Some Identities for Stirling Numbers

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Abstract

We Study The Identities For Stirling Numbers Obtained By Wildon, And Yuluklu *Et Al.*

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1. Introduction

Yuluklu-Simsek-Komatsu [1] Deduced The Identity:

$$A \equiv \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \, 2^{n-j} \, j! \; S_n^{(k)} \, S_k^{[j]} = (-1)^n \, n! \, , \tag{1}$$

Where $S_n^{(k)}$ And $S_k^{[j]}$ Are The Stirling Numbers Of The First And Second Kind, Respectively [2]. In Sec. 2 We Exhibit An Elementary Proof Of (1) And We Give An Extension Of It.

Wildon [3] Used The Technique Of Differentiation To Obtain The Following Relations:

$$\sum_{k=0}^{n} \binom{n}{k} S_k^{[m]} = S_{n+1}^{[m+1]} \, , \tag{2}$$

$$\sum_{k=0}^{n} (-1)^k \; k \, S_n^{(k)} = -S_{n+1}^{(2)} \, , \tag{3}$$

$$\sum_{k=0}^{n} (-1)^k \binom{k}{m} S_n^{(k)} = (-1)^m \, S_{n+1}^{(m+1)} \, , \tag{4}$$

$$C \equiv \sum_{k=0}^{n} \binom{n}{k} S_k^{[m]} \, B(n-k) = \sum_{r=0}^{n} \binom{r}{m} S_n^{[r]} \, , \tag{5}$$

With The Participation Of The Bell Numbers [2, 4-6]:

$$B(q) \equiv \sum_{j=0}^{q} S_q^{[j]} \, . \tag{6}$$

In Sec. 3 We Comment That The Identities (2), (3) And (4) Are Known In The Literature, And We Realize A Simple Demonstration Of (5).

2. Yuluklu *Et Al* Expression

We Have The Orthonormality Of The Stirling Numbers [2, 6]:

$$\sum_{k=j}^{n} S_n^{(k)}\, S_k^{[j]} = \delta_{jn}\,, \tag{7}$$

Then:

$$A = \sum_{j=0}^{n} (-1)^j\, 2^{n-j}\, j!\, \sum_{k=j}^{n} S_n^{(k)} S_k^{[j]} \overset{(7)}{=} (1)\;\; q.e.d.$$

Similarly:

$$D \equiv \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{j-k}\, 2^{n-j}\, j!\, S_n^{(k)}\, S_k^{[j]} = (-1)^n \sum_{j=0}^{n} (-1)^j\, 2^{n-j}\, j!\, L_{n,j}\,, \tag{8}$$

With The Presence Of The Lah Numbers [6-8]:

$$L_{n,j} \equiv \sum_{k=j}^{n} (-1)^{n-k}\, S_n^{(k)}\, S_k^{[j]} = \frac{n!}{j!}\binom{n-1}{j-1}, \tag{9}$$

Thus From (8):

$$D = (-2)^{n-1}\, n!\, \sum_{q=0}^{n-1} \binom{n-1}{q} (-\tfrac{1}{2})^q = (-1)^{n+1}\, n!\,. \tag{10}$$

The Identities (1) And (10) Imply The Result:

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{j-\varepsilon k}\, 2^{n-j}\, j!\, S_n^{(k)}\, S_k^{[j]} = \begin{cases} (-1)^n\, n!\,, & \varepsilon = 0, \\ (-1)^{n+1}\, n!\,, & \varepsilon = 1. \end{cases} \tag{11}$$

3. Wildon's Relations

The Property (2) Is The Equation (15.31) In [2], Also See [9]. The Relation (12.17) In [2] Gives The Following Expression For The Harmonic Numbers:

$$H_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k\, k\, S_n^{(k)}\,, \tag{12}$$

Besides, From [10] We Have That:

$$H_n = \frac{(-1)^{n+1}}{n!}\, S_{n+1}^{(2)}\,, \tag{13}$$

Hence (3) Is Consequence Of (12) And (13). The Identity (4) Is Deduced In [10].

From (9.25) In [2]:

$$D \equiv \sum_{k=0}^{n} \binom{n}{k}\, S_k^{[m]}\, S_{n-k}^{[j]} = \binom{m+j}{m}\, S_n^{[m+j]}\,, \tag{14}$$

Which Allows Consider The Left Member Of (5):

$$C \overset{(6)}{=} \sum_{j=0}^{n} D \overset{(14)}{=} \sum_{j=0}^{n} \binom{m+j}{m}\, S_n^{[m+j]} = \sum_{r=m}^{m+n} \binom{r}{m}\, S_n^{[r]}\,,$$

Equivalent To The Right Member Of (5), *Q.E.D.*

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