Determination of the Order and the Error Constant of an Implicit Linear Four-Step Method

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Abstract

The aim of this work is to determine the order and the error constant of an implicit linear four-step method namely "The Quade's method". From the results generated, It is observed that the method is of order six and the error constant is obtained as $C_7 = -9.0\times 10^{-3}$. The Local Truncation Error (LTE) of the general implicit linear four-step method is also obtained.

Indexing terms/Keywords: Error constant, Implicit case, Linear four-step method, Order

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Introduction

Differential equations play a vital role in various fields such as engineering, mathematics, astronomy, chemistry, physics and biology.

Numerical methods for ordinary differential equations are methods used to find numerical approximations to the solutions of ordinary differential equations (ODEs). Their use is also known as "numerical integration", although this term is sometimes taken to mean the computation of integrals .

The numerical method forms an important part of solving initial value problem in ordinary differential equation, most especially in cases where there is no closed form analytic formula or difficult to obtain exact solutions [Fadugba et al. (2012)].

In this paper, the order and the error constant of an implicit four-step method "The Quade's method" were obtained. The rest of the paper is outlined as follows; Section Two presents definition of some concepts. In Section Three, the order and the error constant of the method were obtained. The Local Truncation Error (LTE) of the general implicit linear four-step method is obtained in Section Four. Finally, Section Five concludes the paper.

Definition of some basic concepts

This section presents some basic concepts as follows; Fatunla (1988), Lambert (1973, 1991).

Definition 2.1: General linear multistep method

The General Linear Multistep Method (GLMM) for the solution of the Initial Value Problem (IVP)

$$y' = f(x,y),\ y(x_0) = y_0,\ x\in[x_0,b],\ -\infty < y < \infty \tag{2.1}$$

is defined as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2.2}$$

It is always the case that $\alpha_k \neq 0$. Also, at least one of α_0 and β_0 will be non-zero. Equation (2.2) is said to be explicit if $\beta_k = 0$, otherwise for $\beta_k \neq 0$, (2.2) becomes an implicit GLMM.

Definition 2.2: Order and error constant

With the GLMM (2.2), the associated linear difference operator *L* is given by

$$L[y(x);h] = \sum_{j=0}^{k} \left(\alpha_j y(x+jh) - h\beta_j y'(x+jh) \right) \tag{2.3}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding the test function $y(x+jh)$ and its derivative $y'(x+jh)$ as Taylor series about x and collecting terms in (2.3) gives

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \ldots + C_q h^q y^{(q)}(x) + \ldots \tag{2.4}$$

where C_q are constants. The difference operator (2.3) and the associated GLMM (2.2) are said to be of order p if in (2.4), $C_0 = C_1 = \ldots = C_p = 0$ and $C_{p+1} \neq 0$. C_{p+1} is called the error constant and the above definition implies that the Local Truncation Error (LTE) denoted by t_{n+1} is given by

$$t_{n+1} = C_{p+1} h^{p+1} y^{(p+1)}(x_n) + O\left(h^{p+2}\right) \tag{2.5}$$

The first term in (2.5) is called the principal LTE.

Definition 2.3: Consistency

The GLMM (2.2) is said to be consistent if it has at least order $p = 1$. In other words the GLMM (2.2) is said to be consistent if and only if

$$C_0 = \sum_{j=0}^{k} \alpha_j = 0 \tag{2.6}$$

and

$$C_1 = \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) = 0 \tag{2.7}$$

Definition 2.4: Zero stability

The GLMM (2.2) is said to be zero stable if the zeros of the first characteristic polynomial

$$\rho(a) = \sum_{j=0}^{k} \alpha_j a^j \tag{2.8}$$

are such that

(i) none is greater than 1 in magnitude.

(ii) any zero equal to 1 in magnitude is simple (that is, not repeated).

Definition 2.5: Convergence

The GLMM (2.2) for the solution of (2.1) is said to be convergent if

(i) It is zero stable

(ii) It is consistent

In other words, the numerical approximation (2.2) to (2.1) converges to the actual solution as $h \to 0$ if (i) and (ii) above hold.

Remark 2.1: The necessary and sufficient conditions for GLMM (2.2) to be convergent are consistent and zero stability.

Definition 2.6: Global error

The global error denoted by e_{n+1} is defined as the difference between the exact/actual solution $y(x_{n+1})$ and the numerical solution y_{n+1}.

$$e_{n+1} = \left| y(x_{n+1}) - y_{n+1} \right| \tag{2.9}$$

Definition 2.7: Local truncation error (LTE)

The LTE at x_{n+k} of the method (2.2) is defined to be the expression

$$L[y(x_n);h] = \sum_{j=0}^{k} \left(\alpha_j y(x_n + jh) - h\beta_j y'(x_n + jh) \right) \tag{2.10}$$

where $y(x)$ is the theoretical solution of the IVP (2.1).

Definition 2.8: Lipschitz condition

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region T defined by $x \in [a,b], -\infty < y < \infty$, *a* and *b* finite and let $\exists$ a constant *L* $\ni$, for every x, y, y^{*} $\ni$ (x, y) and (x, y^{*}) are both in *T*.

$$\left| f(x, y) - f(x, y^{*}) \right| \le L \left| y - y^{*} \right| \tag{2.11}$$

Then if y_0 is any given number, $\exists$ a unique solution $y(x)$ of the IVP (2.1), where $y(x)$ is continuous and differentiable $\forall$ $(x, y) \in T$.

Remark 2.2: The requirement (2.11) is known as the Lipschitz condition and the constant *L* as the Lipschitz constant. This condition ensures the uniqueness and the existence of a solution to the IVP (2.1). In other words, this condition may be thought of as being intermediate between differentiability and continuity in the sense that $f(x, y)$ continuously differentiable with respect to y $\forall$ $(x, y) \in T$.

Order and error constant of the Quade's method

Consider the Quade's method of the form [Lambert (1973a), Ex. 5, pg. 27]

$$y_{n+4}-\frac{8}{19}\left(y_{n+3}-y_{n+1}\right)-y_n=\frac{6h}{19}\left(f_{n+4}+4f_{n+3}+4f_{n+1}+f_n\right) \tag{3.1}$$

From (3.1), and by means of (2.2) with $k=4$, the following choices were obtained as follows

$$\left.\begin{aligned}&\alpha_0=-1,\alpha_1=\frac{8}{19},\alpha_2=0,\alpha_3=-\frac{8}{19},\alpha_4=1\\&\beta_0=\frac{6}{19},\beta_1=\frac{24}{19},\beta_2=0,\beta_3=\frac{24}{19},\beta_4=\frac{6}{19}\end{aligned}\right\} \tag{3.2}$$

Order of the Quade's method

To determine the order of the Quade's method (Implicit linear four-step method), Definition 2.2 and the C_q constants will be used. Recall that

$$C_q=\sum_{j=0}^{4}\left(\frac{j^q}{q!}\alpha_j-\frac{j^{(q-1)}}{(q-1)!}\beta_j\right) \tag{3.3}$$

Using (3.2), then the following values were obtained

$$C_0=\sum_{j=0}^{4}\left(\alpha_j\right)=\frac{8}{19}-\frac{8}{19}=0 \tag{3.4}$$

$$C_1=\sum_{j=0}^{4}\left(j\alpha_j-\beta_j\right)=\frac{60}{19}-\frac{60}{19}=0 \tag{3.5}$$

$$C_2=\sum_{j=0}^{4}\left(\frac{j^2}{2!}\alpha_j-j\beta_j\right)=\frac{120}{19}-\frac{120}{19}=0 \tag{3.6}$$

$$C_3=\sum_{j=0}^{4}\left(\frac{j^3}{3!}\alpha_j-\frac{j^2}{2!}\beta_j\right)=\frac{168}{19}-\frac{168}{19}=0 \tag{3.7}$$

$$C_4=\sum_{j=0}^{4}\left(\frac{j^4}{4!}\alpha_j-\frac{j^3}{3!}\beta_j\right)=\frac{176}{19}-\frac{176}{19}=0 \tag{3.8}$$

$$C_5=\sum_{j=0}^{4}\left(\frac{j^5}{5!}\alpha_j-\frac{j^4}{4!}\beta_j\right)=\frac{146}{19}-\frac{146}{19}=0 \tag{3.9}$$

$$C_6=\sum_{j=0}^{4}\left(\frac{j^6}{6!}\alpha_j-\frac{j^5}{5!}\beta_j\right)=\frac{100}{19}-\frac{100}{19}=0 \tag{3.10}$$

$$C_7 = \sum_{j=0}^{4}\left(\frac{j^7}{7!}\alpha_j - \frac{j^6}{6!}\beta_j\right) = 3.0682 - 3.0772 = -0.0090 \neq 0 \tag{3.11}$$

Since $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0$ and $C_7 = -0.0090 \neq 0$, it is observed that the Quade's method is of order six.

Error constant of the Quade's method

The differential operator (2.10) and the associated GLMM (2.2) are said to be of order p if in (2.4), $C_0 = C_1 = ... = C_p = 0$ and $C_{p+1} \neq 0$. C_{p+1} is called the error constant. From the Quade's method, it is observed that the method is of order six, since $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0$. Hence the error constant for the method is obtained as

$$\begin{aligned} C_7 &= \sum_{j=0}^{4}\left(\frac{j^7}{7!}\alpha_j - \frac{j^6}{6!}\beta_j\right) \\ &= 3.0682 - 3.0772 \\ &= -0.0090 \end{aligned}$$

The LTE of the general implicit linear four-step method under the localizing assumption

The application of the general implicit four-step method to yield y_{n+4} under the localizing assumption is given by the following result.

Theorem 4.1: If the real valued function y is of class $C^2[a,b]$, prove that the local truncation error $L[y(x_n);h]$ associated with the general linear four-step method

$$\sum_{j=0}^{4}\alpha_j y_{n+j} = h\sum_{j=0}^{4}\beta_j f_{n+j} \tag{4.1}$$

for the numerical solution of the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0,\ x_0 \le x \le b,\ y \in \Re \tag{4.2}$$

satisfies

$$L\left[y(x_n);h\right] = \left[1 - h\beta_4 \frac{\partial f\left(x_{n+4},\theta_{n+4}\right)}{\partial y}\right]\left(y\left(x_{n+4}\right) - y_{n+4}\right) \tag{4.3}$$

where $\theta_{n+4} \in \left(y_{n+4}, y(x_{n+4})\right)$.

Proof: Here, it is assumed that no previous errors have been made. In particular, assume that $y_{n+j} = y\left(x_{n+j}\right)$, $j = 0,1,2,3$. From (2.10), for $k = 4$, one gets

$$L[y(x_n);h]=\sum_{j=0}^{4}\left(\alpha_j y(x_n+jh)-h\beta_j y'(x_n+jh)\right) \tag{4.4}$$

Using the fact that $y'=f(x,y)$, then (4.4) yields

$$\begin{aligned}\sum_{j=0}^{4}\alpha_j y(x_n+jh)&=h\sum_{j=0}^{4}\beta_j y'(x_n+jh)+L[y(x_n);h]\\ &=h\sum_{j=0}^{4}\beta_j f\left(x_n+jh,y(x_n+jh)\right)+L[y(x_n);h]\end{aligned} \tag{4.5}$$

Since, in this scenario, $y(x)$ is taken to be the exact solution of the IVP (4.2). The value for y_{n+4} given by (4.1) satisfies

$$\sum_{j=0}^{4}\alpha_j y_{n+j}=h\sum_{j=0}^{4}\beta_j f\left(x_{n+j},y_{n+j}\right) \tag{4.6}$$

The exact solution $y(x_{n+4})$ and numerical solution y_{n+4} are obtained as

$$y(x_{n+4})=h\beta_4 f\left(x_{n+4},y(x_{n+4})\right) \tag{4.7}$$

and

$$y_{n+4}=h\beta_4 f\left(x_{n+4},y_{n+4}\right) \tag{4.8}$$

respectively.

Subtracting (4.8) from (4.7) and by means of the localizing assumption stated above, gives

$$y(x_{n+4})-y_{n+4}=h\beta_4\left[f\left(x_{n+4},y(x_{n+4})\right)-f\left(x_{n+4},y_{n+4}\right)\right]+L\left[y(x_n;h)\right] \tag{4.9}$$

By means of the Mean Value Theorem given by

$$f(b)-f(a)=f'(\xi)(b-a) \tag{4.10}$$

The term

$f\left(x_{n+4},y(x_{n+4})\right)-f\left(x_{n+4},y_{n+4}\right)$ in (4.9) yields

$$f\left(x_{n+4},y(x_{n+4})\right)-f\left(x_{n+4},y_{n+4}\right)=\left(y(x_{n+4})-y_{n+4}\right)\frac{\partial f(x_{n+4},\theta_{n+4})}{\partial y} \tag{4.11}$$

where θ_{n+4} is an interior point of the interval whose endpoints are y_{n+4} and $y\left(x_{n+4}\right)$.

Using (4.9) and (4.11) and rearranging terms, yields (4.3). This completes the proof.

Remark 4.1: For an explicit case, that is $\beta_4 = 0$, the LTE is the difference between the exact solution and the numerical solution. This can be expressed as

$$T_{n+4} = L(y(x_n);h) = y(x_{n+4}) - y_{n+4}$$

Concluding remarks

In this paper, an implicit linear four-step method "The Quade's method" has been considered. By means of the selected choices of the method, the order and the error constant were determined. Moreover, the LTE of the general implicit four-step method is obtained by means of the localizing assumption. Hence, from the results obtained, it is observed that the Quade's method is of order six and the error constant for the method is obtained as $C_7 = -9.0 \times 10^{-3}$.

Conflicts of Interest

Author declares that there is no conflict of interest.

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