MISSING AT RANDOM IN NONPARAMETRIC REGRESSION FOR FUNCTIONAL STATIONARY ERGODIC DATA IN THE FUNCTIONAL INDEX MODEL

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ABSTRACT. The main objective of this paper is to estimate non-parametrically the the estimator for the regression function operator when the observations are linked with a single-index. The functional stationary ergodic data with missing at random (MAR) are considered. In particular, we construct the kernel type estimator of the regression operator, some asymptotic properties such as the convergence rate in probability as well as the asymptotic normality of the estimator are established under some mild conditions respectively. As an application, the asymptotic $(1-\zeta)$ confidence interval of the regression operator is also presented for $0 < \zeta < 1$.

1. INTRODUCTION

The focal point of this article is to study a nonparametric regression model in the case where the variable of interest Y (called response variable) is a scalar response variable and the explanatory variable X is of functional nature which takes values in some abstract infinite dimensional space $(\mathcal{H}, < \cdot, \theta >)$, and is linked with a single-index θ.

Let us consider the following functional nonparametric regression model:

$$Y = r(\theta, X) + \varepsilon \tag{1.1}$$

where $r(\theta, .)$ is an unknown smooth functional regression operator from $\mathcal{H}$ to $\mathbb{R}$, and ε is the random error with $\mathbb{E}(\varepsilon) = 0$ and $0 < Var(\varepsilon) < \infty$.

Compared with the classical nonparametric regression model,

$$Y = r(X) + \varepsilon,$$

that the explanatory variable is a real or finite dimensional case, where the explanatory variables X are often curves or surfaces, is widely applied in many fields such as in medicine, economics, environmetrics, chemometrics and others, The reason is that the data we observed or collected in these fields are exceptionally high-dimensional or even functional.

Let's not that the classical model was widely studied in Ferraty and Vieu (2000), Ferraty and Vieu (2002, 2003, 2004) and Ferraty *et al.* (2006), and the references therein, in the case that the samples are observed completely.

However, in many practical works such as sampling survey, pharmaceutical tracing test and reliability test and so on, some pairs of observations may be incomplete, which is often called the case of missing data. Many examples of missing data

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and its statistical inferences for regression model can be found in statistical literature when explanatory variables are of finite dimensionality (Cheng (1994), Little and Rubin (2002), Nittner (2003), Tsiatis (2006), Liang *et al.* (2007), Efromovich (2011a,b)) and references therein for details. When explanatory variables are in the case of infinite dimensionality or it is of functional nature, only very few literature was reported to investigate the statistical properties of functional nonparametric regression model for missing data.

Recently, Ferraty *et al.* (2013) first proposed to estimate the mean of a scalar response based on an i.i.d. functional sample in which explanatory variables are observed for every subject, while the response variables are missing at random by happenstance for some of them. It generalized the results in Cheng (1994) to the case where the explanatory variables are of functional nature.

The single-index models are becoming increasingly popular because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial econometric and so on. The single-index model, a special case of projection pursuit regression, has proven to be a very efficient way of coping with the high dimensional problem in nonparametric regression. Hardle *et al.* (1993), Hristache *et al.* (2001). Delecroix *et al.* (2003) have studied the estimation of the single-index approach of regression function and established some asymptotic properties. The first work in the fixed functional single-model was given by Ferraty *et al.* (2003), where authors obtained almost complete convergence (with the rate) of the regression function in the i.i.d. case. Their results have been extended to dependent case by Ait Saidi *et al.* (2005). Ait Saidi *et al.* (2008) studied the case where the functional single-index is unknown. The authors have proposed for this parameter an estimator, based on the the cross-validation procedure.

The goal of this paper is establish a nonparametric estimation on functional nonparametric regression model (1.1). At first an estimator of the regression operator in the functional single index, and of a scalar response and the functional covariate which are assumed to be sampled from a stationary and ergodic process is constructed. Meanwhile, the response variables are MAR but not the covariates are missing. Then, the asymptotic properties of the estimator are obtained under some mild conditions. To the best of our knowledge, the estimation of the nonparametric regression operator in the functional single index structure combining missing data and stationary ergodic processes with functional nature has not been studied in the statistical literature.

2. The model and the estimates

2.1. **The functional nonparametric framework.**

2.1.1. *The estimators.* The kernel estimator $r_n(\theta, x)$ of $r(\theta, x)$ is presented as follows:

$$\widetilde{r}_n(\theta, x) = \frac{\sum_{i=1}^{n} Y_i K\left(h_K^{-1}(< x - X_i, \theta >)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1}(< x - X_i, \theta >)\right)}, \tag{2.1}$$

where K is a kernel function, $h_K = h_{K,n}$ a sequence of positive real numbers.

$$\widehat{r}_n(\theta,x)=\frac{\sum_{i=1}^{n}\delta_i Y_i K\left(h_K^{-1}(<x-X_i,\theta>)\right)}{\sum_{i=1}^{n}\delta_i K\left(h_K^{-1}(<x-X_i,\theta>)\right)}=\frac{\widehat{r}_{n,2}(\theta,x)}{\widehat{r}_{n,1}(\theta,x)} \tag{2.2}$$

With

$$\widehat{r}_{n,j}(\theta,x) \;=\; \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\delta_i Y_i^{j-1}K_i(\theta,x) \tag{2.3}$$

Let

$$\overline{r}_{n,j}(\theta,x) \;=\; \frac{1}{n\,\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}\mathbb{E}(\delta_i Y_i^{j-1}K_i(\theta,x)/\mathcal{F}_{i-1}),\ \ j=1,2, \tag{2.4}$$

$$C_n(\theta,x)=\frac{\overline{r}_{n,2}(\theta,x)}{\overline{r}_{n,1}(\theta,x)} \tag{2.5}$$

and

$$B_n(\theta,x)=C_n(\theta,x)-r_n(\theta,x) \tag{2.6}$$

then

$$\hat{r}_n(\theta,x)-C_n(\theta,x)=\frac{Q_n(\theta,x)+R_n(\theta,x)}{\hat{r}_{n,1}(\theta,x)} \tag{2.7}$$

where

$$Q_n(\theta,x)=(\hat{r}_{n,2}(\theta,x)-\bar{r}_{n,2}(\theta,x))-r(\theta,x)(\hat{r}_{n,1}(\theta,x)-\bar{r}_{n,1}(\theta,x)) \tag{2.8}$$

and

$$R_n(\theta,x)=-B_n(\theta,x)(\hat{r}_{n,1}(\theta,x)-\bar{r}_{n,1}(\theta,x)) \tag{2.9}$$

Our results are stated under some mild assumptions we gather below for easy references. Throughout the paper, when no confusion will be possible, we will denote by C, C_0 some positive generic constants whose values are allowed to change.

(A1) Assumptions on the kernel function K

K is a nonnegative bounded kernel function with support $[0,1]$, and the derivativeK' exists on $[0,1]$ with $K'(t)<0$ for all $t\in[0,1]$ and $|\int_0^1(K^j)'(t)dt|<\infty$, for $j=1,2$.

(A2) Assumptions on the stationary ergodic nature

For x$\in\mathcal{H}$,there exist a sequence of nonnegative bounded random functions $(f_{i,1})_{i\geq 1}$, a sequence of random functions $(g_{i,x,\theta})_{i\geq 1}$ 1, a deterministic nonnegative bounded function f_1 and a nonnegative real function $\phi_\theta(\cdot)$ tending to zero, as its argument tends to 0, such that

(i) $F_{x,\theta}(t)=\phi_\theta(t)f_1(\theta,x)+\circ(\phi_\theta(t))$ as $t\to 0$.

(ii) For any i $\in\mathbb{N}$, $F_{x,\theta}^{\mathcal{F}_{i-1}}(t)=\phi_\theta(t)f_{i,1}(\theta,x)+g_{i,x,\theta}(t)$ with $g_{i,x,\theta}=\circ_{a.s}(\phi(t))$ as $t\to 0$, $\frac{g_{i,x,\theta}(t)}{\phi_\theta(t)}$ almost surely bounded and $n^{-1}\sum_{i=1}^{n}g_{i,x,\theta}^j(t)=o_{a.s}(\phi_\theta^j(t))$ $asn\to\infty$ for $j=1,2$.

(iii) $n^{-1}\sum_{i=1}^{n}f_{i,1}^j(\theta,x)\to f_1^j(\theta,x)$ almost surely as $n\to\infty$ for $j=1,2$.

(iv) There exists a nondecreasing bounded function τ_0 such that, uniformly in t $\in [0,1]$, $\frac{\phi_\theta(ht)}{\phi_\theta(h)} = \tau_0 + o(1)$,as $h \downarrow 0$.and $\int_0^1 (K^j)' \tau_0(t)dt < \infty$ for $j \geq 1$.

(A3) Assumptions on the conditional moments

(i) The conditional mean of Y_i given the σ-field g_{i-1} depends only on X_i, i.e, for any $i \geq 1$, $\mathbb{E}(Y_i|g_{i-1}) = \mathbb{E}(Y_i|X_i) = r(X_i)$, a.s.

(ii) For any $i \geq 1, E[(Y_i - r(X_I))^2|g_{i-1}] = E[(Y_i - r(X_I))^2|X_i] = V(X_i)$,a.s.

(A4) Local smoothness and continuous conditions

(i) $\exists\beta > 0$ and a constant $C > 0$ such that $|r(u) - r(v)| \leq Cd(u,v)^\beta$ for all $(u,v) \in \mathcal{H} \times \mathcal{H}$

(ii) $V(\cdot)$ and $P(\cdot)$ are continuous in a neighborhood of x respectively, that is as $h \to 0$

$$\sup_{u:<x-u,\theta>\leq h} |V(u) - V(\theta,x)| = o(1),$$
$$\sup_{u:<x-u,\theta>\leq h} |p(u) - p(\theta,x)| = o(1).$$

(iii) $\exists\delta > 0$: $E|Y_1|^{2+\delta} < \infty$, and let $\overline{W}_{2+\delta}(u) = E\big(|\big(Y_1 - r(\theta,x)\big)|^{2+\delta}|X_1 = u\big)$ be continuous in a neighborhood of (θ,x) for $u \in \mathcal{H}$

3. Asymptotic properties

In this section, we show some asymptotic properties of the estimator $\hat{r}_n(\theta,x)$ for the regression operator in the model (2.1) based on the functional stationary ergodic data with MAR. More precisely, Theorem (3.1) shows the convergence rate in probability of the estimator. The asymptotic distribution of the estimator is presented in Theorem (3.2).

Theorem 3.1. *Under assumptions (A1)-(A4)(i),*

(a) *If*

$$\frac{n\phi_\theta(h)}{loglog(n)} \to \infty, asn \to \infty \tag{3.1}$$

for any $x \in \mathcal{H}$ such that$f_1(\theta,x) > 0$,then we have

$$\left(\frac{n\phi_\theta(h)}{loglog(n)}\right)^{\frac{1}{2}} (\hat{r}_n(\theta,x) - C_n(\theta,x)) \xrightarrow{p} 0. \tag{3.2}$$

(b) *In addition, if*

$$\frac{n\phi_\theta(h)h^{2\beta}}{loglog(n)} \to 0.asn \to \infty, \tag{3.3}$$

where is satisfied in (A4)(i),then we have

$$\left(\frac{n\phi_\theta(h)}{loglog(n)}\right)^{\frac{1}{2}} (\hat{r}_n(\theta,x) - r(\theta,x)) \xrightarrow{p} 0 \tag{3.4}$$

where $\xrightarrow{p}$means the convergence in probability.

Theorem 3.2. *Under assumptions (A1)-(A4),*

(a) *If*

$$n\phi_\theta(h) \to \infty, as n \to \infty, \tag{3.5}$$

for any $x \in \mathcal{H}$ *such that* $f_1(\theta, x) > 0$*, then we have*

$$\sqrt{n\phi_\theta(h)}(\hat{r}_n(\theta, x) - C_n(\theta, x)) \xrightarrow{D} N(0, \sigma^2(\theta, x)) \tag{3.6}$$

where $\xrightarrow{D}$ *means the convergence in distribution and* $\sigma^2(\theta, x) = \frac{M_2}{M_1^2} \frac{V(\theta,x)}{P(\theta,x) f_1(\theta,x)}$ *with* $M_j = K^j(1) - \int_0^1 (K^j)'(u)\tau_0(u)du$ *for* $j = 1, 2$.

(b) *In addition ,if*

$$h^\beta \big(n\phi_\theta(h)\big)^{\frac{1}{2}} \to 0, \quad as n \to \infty, \tag{3.7}$$

where β *is specified in (A4)(i),then we have*

$$\sqrt{n\phi_\theta(h)}(\hat{r}_n(x, \theta) - r(x, \theta)) \xrightarrow{D} N(0, \sigma^2(x)) \tag{3.8}$$

3.1. **Remarks on the assumptions.** Similar to the discussions in Laib and Louani (2010, 2011), (A1), (A4)(i) are the quite usual conditions on the kernel function and regression operator for nonparametric functional data analysis. (A2) shows the ergodic nature of the data and the small ball techniques used in this paper. Assumption (A3) on condition moment shows the Markovian nature of the functional stationary ergodic data. (A4)(ii) and (A4)(iii) stand as local continuous conditions, which is necessary to establish the main results and make the results concise in this paper.

It is worth being noted that the results in our work extend the complete data in Laib and Louani (2010, 2011) to MAR case. On the other hand, as for the asymptotic normality, we also solve the second important open issue in MAR modeling proposed by Ferraty *et al.* (2013). In fact, the limiting variance in Theorem 3.2 contains the unknown function operator $f_1(\cdot), V(\cdot), P(\cdot)$ and unknown parameter M_j for $j = 1, 2$. respectively. Meanwhile, the normalization depends on the function $\phi_\theta(\cdot)$which is also not identifiable explicitly. Therefore, we have to estimate them respectively so as to obtain asymptotic confidence interval of $r(\theta, x)$ in practice. First, the estimator of the conditional variance $V(\theta, x)$ can be defined as:

$$\begin{aligned} V_n(\theta, x) &= \frac{\sum_{i=1}^n (\delta_i Y_i - \hat{r}_n(\theta, x))^2 K\left(\frac{< x - X_i, \theta >}{h}\right)}{\sum_{i=1}^n \delta_i K\left(\frac{< x - X_i, \theta >}{h}\right)} \\ &= \frac{\sum_{i=1}^n \delta_i Y_i^2 K\left(\frac{< x - X_i, \theta >}{h}\right)}{\sum_{i=1}^n \delta_i K\left(\frac{< x - X_i, \theta >}{h}\right)} - \big(\hat{r}_n(\theta, x)\big)^2 \\ &= \hat{g}_n(\theta, x) - \big(\hat{r}_n(\theta, x)\big)^2. \end{aligned} \tag{3.9}$$

Second, by the assumptions (A2)(i) and (A2)(iv), the estimator of $\tau_0(x)$is defined as

$$\tau_n(u) = \frac{F_{\theta,x,n}(uh)}{F_{\theta,x,n}(h)},$$

where

$$F_{\theta,x,n}(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{<x-X_i,\theta>\leq u\}}.$$

Can be used to estimate $\phi_\theta(h)$). Therefore, for a given kernel K, the estimator of M_1 and M_2, namely $M_{1,n}$ and $M_{2,n}$ respectively, is obtained by replacing τ_0 with τ_n in their respective expressions. Finally, the estimator of P(x) is denoted by

$$P_n(\theta, x) = \frac{\sum_{i=1}^{n} \delta_i K\left(\frac{< x - X_i, \theta >}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{< x - X_i, \theta >}{h}\right)} \tag{3.10}$$

Then, the following Corollary is obtained immediately.

Corollary 3.1. *Under assumption (A1)-(A4), let K' and $(K^2)'$ be integral functions and*

$$nF_{x,\theta}(h) \longrightarrow \infty,\ h^{\beta}\big(nF_{\theta,x}(h)\big)^2 \longrightarrow 0,\ \text{as } n \to \infty \tag{3.11}$$

for any $x \in \mathcal{H}$ such that $f_1(\theta, x) > 0$,then we have

$$\frac{M_{1,n}}{\sqrt{M_{2,n}}}\sqrt{\frac{nF_{x,\theta}(h)}{V_n(\theta, x), \theta}}\big(\hat{r}_n(\theta, x) - r(\theta, x)\big) \xrightarrow{D} \mathcal{N}(0, 1) \tag{3.12}$$

thus, by (3.12),the asymptotic $(1-\zeta)$ confidence interval for the regression function operator $r(x)$ is given by

$$\hat{r}_n(\theta, x) \pm \mu_{\frac{\zeta}{2}} \frac{\sqrt{M_{2,n}}}{M_{1,n}}\sqrt{\frac{V_n(\theta, x)}{nF_{\theta,x}(h)P_n(\theta, x)}},$$

where $\mu_{\frac{\zeta}{2}}$ is the upper $\frac{\zeta}{2}$ quantile of the Normal distribution $\mathcal{N}(0, 1)$.

4. Proofs of some lemmas and main results

In this section, we first present some lemmas and their proofs which are necessary to establish our main results.

Lemma 4.1. *Assume that assumptions (A1) and (A2)(i)(ii)(iv) hold true. For any real numbers $1 \leq j \leq 2+\delta$ and $1 \leq k \leq 2+\delta$ with $\delta > 0$, as $n \to \infty$, we have*

(i) $\frac{1}{\phi_\theta(h)} E\big[K_i^j(\theta, x)|\mathcal{F}_{i-1}\big] = M_j f_{i,1}(\theta, x) + O_{a.s}\left(\frac{\mathrm{gi}_{\theta,\mathrm{x}}(\theta,\mathrm{x})}{\phi_\theta(h)}\right)$.
(ii) $\frac{1}{\phi_\theta(h)} E\big[K_i^j(\theta, x)\big] = M_j f_1(\theta, x) + o(1)$.
(iii) $\frac{1}{\phi_\theta^k}\big(E(K_1^j(\theta, x))\big)^k = M_1^k f_1^k(\theta, x) + o(1)$.

Proof of Lemme 4.1. See the proof of Lemma 1 in Laib and Louani (2010). □

Lemma 4.2. *Under the assumptions (A1)-(A2) and the condition (3.5), for any* $(\theta, x) \in \mathcal{H} \times \times$ *such that* $f_1(\theta, x) > 0$ *we have*

$$\hat{r}_{n,1}(\theta, x) \xrightarrow{P} P(\theta, x), as n \to \infty. \tag{4.1}$$

Proof of Lemme 4.2. By (2.3) we have the decomposition as follows

$$\hat{r}_{n,1}(\theta, x) = R_{n,1}(\theta, x) + \bar{r}_{n,1}(\theta, x). \tag{4.2}$$

where

$$R_{n,1}(\theta, x) = \frac{1}{nE(K_1(\theta, x))} \sum_{i=1}^{n} \big(\delta_i K_i(\theta, x) - E\big[\delta_i K_i(\theta, x)|\mathcal{F}_{i-1}\big]\big)$$

and

$$\bar{r}_{n,1}(\theta, x) = \frac{1}{nE\big(K_1(\theta, x)\big)} \sum_{i=1}^{n} E\big[\delta_i K_i(\theta, x)|\mathcal{F}_{i-1}\big]$$

First, we need to establish

$$\bar{r}_{n,1}(\theta, x) \xrightarrow{P} P(\theta, x), as n \to \infty. \tag{4.3}$$

By the properties of conditional expectation and the mechanism of MAR, combining the assumptions (A2)(ii)(iii), (A3) and the continuous property of $p(\theta, x)$ with Lemma (4.1), we have

$$\begin{aligned}
\bar{r}_{n,1}(\theta, x) &= \tfrac{1}{nE\left(K_1(\theta,x)\right)} \sum_{i=1}^{n} E\big[E[(\delta_i K_i(\theta, x)|\mathcal{F}_{i-1}]|\mathrm{g}_{i-1}\big] \\
&= \frac{1}{nE\big(K_1(\theta, x)\big)} \sum_{i=1}^{n} E\big[P(\theta, x) + o(1)\delta_i(\theta, x)|\mathcal{F}_{i-1}\big] \\
&=\big(P(\theta, x) + o(1)\big) \frac{1}{nE\big(K_1(\theta, x)\big)} \sum_{i=1}^{n} E[K_i(\theta, x)|\mathcal{F}_{i-1}] \\
&=\big(P(\theta, x) + o(1)\big) \frac{1}{nE\big(K_1(\theta, x)\big)} \sum_{i=1}^{n} \Big(\phi_\theta(h) M_1 f_{i1}(\theta, x) + 0_{a.s}\big(\mathrm{g}i_{\theta,x}\big)\Big) \\
&=\big(P(\theta, x) + o(1)\big) \frac{\phi_\theta(h)}{E\big(K_1(\theta, x)\big)} \left(\frac{1}{n} \sum_{i=1}^{n} M_1 f_{i1}(\theta, x) + \frac{1}{n} \sum_{i=1}^{n} O_{a.s}\left(\frac{\mathrm{g}i_{\theta,x}(h)}{\phi_\theta(h)}\right)\right) \\
&=\big(P(\theta, x) + o(1)\big) \frac{1}{M_1 f_1(\theta, x) + o(1)} \Big(M_1\big(f_1(\theta, x) + o(1)\big) + O_{a.s}(1)\Big) \\
&\to P(\theta, x) a.s, as n \to \infty
\end{aligned}$$

Second, we will prove that

$$R_{n,1}(\theta, x) \xrightarrow{P} 0 as\, n \to \infty \tag{4.4}$$

On the one hand, we denote $\eta_{n.i}(\theta, x) = \delta_i K_i(\theta, x) - E\big(\delta_i K_i(\theta, x)|\mathcal{F}_{i-1}\big)$.

Then,$(\eta_{n.i},\ 1 \le i \le n)$ forms a triangular array of martingale differences with respect to the $\sigma - field\ \mathcal{F}_{i-1}$ and

$$R_{n,1}(\theta, x) = \frac{1}{nE\big(K_1(\theta, x)\big)} \sum_{i=1}^{n} \eta_{n,i}(\theta, x).$$

On the other hand, by Burkholders inequality of martingale differences (Hall and Heyde, 1980), we have, as $n \to \infty$

$$\begin{aligned} P\big(|R_{n,1}(\theta,x)| > \varepsilon\big) &= P\bigg(\bigg|\sum_{i=1}^{n} \eta_{n.i}(\theta,x)\bigg| > \varepsilon n E\big(K_1(\theta,x)\big)\bigg) \\ &\leq C_0 \frac{E\eta_{n.i}^2(\theta,x)}{\varepsilon^2 n\big(E(K_1(\theta,x)\big)} \\ &< C_0 \frac{E\big(\delta_1 K_1^2(\theta,x)\big)}{\varepsilon^2 n E(K_1^2(\theta,x))} \to 0, \end{aligned}$$

which means that (4.4) is correct. Finally, (4.1) follows from (4.2) to (4.4).

□

Lemma 4.3. *Under the assumptions (A1)-(A2), (A3)(i), (A4)(i) and the condition (3.5), for any for any $x \in \mathcal{H}$ such that $f_1(\theta,x) > 0$, we have*

$$B_n(\theta,x) = 0_p(h^{\beta}) \tag{4.5}$$

and

$$\sqrt{n\phi_\theta(h)} R_n(\theta,x) \xrightarrow{P} 0, asn \to \infty \tag{4.6}$$

Proof of Lemme 4.3. First, by (2.5) and (2.6), we have

$$B_n(\theta,x) = \frac{\bar{r}_{n,2}(\theta,x) - r(\theta,x)\bar{r}_{n,1}(\theta,x)}{\bar{r}_{n,1}} := \frac{B_n(\bar{\theta},x)}{\bar{r}_{n,1}(\theta,x)}$$

Then by (4.3), we need to show that

$$\overline{B}_n(\theta,x) = \bar{r}_{n,2}(\theta,x) - r(\theta,x)\bar{r}_{n,1}(\theta,x) = 0_{a.s}(h^{\beta}) \tag{4.7}$$

In fact, by the assumptions (A3)(i) and (A4)(i), similar to the proof of Lemma (4.2), it follows that

$$\begin{aligned} |\overline{B}_n(\theta,x)| &= \bigg|\frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} E\big[\big(Y_i - r(\theta,x)\big)\delta_i K_i(\theta,x)|\mathcal{F}_{i-1}\big]\bigg| \\ &= \bigg|\frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} E\Big[E\big[(Y_i - r(\theta,x))\delta_i K_i(\theta,x)|\mathrm{g}_{i-1}\big]|\mathcal{F}_{i-1}\Big]\bigg| \\ &= \bigg|\frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} E\Big[E\big[(Y_i - r(\theta,x))\delta_i K_i(\theta,x)|X_i\big]|\mathcal{F}_{i-1}\Big]\bigg| \\ &= \bigg|\frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} E\big[(r(\theta,X_i) - r(\theta,x))P(X_i)K_i(\theta,x)|X_i\big]|\mathcal{F}_{i-1}\bigg| \\ &\leq \sup_{u\in B(\theta,x,h)} |r(u) - r(\theta,x)|.\bigg|\frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} E\bigg(P(X_i)K_i(\theta,x)|\mathcal{F}_{i-1}\bigg)\bigg| \\ &= O_{a.s}(h^{\beta}). \end{aligned}$$

Thus, (4.5) follows from (4.3) and (4.7).

Finally, in order to establish (4.6), observe that

$$\hat{r}_{n,1}(\theta,x) - \bar{r}_{n,1}(\theta,x) = \frac{1}{nE\big(K_1(\theta,x)\big)} \sum_{i=1}^{n} \eta_{n.i}(\theta,x)$$

is a summation of a martingale difference $\eta_{n.i}, 1 \le i \le n$. Following the same steps as that in Laib and Louani (2010), if we establish that

$$\sqrt{n\phi_\theta(h)}\big(\hat{r}_{n,1}(\theta,x) - \bar{r}_{n,1}(\theta,x)\big) \xrightarrow{D} N(0, \rho^2(\theta,x)), \tag{4.8}$$

where $\rho(\theta,x) = \frac{M_2}{M_1^2}\frac{P(\theta,x)}{f_1(\theta,x)}$,then by (4.8), (4.5) and (2.9), (4.6) is follows.

In fact, the proof of (4.8) is similar to that of Lemma (4.4) which establishes the asymptotic normality of $Q_n(\theta,x)$. □

Lemma 4.4. *Under the assumptions (A1)-(A4) and the condition (3.5), for any $(\theta,x) \in \mathcal{H}$ such that $f_1(\theta,x) > 0$, we have*

$$\sqrt{n\phi_\theta(h)}Q_n(\theta,x) \xrightarrow{D} N(0, \sigma_0^2(\theta,x)). \tag{4.9}$$

where

$$\sigma_0^2 = \frac{M_2}{M_1^2}\frac{P(\theta,x)V(\theta,x)}{f_1(\theta,x)}.$$

Proof of Lemme 4.4. Let's denote

$$\zeta_{ni} = \big(\frac{\phi(h)}{n}\big)^{\frac{1}{2}}\delta_i(Y_i - r(\theta,x))\frac{K_i(\theta,x)}{E\big(K_1(\theta,x)\big)}$$

and

$$\xi_{ni} = \zeta_{ni} - E[\zeta_{ni}^2|\mathcal{F}_{i-1}].$$

It is easy to see that

$$\big(n\phi_\theta(h)\big)^2 Q_n(\theta,x) = \sum_{i=1}^{n}\xi_{ni}. \tag{4.10}$$

Thus the $\xi_{ni}, 1 \le i \le n$ forms a triangular array of stationary martingale differences with respect to the $\sigma - field\ \mathcal{F}_{i-1}$. Similar to the proof of Lemma 2.4 in Laib and Louani (2010), we apply the central limit theorem for discrete-time arrays of real-valued martingales (Hall and Heyde, 1980) to obtain the asymptotic normality of $Q_n(\theta,x)$. Therefore, we have to establish the following statements:

(a) $\sum_{i=1}^n E[\xi_{ni}|\mathcal{F}_{i-1}] \xrightarrow{P} \sigma_0^2(\theta,x)$

(b) $nE\big[\xi_{ni}^2 I_{[|\xi_{ni}|>\varepsilon]}\big] = o(1) for \forall\varepsilon > 0.$

proof of part (a) Observe that

$$\Big|\sum_{i=1}^{n}E\big[\zeta_{ni}^2|\mathcal{F}_{i-1}\big] - \sum_{1}^{n}E\big[\xi_{ni}^2|\mathcal{F}_{i-1}\big]\Big| \le \sum_{1}^{n}\big(E\big[\zeta_{ni}|\mathcal{F}_{i-1}\big]\big)^2 \tag{4.11}$$

By (A4)(i), the continuous condition of $P(\theta,x)$ and Lemma 4.1, we obtain that

$$\begin{aligned}
\big|E[\zeta_{ni}|\mathcal{F}_{i-1}]\big| &= \frac{\big(\frac{\phi_\theta(h)}{n}\big)^{\frac{1}{2}}}{E\big(K_1(\theta,x)\big)}\Big|E\big[(r(X_i) - r(\theta,x))P(X_i)K_i(\theta,x)|\mathcal{F}_{i-1}\big]\Big| \\
&\le \frac{\big(\frac{\phi_\theta(h)}{n}\big)^{\frac{1}{2}}}{E\big(K_1(\theta,x)\big)}\sup_{u\in B(\theta,x,h)}\big|r(u) - r(\theta,x)\big|E\big(K_i(\theta,x)|\mathcal{F}_{i-1}\big)h^\beta\big(o(1) + P(\theta,x)\big) \\
&\le C\Big(\frac{\phi_\theta(h)}{n}\Big)^{\frac{1}{2}}\Big(\frac{f_{i1}(\theta,x)}{f_1(\theta,x)} + O_{a.s}\Big(\frac{\mathrm{gi}_{(\theta,x)}}{\phi_n(h)}\Big)\Big)
\end{aligned}$$

Thus, by (A2)(ii) and (A2)(iii), we have
$\sum_{i=1}^{n}\big(E[\zeta_{ni}|\mathcal{F}_{i-1}]\big)^{\frac{1}{2}} \leq O_{a.s}\big(h^{2\beta}\phi_\theta(h)\big)\bigg(\frac{1}{f_1^2(\theta,x)}\frac{1}{n}\sum_{i=1}^{n} f_{i1}^2(\theta,x) + O_{a.s}(1)\bigg)\big(o(1) + P(\theta,x)\big)^2.$

Hence, the statement (a) follows if we show that

$$\sum_{i=1}^{n} E[\zeta_{ni}^2 F_{i-1}] \xrightarrow{P} \sigma_0^2(\theta,x) \tag{4.12}$$

To establish (4.12), we have the decomposition as follows

$$\sum_{i=1}^{n} E[\zeta_{ni}^2|\mathcal{F}_{i-1}] = \frac{\phi_\theta(h)}{n\big(EK_i(\theta,x)\big)^2}\sum_{i=1}^{n} E\big[\big(Y_i - r(\theta,x)\big)^2\delta_i K_i^2(\theta,x)|\mathcal{F}_{i-1}\big] = J_{1n} + J_{2n}, \tag{4.13}$$

where

$$J_{1n} = \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\big[\big(Y_i - r(X_i)\big)^2\delta_i K_i^2(\theta,x)|\mathcal{F}_{i-1}\big]$$

and

$$J_{2n} = \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\big[\big(r(X_i) - r((\theta,x)\big)^2\delta_i K_i^2(\theta,x)|\mathcal{F}_{i-1}\big]$$

Thus, by the properties of conditional expectation, we obtain that

$$\begin{aligned} J_{1n} &= \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\Big[E\big[\big(Y_i - r(X_i)\big)^2\delta_i K_i^2(\theta,x)|\mathrm{g}_{i-1}\big]|\mathcal{F}_{i-1}\Big] \\ &= \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\Big[K_i^2(\theta,x)E\big[\big(Y_i - r(X_i)\big)^2\delta_i|X_i\big]|\mathcal{F}_{i-1}\Big] \\ &= \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\big[V(X_i)P(X_i)K_i^2(\theta,x)|\mathcal{F}_{i-1}\big] \end{aligned}$$

Then, by A2(ii) and smoothness conditions (A4) as well as Lemma (4.1), we have that

$$\begin{aligned} J_{1n} &= \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\Big[\big(o(1)+V(\theta,x)\big)\big(o(1)+P(\theta,x)\big)K_i^2|\mathcal{F}_{i-1}\Big] \\ &= \frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n}\big(o(1)+V(\theta,x)\big)\big(o(1)+P(\theta,x)\big)\big(M_2\phi_\theta(h)f_{i1}(\theta,x) + O_{as}(\mathrm{gi}_{(\theta,x)}(h))\big) \\ &\to \frac{M_2V(\theta,x)P(\theta,x)}{M_1^2 f_{i1}(\theta,x)} =: \sigma_0^2(\theta,x) \end{aligned} \tag{4.14}$$

Similarly, by the assumptions (A2)(ii)(iii) and (A4)(i) together with Lemma **??** again, it follows that

$$\begin{aligned} J_{2n} &= O(h^{2\beta})\frac{\phi_\theta(h)}{n\big(EK_1(\theta,x)\big)^2}\sum_{i=1}^{n} E\big[\delta_i K_i^2(\theta,x)|\mathcal{F}_{i-1}\big] \\ &\leq O(h^{2\beta})\bigg(\frac{M_2}{M_1^2}\frac{1}{f_1(\theta,x)} + 0_{a.s}(1)\bigg) \longrightarrow 0 \text{ a.s } n\to\infty \end{aligned} \tag{4.15}$$

Finally, by (4.13)-(4.15), (4.12) is valid.

Proof of part (b).

The proof of this part is also similar to that in Laib and Louani (2010). In fact, by the definition of ξ_{ni}, we have $nE\xi_{ni}^2 I_{[}|\xi_{ni}| > \varepsilon] \leq 4nE\big[\zeta_{ni}^2 I_{[}|\zeta_{ni}| > \frac{\varepsilon}{2}\big]$, where I_A is an indicator function of a set A. Let $a > 1$ and $b > 1$ such that $\frac{1}{a} + \frac{1}{b} = 1$. By Hölder and Markov inequalities, one can write, for all $\varepsilon > 0$,

$$E\left[\zeta_{ni}^2 I_{\left(|\zeta_{ni}|>\frac{\varepsilon}{2}\right)}\right] \leq \frac{E|\zeta_{ni}|^{2a}}{\left(\frac{\varepsilon}{2}\right)^{\frac{2a}{b}}} \tag{4.16}$$

Taking C_0 a positive constant and $2a = 2 + \delta$ (with δ as in (A4)(iii)), by the local continuous condition, we can obtain

$$4nE\big[\zeta_{ni}^2 I_{[}|\zeta_{ni}| > \tfrac{\varepsilon}{2}\big] \leq C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}} \frac{n}{\Big(E\big(K_1(\theta,x)\big)\Big)^{2+\delta}} E\Big(\big[|Y_i - r(X_i|\big)^2 \delta_i K_i^2(\theta,x)\big]^{2+\delta}\Big)$$

$$\leq C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}} \frac{n}{\Big(E\big(K_1(\theta,x)\big)\Big)^{2+\delta}} E\Big(E\big[|Y_i - r(X_i)|^{2+\delta}\delta_i\big(K_i(\theta,x)\big)^{2+\delta}|X_i\big]\Big)$$

$$\leq C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}} \frac{n}{\Big(E\big(K_1(\theta,x)\big)\Big)^{2+\delta}} E\big[\big(K_1(\theta,x)\big)^{2+\delta} P(X_i)\overline{W}_{2+\delta}(X_i)\big]$$

$$= C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}} \frac{n}{\Big(E\big(K_1(\theta,x)\big)\Big)^{2+\delta}} E\Big[\big(K_1(\theta,x)\big)^{2+\delta}\big(P(\theta,x)+o(1)\big)\big(\overline{W}_{2+\delta}(\theta,x)+o(1)\big)\Big]$$

$$\leq C_0\left(\frac{\phi_\theta(h)}{n}\right)^{\frac{2+\delta}{2}} \frac{nE\big(K_1(\theta,x)\big)^{2+\delta}}{\big(E(K_1(\theta,x))\big)^{2+\delta}}\big(P(\theta,x)\overline{W}_{2+\delta}(\theta,x) + o(1)\big)$$

Thus, by Lemma 2.1, it follows that

$$\begin{aligned} 4nE\big[\zeta_{ni}^2 I_{[}|\zeta_{ni}| > \frac{\varepsilon}{2}\big] \quad &\leq \quad C_0\big(n\phi_\theta(h)\big)^{-\frac{\delta}{2}} \frac{M_{2+\delta} f_1(\theta,x) + o(1)}{M_1^{2+\delta} f_1^{2+\delta}(\theta,x) + o(1)}\big(P(\theta,x)\overline{W}_{2+\delta}(\theta,x) + o(1)\big) \\ &= \quad O\big(n\phi_\theta(h)\big)^{-\frac{\delta}{2}} \end{aligned}$$

Finally, by (3.5), the proof of part (b) is completed. Then, (4.9) is valid.

□

Proof of Theorem 3.1. First, we present the proof of (3.2). By Lemma 4.4, it follows that$\big(n\phi_\theta(h)\big)^{\frac{1}{2}} Q_n(\theta,x) = O_p(1)$, which leads to

$$\left(\frac{n\phi_\theta(h)}{loglogn}\right)^{\frac{1}{2}} Q_n(\theta,x) = 0_p(1) \tag{4.17}$$

On the other hand, by Lemma 4.3, we have

$$\left(\frac{n\phi_\theta(h)}{loglogn}\right)^{\frac{1}{2}} R_n(\theta,x) = O_p(1) \tag{4.18}$$

Thus, by Lemma 4.2 and (2.7), (3.2) is valid.

Second, we give the proof of (3.4). Note the fact that

$$\hat{r}_n(\theta,x) - r(\theta,x) = \hat{r}_n(\theta,x) - C_n(\theta,x) + B_n(\theta,x) \tag{4.19}$$

Hence, by (4.19) together with (2.2), (2.3) and (3.5), (3.4) follows.

□

Proof of Theorem 3.2. On the one hand, (3.6) follows directly from (2.7), (4.1), (4.6), (4.9) and the Slulsky Theorem. On the other hand, by (3.6), (3.7), (4.5), (4.19) and the Slulsky Theorem again, (3.8) is also obtained. □

Proof of Corollary 3.1. First, one can observe that

$$\frac{M_{1.n}}{\sqrt{M_{1.n}}}\sqrt{\frac{nP_n(\theta,x)F_{(\theta,x).n}(h)}{V_n(\theta,x)}}\big(\hat{r}_n(\theta,x)-r(\theta,x)\big)$$

$$=\frac{M_{1.n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2.n}}}\sqrt{\frac{nF_{(\theta,x).n}(h)P_n(\theta,x)V(\theta,x)}{P(\theta,x)V_n(\theta,x)n\phi_\theta(h)f_1(\theta,x)}}\times\frac{M_1}{\sqrt{M_2}}\sqrt{\frac{n\phi_\theta(h)f_1(\theta,x)P(\theta,x)}{V(\theta,x)}}\big(\hat{r}_n(\theta,x)-r(\theta,x)\big).$$

By (3.8), we have

$$\frac{M_1}{\sqrt{M_2}}\sqrt{\frac{P(\theta,x)n\phi_\theta(h)f_1(\theta,x)}{V(\theta,x)}}\big(\hat{r}_n(\theta,x)-r(\theta,x)\big)\xrightarrow{D}N(0,1), asn\to\infty$$

Therefore, we need to establish the following statement

$$\frac{M_{1.n}}{M_1}\frac{\sqrt{M_2}}{\sqrt{M_{2.n}}}\sqrt{\frac{nF_{(\theta,x).n}(h)V(\theta,x)P_n(\theta,x)}{P(\theta,x)V_n(\theta,x)n\phi_\theta(h)f_1(\theta,x)}}\xrightarrow{P}1,\ \text{ as } n\to\infty \tag{4.20}$$

Similar to the proof of Corollary 1 in Laib and Louani (2010), we have

$$M_{1.n}\xrightarrow{p}M_1, M_{2.n}\xrightarrow{p}M_2, \frac{F_{(\theta,x).n}(h)}{\phi_\theta(h)f_1(\theta,x)}, asn\to\infty \tag{4.21}$$

In addition, by (3.1) and (3.4), it follows that

$$\hat{r}_n(\theta,x)\xrightarrow{p}r(\theta,x), asn\to\infty \tag{4.22}$$

On the other hand, by the same steps as in the proof of Theorem 4.1, we have

$$\hat{\mathrm{g}}_n(\theta,x)\xrightarrow{p}E(Y^2|X=(\theta,x)), asn\to\infty \tag{4.23}$$

Then, by (3.9), we obtain

$$V_n(\theta,x)\xrightarrow{p}V(\theta,x), asn\to\infty \tag{4.24}$$

Finally, by Proposition 2 in Laib and Louani (2010), it follows that

$$P_n(\theta,x)\xrightarrow{p}E(\delta|X=(\theta,x)=p(\delta=1|X=(\theta,x)=P(\theta,x), asn\to\infty \tag{4.25}$$

Hence, (4.20) follows from (4.21)-(4.25). □

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