g, \xi, -\frac{1}{2k}h(\xi,\xi), \mu)$ yields an η-Einstein soliton.*

Proof. Now, we can calculate

$$h(\xi,\xi) = \mathcal{L}_\xi g(\xi,\xi) + 2\bar{S}(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2k + \lambda + scal \tag{4.13}$$

so $\lambda = -\frac{1}{2k}[h(\xi,\xi) - sacl]$. From (5.9) we conclude that

$$h(X,Y) = -(2k + \lambda - scal)g(X,Y)$$

for any $X, Y \in \chi(M)$. Therefore

$$\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta = -(2k + \lambda - scal)g$$

. □

For $\mu = 0$ follows $\mathcal{L}_\xi g + 2S - S(\xi,\xi)g = 0$ and this gives

Corollary 4.3. *On a $N(k)$-paracontact manifold (M, ϕ, ξ, η, g) with property that the symmetric $(0,2)$-tensor field $h = \mathcal{L}_\xi g + 2S$ is parallel associated to g, the relation (4.1), for $\mu = 0$, defines a Ricci soliton.*

Conversely, we shall study the consequences of the existence of η-Einstein solitons on a $N(k)$-paracontact manifold . From (5.10) we give the conclusion:

Theorem 4.4. *If equation (4.1) define an η-Eintein soliton on a $N(k)$-paracontact manifold (M, ϕ, ξ, η, g) , then (M, g) is quasi-Einstein* [4].

Recall that the manifold is called *quasi-Einstein* if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η satisfying $\eta = g(X,\xi)$, for ξ a unit vector field and respectively, *Einstein* if S is collinear with g.

Theorem 4.5. *If (ϕ, ξ, η, g) is a $N(k)$-paracontact manifold on M and (4.1) defines an η-Einstein soliton on M, then*

(1) $Q \circ \phi = \phi \circ Q$
(2) *Q and S are parallel along ξ.*

Proof. The first statement follows from a direct computation and for the second one, note that

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X) \tag{4.14}$$

and

$$(\bar{\nabla}_\xi S)(X,Y) = \xi(S(X,Y)) - S(\bar{\nabla}_\xi X, Y) - S(X, \bar{\nabla}_\xi Y). \tag{4.15}$$

Replacing Q and S from (5.12) and (5.11) we get the conclusion. □

A particular case arise when the manifold is ϕ-Ricci symmetric, which means that $\phi^2 \circ \nabla Q = 0$, that fact stated in the next theorem.

Theorem 4.6. *Let (M, ϕ, ξ, η, g) be a $N(k)$-paracontact manifold . If M is ϕ-Ricci symmetric and (4.1) defines an η-Einstein soliton on M, then $\mu = 1$ and (M, g) is Einstein manifold.*

Proof. Replacing Q from (5.12) in (5.14) and applying ϕ^2 we obtain

$$(\mu - 2k - \frac{scal}{2})\eta(Y)[X - \eta(X)\xi] = 0, \tag{4.16}$$

for any $X, Y \in \chi(M)$. Follows $\mu = 2k + \frac{scal}{2}$ and $S = -(2k + \lambda + \mu + 1 - \frac{scal}{2})g$. □

Remark 4.7. *In particular, the existence of an η-Einstein soliton on a $N(k)$-paracontact manifold which is Ricci symmetric (i.e. $\nabla S = 0$) implies that M is Einstein manifold* [**?**]. *The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which belong also the locally symmetric manifold (i.e. satisfying $\nabla R = 0$). The condition $\nabla S = 0$ implies $\bar{R}.\bar{S} = 0$ and the manifolds satisfying this condition are called Ricci semi-symmetric.*

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