Abstract

Coupled systems of integral and differential equations are studied in many papers [5], [6], [10] and [11]. Especially, the investigation for coupled systems of fractional differential equations appears in many literatures, for example [12], [13] and [14]. Here, we are concerned with the existence of solutions of some coupled systems of functional equations, differential equations of fractional orders, two-point boundary-value problems of fractional orders.

Keywords: Fractional-calculus; functional equation; Coupled systems; Continuous solutions.

1 Introduction and preliminaries

Systems appear in different problems of applied nature, for instance, see ([4]-[6], [15], [16] and [18]). Recently, Su [22] studied a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [23] studied the solutions of coupled nonlinear fractional reaction-diffusion equations. The solvability of the coupled systems of integral equations in reflexive Banach space proved in [10]-[12]. Also, a comparison between the classical method of successive approximations (Picard) method and Adomian decomposition method of coupled system of quadratic integral equations proved in [13].

Let $L_1(J)$ be the space of Lebesgue integrable functions defined on the interval $J = [0,1]$. Let $C(J)$ be the space of all continuous functions on $J$ with sup-norm.

Let $X = C(J) \times C(J) = \{u(t) = (x(t), y(t)) : x, y \in C(J), t \in J\}$ which is a Banach space with the norm defined as $\| (x,y) \|_X = \max\{ \| x \|_{C(J)} + \| y \|_{C(J)} \} \forall (x,y) \in X$ ([4]).

Let $AC(J)$ be the space of all absolutely continuous functions on $J$ and denote $Y = AC(J) \times AC(J) = \{u(t) = (x(t), y(t)) : x, y \in AC(J), t \in J\}$.

The functional equations have been studied in several papers and monographs (see for examples [1]-[3], [8] and [9]). Banas [1] proved the existence of monotonic integrable solution for the functional equation

$$y(t) = f(t, y(t)), \quad t \in J$$

under certain monotonicity condition by using the technique of measure of noncompactness. Here, we shall prove the existence of continuous solution of the coupled system of functional equations

$$x(t) = f_1(t, y(t)), \quad t \in J$$

$$y(t) = f_2(t, x(t)), \quad t \in J,$$

and the coupled system of differential equations
\[ \frac{dx(t)}{dt} = f_1(t, y(t)), \quad t \in J \] (3)

\[ \frac{dy(t)}{dt} = f_2(t, x(t)), \quad t \in J, \]

With the boundary conditions
\[ x(0) = a x(\eta), \quad y(0) = b y(\tau), \quad \eta, \tau \in J \]

then we extend our result to the coupled system of differential equations of fractional orders

\[ \frac{dx(t)}{dt} = f_1(t, D^\alpha y(t)), \quad t \in J, \quad \alpha \in (0,1] \] (4)

\[ \frac{dy(t)}{dt} = f_2(t, D^\beta x(t)), \quad t \in J, \quad \beta \in (0,1]. \]

Also, the coupled of Cauchy system problems
\[ R^D\alpha x(t) = f_1(t, y(t)), \quad t \in J, \quad \alpha \in (0,1) \]

\[ R^D\beta y(t) = f_2(t, x(t)), \quad t \in J, \quad \beta \in (0,1) \] (5)

With the initial conditions
\[ I^{1-\alpha} x(t)|_{t=0} = I^{1-\beta} y(t)|_{t=0} = 0 \]

will be studied.

The existence results will be based on the following fixed-point theorems and definitions.

**Theorem 1. (Schauder Fixed Point Theorem)**[7]. Let Q be a nonempty, convex, compact subset of a Banach space X, and T : Q → Q be a continuous map. Then T has at least one fixed point in Q.

Let \( \beta \) be a positive real number

**Definition 1.** The fractional-order integral of order \( \beta \) of the function \( f \) is defined on \([a,b]\) by (see [17], [19], [20] and [21])

\[ I^\beta_a f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \, ds, \] (6)

and when \( a = 0 \), we have \( I^\beta f(t) = I^\beta_0 f(t) \).

**Definition 2.** The Caputo-fractional-order derivative of order \( \beta \in (0,1] \) of the absolutely continuous function \( f \) is given by (see [17], [19], [20] and [21])

\[ D^\beta f(t) = I^{1-\beta} \frac{d}{dt} f(t). \]

**Definition 3.** The Riemann-Liouville fractional-order derivative of order \( \beta \in (0,1) \) of the function \( f \) is given by (see [17], [19], [20] and [21])

\[ R^D^\beta f(t) = \frac{d}{dt} I^{1-\beta} f(t). \]

For the properties of fractional calculus see [17], [19], [20] and [21] for example.
1 Coupled system of functional equations

Consider the following assumptions:

(i) \( f_i : J \times \mathbb{R} \to \mathbb{R}, \ i = 1, 2 \) is continuous and bounded with \( K_i = \sup_{(t,x) \in J \times \mathbb{R}} |f_i(t,x)|, \ i = 1, 2. \)

(ii) There exist two constants \( l_i, h_i, \ i = 1, 2 \) such that

\[
|f_i(t,x) - f_i(s,y)| \leq l_i |t - s| + h_i |x - y|
\]

for all \( t, s \in J \) and \( x, y \in \mathbb{R} \).

Define an operator \( T : X \to X \) as

\[
T(x,y)(t) = (f_1(t,y(t)), f_2(t,x(t)))
\]

where

\[
T_1y(t) = f_1(t,y(t)), \quad t \in J
\]

\[
T_2x(t) = f_2(t,x(t)), \quad t \in J.
\]

Then the coupled system (2) may be written as:

\[
x(t) = T_1y(t)
\]

\[
y(t) = T_2x(t).
\]

**Theorem 2.** Let the assumptions (i)-(ii) be satisfied. Then the coupled system of functional equations (2) has at least one solution in \( X \).

**Proof.**

Define

\[
U = \{ u = (x(t), y(t)) | (x(t), y(t)) \in X : ||(x,y)||_X \leq \max\{ K_1, K_2 \} \}.
\]

For \( (x,y) \in U \), we have

\[
|T_1y(t)| \leq |f_1(t,x(t))| \leq K_1.
\]

Then

\[
\| T_1y(t) \| \leq K_1.
\]

and

\[
\| T_2x(t) \| \leq K_2.
\]

Therefore,

\[
\| Tu(t) \| = ||T(x,y)(t)|| = ||(T_1y(t), T_2x(t))|| = \max_{t \in J} \{ \| T_1y(t) \|, \| T_2x(t) \| \}
\]

\[
\leq \max_{t \in J} \{ K_1, K_2 \}.
\]

Then, for every \( u = (x,y) \in U \) we have \( Tu \in U \) and hence \( TU \subset U \).

It is clear that the set \( U \) is closed and convex.

Assumption (i) implies that \( T : U \to X \) is a continuous operator. Now, for \( u = (x,y) \in U, \) and for each \( t_1, t_2 \in J \) (without loss of generality assume that \( t_1 < t_2 \)), we get

\[
| T_1y(t_2) - T_1y(t_1) | \leq |f_1(t_2,y(t_2)) - f_1(t_1,y(t_1))| \leq l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)|
\]

Then

\[
\| T_1y(t_2) - T_1y(t_1) \|_{C(J)} \leq l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)|
\]

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Similarly,

$$\| T_2 x(t_2) - T_2 x(t_1) \|_{C(J)} \leq l_2 |t_2 - t_1| + h_2 |x(t_2) - x(t_1)|$$

Now, from the definition of the operator \( T \), we get

$$Tu(t_2) - Tu(t_1) = T(x, y)(t_2) - T(x, y)(t_1)$$

$$= (T_1 y(t_2), T_2 x(t_2)) - (T_1 y(t_1), T_2 x(t_1))$$

$$= (T_1 y(t_2) - T_1 y(t_1), T_2 x(t_2) - T_2 x(t_1)),$$

and

$$\| Tu(t_2) - Tu(t_1) \| = \max_{t_1, t_2 \in J} \{ \| T_1 y(t_2) - T_1 y(t_1) \| + \| T_2 x(t_2) - T_2 x(t_1) \| \}$$

$$\leq l_1 |t_2 - t_1| + h_2 |x(t_2) - x(t_1)| + l_2 |t_2 - t_1| + h_1 |y(t_2) - y(t_1)|$$

Hence

$$| t_2 - t_1 | < \delta \implies \| Tu(t_2) - Tu(t_1) \| < \varepsilon(\delta),$$

This means that the functions of \( TU \) are equi-continuous on \( J \). Then by the Arzela-Ascoli Theorem [7] the closure of \( TU \) is compact.

Since all conditions of the Schauder fixed-point theorem hold, then \( T \) has a fixed point in \( U \) which completes the proof. ■

**Example 1**

Consider the following coupled system of functional equations

$$x(t) = \sqrt{t^2 + 5} + t(|\log(y(t)) + 3| + 1), \ t \in J$$

$$y(t) = \frac{1 + 2t}{10} + e^{-t^2/30}, \ t \in J.$$  \(7\)

Set

$$f_1(t, y) = \sqrt{t^2 + 5} + t(|\log(y(t)) + 3| + 1), \ t \in J$$

$$f_2(t, x) = \frac{1 + 2t}{10} + e^{-t^2/30}.$$

Then easily we can deduce that:

$$|f_1(t, z) - f_1(s, y)| = |\sqrt{t^2 + 5} + t(|\log(z(t)) + 3| + 1) - \sqrt{s^2 + 5} - s(|\log(y(s)) + 3| + 1)|$$

$$\leq |\sqrt{t^2 + 5} - \sqrt{s^2 + 5}| + t(|\log(z(t)) + 1) - (|\log(y(s)) + 3| + 1)|$$

$$+ |t(|\log(y(s)) + 3| + 1) - s(|\log(y(s)) + 3| + 1)|$$

$$\leq \frac{2}{5} |t - s| + \frac{1}{10} |z - y| + |t - s| + 3 |t - s|$$

$$\leq \frac{11}{5} |t - s| + \frac{1}{10} |z - y|$$
and

\[ |f_2(t, z) - f_2(s, x)| = \left| \frac{1 + t}{10} + e^{-t} \cdot \frac{z^2}{30} - \frac{1 + s}{10} - e^{-s} \cdot \frac{x^2}{30} \right| \]

\[ \leq \frac{1}{10} |t - s| + \frac{1}{30} |e^{-t} z^2 - e^{-s} x^2| + \frac{1}{30} |e^{-t} x^2 - e^{-s} z^2| \]

\[ \leq \frac{1}{10} |t - s| + \frac{2}{30} |x + z| |x - z| + \frac{1}{30} |e^{-t} - e^{-s}| \]

\[ \leq \frac{1}{10} |t - s| + \frac{2}{30} |x + z| |x - z| + \frac{1}{30} |e^{-s} - e^{-t}| \]

Then all the assumptions of Theorem 2 are satisfied so the coupled system of the functional equations (7) possesses at least one solution in \( X \).

**Example:2**

Consider the following coupled system of functional equations

\[ x(t) = t + \frac{1}{3} |y(t)|, \quad t \in J \]

\[ y(t) = t + \sin x(2t), \quad t \in J. \]  

(8)

Set

\[ f_1(t, y) = t + \frac{1}{3} |y(t)|, \quad t \in J \]

\[ f_2(t, x) = t + \sin x(2t). \]

Then easily we can deduce that:

\[ |f_1(t, z) - f_1(s, y)| \leq |t - s| + \frac{1}{3} |z - y| \]

and

\[ |f_2(t, z) - f_2(s, x)| \leq |t + \sin z(2t) - s - \sin x(2s)| \]

\[ \leq |t - s| + |\sin z(2t) - \sin x(2s)| \]

\[ \leq |t - s| + |z - x| \]

**Example:3**

Consider the following coupled system of functional equations

\[ x(t) = \frac{1}{t + 1} + \sin \left( \frac{y(t)}{4} \right), \quad t \in J \]

\[ y(t) = \frac{1}{\ln(5)} \ln \left( \frac{20 + \sqrt{x(t)}}{1 + t} \right), \quad t \in J. \]  

(9)

Set

\[ f_1(t, y) = \frac{1}{t + 1} + \sin \left( \frac{y(t)}{4} \right), \quad t \in J \]

\[ f_2(t, x) = \frac{1}{\ln(5)} \ln \left( \frac{20 + \sqrt{x(t)}}{1 + t} \right). \]
Then easily we can deduce that:

\[ |f_1(t, z) - f_1(s, y)| \leq \frac{1}{t+1} + \left| \frac{\sin(z(t))}{4} - \frac{\sin(y(t))}{4} \right| \]

\[ \leq \frac{1}{4} |t - s| + \frac{1}{4} |z - y| \]

and

\[ |f_2(t, z) - f_2(s, x)| \leq \frac{1}{\ln(5)} \left| \ln(1+t) - \ln(1+s) \right| + \frac{1}{\ln(5)} \left| \ln(20 + \sqrt{z(t)}) - \ln(20 + \sqrt{x(s)}) \right| \]

\[ \leq \frac{1}{\ln(5)} |t - s| + \frac{1}{2\sqrt{\xi}\ln(5)(20 + \sqrt{\xi})} \left| \sqrt{z(t)} - \sqrt{x(s)} \right| \]

\[ \leq \frac{1}{\ln(5)} |t - s| + \frac{1}{4\xi\ln(5)(20 + \sqrt{\xi})} \left| \sqrt{z(t)} - \sqrt{x(s)} \right| \left| \sqrt{z(t)} + \sqrt{x(s)} \right| \]

\[ \leq \frac{1}{\ln(5)} |t - s| + \frac{100}{804\ln(5)} |z - x| \]

2 Coupled system of Two-points boundary value problems

Now, let \( z(t) = \frac{dx(t)}{dt} \) and \( w(t) = \frac{dw(t)}{dt} \), using the boundary conditions then we get

\[
\begin{align*}
x(t) &= x(0) + Iz(t) \\
x(\eta) &= x(0) + Iz(\eta) \\
x(\eta)(1-b) &= Iz(\eta) \\
x(\eta) &= \frac{1}{1-b} Iz(\eta) \\
x(0) &= b x(\eta) = \frac{b}{1-b} Iz(\eta) \\
x(t) &= \frac{b}{1-b} Iz(\eta) + Iz(t).
\end{align*}
\]

By a similar way, we have

\[ y(t) = \frac{a}{1-a} Iw(\eta) + Iw(t) \]

Therefore, the coupled system (3) has the form:

\[
\begin{align*}
z(t) &= f_1(t, \frac{a}{1-a} Iw(\eta) + Iw(t)), \quad t \in J \\
w(t) &= f_2(t, \frac{b}{1-b} Iz(\eta) + Iz(t)), \quad t \in J,
\end{align*}
\]

Definition 4. A pair of functions \((x, y)\) is a solution of (3), if the functions \( x \) and \( y \) are absolutely continuous on \( J \) and satisfy the coupled system (3).

Then we can deduce the following theorem.

Theorem 3. Let the assumptions of Theorem 2 be satisfied, then the coupled system (3) has at least one solution \((x, y) \in Y\).
3 Coupled system of differential equations of fractional order

Now, let $z(t) = \frac{dx(t)}{dt}$ and $w(t) = \frac{dy(t)}{dt}$, then we get

$$I^{1-\beta} \frac{dx(t)}{dt} = I^{1-\beta} z(t) = D^\beta x(t)$$

and similarly

$$I^{1-\alpha} \frac{dy(t)}{dt} = I^{1-\alpha} w(t) = D^\alpha y(t).$$

Then the coupled system of differential equations of fractional order (4) has the form:

$$z(t) = f_1(t, I^{1-\alpha} w(t)), \quad t \in J$$

$$w(t) = f_2(t, I^{1-\beta} z(t)), \quad t \in J.$$ 

**Definition 5.** A pair of functions $(x, y)$ is a solution of (4), if the functions $x$ and $y$ are absolutely continuous on $J$ and satisfy the coupled system (4).

Then we can deduce the following theorem.

**Theorem 4.** Let the assumptions of Theorem 2 be satisfied, then the coupled system (4) has at least one solution $(x, y) \in Y$.

Now, letting $\alpha, \beta \to 1$, then as a particular case of Theorem 4 we can obtain existence result of the following coupled system of functional equations

$$\frac{dx(t)}{dt} = f_1(t, \frac{dy(t)}{dt}), \quad t \in J$$

$$\frac{dy(t)}{dt} = f_2(t, \frac{dx(t)}{dt}), \quad t \in J.$$ 

Letting $z(t) = \frac{dx(t)}{dt}$ and $w(t) = \frac{dy(t)}{dt}$, then we get

$$z(t) = f_1(t, w(t)), \quad t \in J$$

$$w(t) = f_2(t, z(t)), \quad t \in J.$$ 

4 Coupled system of Cauchy problems of fractional orders

Now, let $z(t) = R^{\alpha} x(t)$ and $w(t) = R^{\beta} y(t)$, then we get

$$\frac{d}{dt} I^{1-\alpha} z(t) = z(t)$$

$$I^{1-\alpha} x(t) - I^{1-\alpha} x(t)|_{t=0} = I z(t);$$

$$I^{1-\alpha} x(t) = I z(t);$$

$$I x(t) = I^{\alpha+1} x(t);$$

$$x(t) = I^\alpha x(t).$$

Similarly

$$y(t) = I^\beta w(t).$$

Then the coupled system of differential equations of fractional order (5) has the form:

$$z(t) = f_1(t, I^\beta w(t));$$

$$w(t) = f_2(t, I^\alpha z(t)).$$
References


