Identities of Choi-Lee-Srivastava involving the Euler-Mascheroni's constant

C. Hernández-Aguilar, J. López-Bonilla, R. López-Vázquez,

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 3er. Piso, Col. Lindavista CP 07738, CDMX, México; jlopezb@ipn.mx

Abstract:

We give an elementary deduction of the Choi-Lee-Srivastava's identities involving the Euler Mascheroni's constant, thus from them is immediate the identity of Wilf.

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1. Introduction

Wilf [1] proposed to prove the identity [2-4]:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\, e^{\gamma} \prod_{k=1}^{\infty} e^{-\frac{1}{k}} \left(1 + \frac{1}{k} + \frac{1}{2\,k^2}\right), \tag{1}$$

where γ =0.5772 1566 4901 5328 6060 … is the Euler-Mascheroni's constant [4, 5].

In Sec. 2 we use properties of the gamma function [4-9] to give an elementary deduction of (1), and in Sec. 3 this process allows generalize (1) to obtain the Choi-Lee-Srivastava's expressions [2]:

$$\cosh(\alpha\,\pi) = \pi\left(\alpha^2 + \frac{1}{4}\right) e^{\gamma} \prod_{j=1}^{\infty} e^{-\frac{1}{j}} \left(1 + \frac{1}{j} + \frac{\alpha^2 + \frac{1}{4}}{j^2}\right), \qquad \alpha \neq \pm\,\frac{i}{2}, \tag{2}$$

which implies (1) for $\alpha = \frac{1}{2}$, and:

$$\sinh(\beta\,\pi) = \beta\,\pi\,(\beta^2 + 1)\, e^{2\gamma} \prod_{j=1}^{\infty} e^{-\frac{2}{j}} \left(1 + \frac{2}{j} + \frac{\beta^2 + 1}{j^2}\right), \quad \beta \neq \pm\, i\,. \tag{3}$$

2. Wilf's formula

In [10] we find the following relation involving an infinite product and the gamma function:

$$\prod_{k=m}^{\infty} \frac{(k+z)^2 - b}{(k+z)^2 - a} = \frac{\Gamma(z+m-\sqrt{a})\,\Gamma(z+m+\sqrt{a})}{\Gamma(z+m-\sqrt{b})\,\Gamma(z+m+\sqrt{b})}, \tag{4}$$

where we can employ $a = -b = \frac{1}{4}$, $m = 1$ and $z = \frac{1}{2}$ to obtain the expression:

$$\prod_{k=1}^{\infty} \frac{(2k+1)^2 + 1}{(2k+1)^2 - 1} \equiv \prod_{k=1}^{\infty} \left[1 + \frac{1}{2k\,(k+1)}\right] = \frac{1}{\Gamma\left(\frac{3-i}{2}\right)\Gamma\left(\frac{3+i}{2}\right)} = \frac{2}{\Gamma\left(\frac{1-i}{2}\right)\Gamma\left(\frac{1+i}{2}\right)}. \tag{5}$$

On the other hand, we know the property [7]:

$$\frac{\pi}{\cosh(\pi\,x)} = \Gamma\left(\frac{1}{2} - i\,x\right)\Gamma\left(\frac{1}{2} + i\,x\right), \tag{6}$$

that we can apply with $x = \frac{1}{2}$ into (5) to deduce the identity:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \prod_{k=1}^{\infty} [1 + \frac{1}{2k\,(k+1)}]. \tag{7}$$

Now we observe the relation:

$$\frac{1 + \frac{1}{k} + \frac{1}{2k^2}}{1 + \frac{1}{k}} = 1 + \frac{1}{2k\,(k+1)}, \tag{8}$$

then (7) is equivalent to:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \frac{\prod_{k=1}^{\infty} e^{-\frac{1}{k}}(1 + \frac{1}{k} + \frac{1}{2k^2})}{\prod_{r=1}^{\infty} e^{-\frac{1}{r}}(1 + \frac{1}{r})}, \tag{9}$$

but the Newman (1848)-Weierstrass (1856) formula [7]:

$$z\, e^{\gamma z} \prod_{r=1}^{\infty} e^{-\frac{z}{r}} (1 + \frac{z}{r}) = \frac{1}{\Gamma(z)}, \tag{10}$$

with $z = 1$ gives the expression:

$$\prod_{r=1}^{\infty} e^{-\frac{1}{r}} \left(1 + \frac{1}{r}\right) = e^{-\gamma}, \tag{11}$$

whose application in (9) implies the Wilf's identity (1) [1-4, 11], q.e.d.

3. Choi-Lee-Srivasatava's relations

The process indicated in Sec. 2 permits to prove (2) and (3), in fact, we use (4) with $a = \frac{1}{4}$, $b = -\alpha^2$, $m = 1$, $z = \frac{1}{2}$, and (6) for $x = \alpha \neq \pm \frac{i}{2}$:

$$\prod_{k=1}^{\infty} \frac{(2k+1)^2 + 4\,\alpha^2}{(2k+1)^2 - 1} = \frac{\cosh(\alpha\,\pi)}{\pi\,(\alpha^2 + \frac{1}{4})}, \tag{12}$$

however, we have the property:

$$\frac{(2k+1)^2 + 4\,\alpha^2}{(2k+1)^2 - 1} = \frac{1 + \frac{1}{k} + \frac{\alpha^2 + \frac{1}{4}}{k^2}}{1 + \frac{1}{k}}, \tag{13}$$

hence the application of (11) and (13) into (12) implies (2), q.e.d.

Similarly, from (4) for $a = m = z = 1$, $b = -\beta^2$, and the companion relation of (6) if $\beta \neq \pm i$:

$$\frac{\pi}{\sinh(\pi\,\beta)} = \beta\, \Gamma(i\beta)\, \Gamma(-i\beta), \tag{14}$$

we deduce the expression:

$$\prod_{k=1}^{\infty} \frac{(k+1)^2 + \beta^2}{(k+1)^2 - 1} = \frac{2}{\beta\,(\beta^2 + 1)} \frac{\sinh(\beta\,\pi)}{\pi}, \tag{15}$$

but we have the decomposition:

$$\frac{(k+1)^2+\beta^2}{(k+1)^2-1} = \frac{1+\frac{2}{k}+\frac{\beta^2+1}{k^2}}{1+\frac{2}{k}} \,, \tag{16}$$

and from (10) with $z = 2$:

$$\prod_{r=1}^{\infty} e^{-\frac{2}{r}} \left(1+\frac{2}{r}\right) = \frac{1}{2}\, e^{-2\gamma} \,, \tag{17}$$

then the use of (16) and (17) into (15) gives (3), q.e.d.

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