On some coupled systems of functional differential equations of fractional orders

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Abstract

Coupled systems of integral and differential equations are studied in many papers [5], [6], [10] and [11]. Especially, the investigation for coupled systems of fractional differential equations appears in many literatures, for example [12], [13] and [14].
Here, we are concerning with the existence of solutions of some coupled systems of functional equations, differential equations of fractional orders, two-point boundary-value problems of fractional orders.

Keywords: Fractional-calculus; functional equation; Coupled systems; Continuous solutions.

1 Introduction and preliminarie

Systems appear in different problems of applied nature, for instance, see ([4]-[6], [15], [16] and [18]). Recently, Su [22] studied a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [23] studied the solutions of coupled nonlinear fractional reaction-diffusion equations.
The solvability of the coupled systems of integral equations in reflexive Banach space proved in [10]-[12]. Also, a comparison between the classical method of successive approximations (Picard) method and Adomian decomposition method of coupled system of quadratic integral equations proved in [13].

Let $L_1(J)$ be the space of Lebesgue integrable functions defined on the interval $J = [0, 1]$. Let $C(J)$ be the space of all continuous functions on J with sup-norm.
Let $X = C(J) \times C(J) = \{u(t) = (x(t),\ y(t)) :\ x,\ y \in C(J),\ t \in J\}$
which is a Banach space with the norm defined as
$||(x, y)||_X = \max\{||x||_{C(J)} + ||y||_{C(J)}\}\ \forall\ (x, y) \in X$ ([4]).
Let $AC(J)$ be the space of all absolutely continuous functions on J and denote $Y = AC(J) \times AC(J) = \{u(t) = (x(t),\ y(t)) :\ x,\ y \in AC(J),\ t \in J\}$.

The functional equations have been studied in several papers and monographs (see for examples [1]-[3], [8] and [9]).
Banas [1] proved the existence of monotonic integrable solution for the functional equation

$$y(t) = f(t, y(t)), \quad t \in J \tag{1}$$

under certain monotonicity condition by using the technique of measure of noncompactness.
Here, we shall prove the existence of continuous solution of the coupled system of functional equations

$$\begin{aligned} x(t) &= f_1(t, y(t)), \quad t \in J \\ y(t) &= f_2(t, x(t)), \quad t \in J, \end{aligned} \tag{2}$$

and the coupled system of differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= f_1(t, y(t)), \quad t \in J \\ \frac{dy(t)}{dt} &= f_2(t, x(t)), \quad t \in J, \end{aligned} \tag{3}$$

With the boundary conditions

$$x(0) = a\, x(\eta), \quad y(0) = b\, y(\tau), \quad \eta, \tau \in J$$

then we extend our result to the coupled system of differential equations of fractional orders

$$\begin{aligned} \frac{dx(t)}{dt} &= f_1(t, D^{\alpha} y(t)), \quad t \in J, \quad \alpha \in (0,1] \\ \frac{dy(t)}{dt} &= f_2(t, D^{\beta} x(t)), \quad t \in J, \quad \beta \in (0,1]. \end{aligned} \tag{4}$$

Also, the coupled of Cauchy system problems

$$\begin{aligned} {}_R D^{\alpha} x(t) &= f_1(t, y(t)), \quad t \in J, \quad \alpha \in (0,1) \\ {}_R D^{\beta} y(t) &= f_2(t, x(t)), \quad t \in J, \quad \beta \in (0,1) \end{aligned} \tag{5}$$

With the initial conditions

$$I^{1-\alpha} x(t)|_{t=0} = I^{1-\beta} y(t)|_{t=0} = 0$$

will be studied.

The existence results will be based on the following fixed-point theorems and definitions.

Theorem 1. *(Schauder Fixed Point Theorem)[7].*
Let Q be a nonempty, convex, compact subset of a Banach space X, and $T : Q \to Q$ be a continuous map. Then T has at least one fixed point in Q.

Let β be a positive real number

Definition 1. *The fractional-order integral of order β of the function f is defined on $[a, b]$ by (see [17], [19], [20] and [21])*

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \tag{6}$$

and when $a = 0$, we have $I^{\beta} f(t) = I_0^{\beta} f(t)$.

Definition 2. *The Caputo- fractional-order derivative of order $\beta \in (0, 1]$ of the of the absolutely continuous function f is given by (see [17], [19], [20] and [21])*

$$D^{\beta} f(t) = I^{1-\beta} \frac{d}{dt} f(t).$$

Definition 3. *The Riemann-Liouville fractional-order derivative of order $\beta \in (0, 1)$ of the function f is given by (see [17], [19], [20] and [21])*

$${}_R D^{\beta} f(t) = \frac{d}{dt} I^{1-\beta} f(t).$$

For the properties of fractional calculus see [17], [19], [20] and [21] for example.

1 Coupled system of functional equations

Consider the following assumptions:

(i) $f_i : J \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$ is continuous and bounded with $K_i = \sup_{(t,x)\in J\times\mathbb{R}} |f_i(t,x)|$, $i = 1, 2$.

(ii) There exist two constants l_i, h_i, $i = 1, 2$ such that

$$|f_i(t,x) - f_i(s,y)| \leq l_i\ |t-s| + h_i\ |x-y|$$

for all $t,\ s \in J$ and $x,\ y \in \mathbb{R}$.

Define an operator $T : X \rightarrow X$ as

$$\begin{aligned} T(x,y)(t) &= (\ f_1(t,y(t)),\ f_2(t,x(t))) \\ &=: (T_1y(t),\ T_2x(t)). \end{aligned}$$

where

$$\begin{aligned} T_1y(t) &= f_1(t,y(t)), \quad t \in J \\ T_2x(t) &= f_2(t,x(t)), \quad t \in J. \end{aligned}$$

Then the coupled system (2) may be written as:

$$\begin{aligned} x(t) &= T_1y(t) \\ y(t) &= T_2x(t). \end{aligned}$$

Theorem 2. *Let the assumptions (i)-(ii) be satisfied. Then the coupled system of functional equations (2) has at least one solution in* X.

Proof.
Define

$$U = \{u = (x(t),y(t)) | (x(t),y(t)) \in X :\ ||(x,y)||_X \leq \max\{K_1,\ K_2\}\}.$$

For $(x,y) \in U$, we have

$$|\ T_1y(t)\ | \ \leq\ |f_1(t,x(t))| \leq K_1.$$

Then

$$||\ T_1y(t)\ || \ \leq\ K_1.$$

and

$$||\ T_2x(t)\ || \ \leq\ K_2.$$

Therefore,

$$\begin{aligned} ||Tu(t)|| \ = \ ||T(x,y)(t)|| \ = \ ||(T_1y(t), T_2x(t))|| \ &= \max_{t\in J}\{\ ||T_1y(t)||\ ,\ ||T_2x(t)||\ \} \\ &\leq \max_{t\in J}\{\ K_1, K_2\ \}. \end{aligned}$$

Then, for every $u = (x,y) \in U$ we have $Tu \in U$ and hence $TU \subset U$.
It is clear that the set U is closed and convex.
Assumption (i) implies that
$T :\ U \rightarrow X$ is a continuous operator. Now, for $u = (x,y) \in U$, and for each $t_1,\ t_2 \in J$ (without loss of generality assume that $t_1 < t_2$),we get

$$\begin{aligned} |\ T_1y(t_2) - T_1y(t_1)\ | \ &\leq\ |f_1(t_2,y(t_2)) - f_1(t_1,y(t_1))| \\ &\leq\ l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \end{aligned}$$

Then

$$||\ T_1y(t_2) - T_1y(t_1)\ ||_{C(J)} \leq l_1|t_2 - t_1| + h_1|y(t_2) - y(t_1)|$$

Similarly,

$$|| \ T_2x(t_2) - T_2x(t_1) \ ||_{C(J)} \leq l_2|t_2 - t_1| + h_2|x(t_2) - x(t_1)|$$

Now, from the definition of the operator T, we get

$$\begin{aligned} Tu(t_2) \ - \ Tu(t_1) &= T(x,y)(t_2) \ - \ T(x,y)(t_1) \\ &= (T_1y(t_2), \ T_2x(t_2)) \ - \ (T_1y(t_1), \ T_2x(t_1)) \\ &= (T_1y(t_2) \ - \ T_1y(t_1), \ T_2x(t_2) \ - \ T_2x(t_1)), \end{aligned}$$

and

$$\begin{aligned} ||Tu(t_2) \ - \ Tu(t_1)|| &= \max_{t_1,t_2 \in J}\{||T_1y(t_2) \ - \ T_1y(t_1)|| + ||T_2x(t_2) \ - \ T_2x(t_1)||\} \\ &\leq \ l_1|t_2 - t_1| + h_2|x(t_2) - x(t_1)| + l_2|t_2 - t_1| + h_1|y(t_2) - y(t_1)| \end{aligned}$$

Hence

$$| \ t_2 - t_1 \ | < \delta \Longrightarrow \ || \ Tu(t_2) \ - \ Tu(t_1) \ || < \varepsilon(\delta),$$

This means that the functions of TU are equi-continuous on J. Then by the Arzela-Ascoli Theorem [7] the closure of TU is compact .
Since all conditions of the Schauder fixed-point theorem hold, then T has a fixed point in U which completes the proof. ■

Example:1
Consider the following coupled system of functional equations

$$\begin{aligned} x(t) &= \sqrt{t^2+5} + t(|log(y(t)+3)|+1), \quad t \in J \\ y(t) &= \frac{1+2t}{10} + e^{-t}\frac{x^2}{30}, \quad t \in J. \end{aligned} \tag{7}$$

Set

$$\begin{aligned} f_1(t,y) &= \sqrt{t^2+5} + \ t(|log(y(t)+3)|+1), \ \ t \in J \\ f_2(t,x) &= \frac{1+2t}{10} + e^{-t}\frac{x^2}{30}. \end{aligned}$$

Then easily we can deduce that:

$$\begin{aligned} |f_1(t,z) - f_1(s,y)| &= |\sqrt{t^2+5} + t(|log(z(t)+3)|+1) - \sqrt{s^2+5} - s(|log(y(s)+3)|+1)| \\ &\leq |\sqrt{t^2+5} - \sqrt{s^2+5}| + t|(|log(z(t)+3)|+1) - (|log(y(s)+3)|+1)| \\ &+ |t(|log(y(s)+3)|+1) - s(|log(y(s)+3)|+1)| \\ &\leq \frac{2}{5}\ |t-s| + \frac{1}{10}\ |z-y| + |t-s| + 3.|t-s| \\ &\leq \frac{11}{5}\ |t-s| + \frac{1}{10}\ |z-y| \end{aligned}$$

and

$$\begin{aligned} |f_2(t,z)-f_2(s,x)| &= |\frac{1+t}{10}+e^{-t}.\frac{z^2}{30}-\frac{1+s}{10}-e^{-s}.\frac{x^2}{30}| \\ &\leq \frac{1}{10}|t-s|+\frac{1}{30}|e^{-t}z^2-e^{-t}x^2|+\frac{1}{30}|e^{-t}x^2-e^{-s}x^2| \\ &\leq \frac{1}{10}|t-s|+\frac{2|x+z|}{30}|x-z|+\frac{1}{30}|e^{-t}-e^{-s}| \\ &\leq \frac{1}{10}|t-s|+\frac{2|x+z|}{30}|x-z|+\frac{e^{-s}}{30}|t-s| \\ &\leq \frac{2}{15}\ |t-s|+\frac{1}{30}\ |x-z| \end{aligned}$$

Then all the assumptions of Theorem 2 are satisfied so the coupled system of the functional equations (7) possesses at least one solution in X.

Example:2
Consider the following coupled system of functional equations

$$\begin{aligned} x(t) &= t+\frac{1}{3}|y(t)|, \quad t\ \in\ J \\ y(t) &= t+\sin x(2t), \quad t\ \in\ J. \end{aligned} \tag{8}$$

Set

$$\begin{aligned} f_1(t,y) &= t+\frac{1}{3}|y(t)|, \quad t\in J \\ f_2(t,x) &= t+\sin x(2t). \end{aligned}$$

Then easily we can deduce that:

$$|f_1(t,z)-f_1(s,y)| \leq\ |t-s|+\frac{1}{3}\ |z-y|$$

and

$$\begin{aligned} |f_2(t,z)-f_2(s,x)| &= |t+\sin z(2t)-s-\sin x(2s)| \\ &\leq |t-s|+|\sin z(2t)-\sin x(2s)| \\ &\leq |t-s|+|z-x| \end{aligned}$$

Example:3
Consider the following coupled system of functional equations

$$\begin{aligned} x(t) &= \frac{1}{t+1}+\sin(\frac{y(t)}{4}), \quad t\ \in\ J \\ y(t) &= \frac{1}{\ln(5)}\ln(\frac{20+\sqrt{x(t)}}{1+t}), \quad t\ \in\ J. \end{aligned} \tag{9}$$

Set

$$\begin{aligned} f_1(t,y) &= \frac{1}{t+1}+\sin(\frac{y(t)}{4}), \quad t\in J \\ f_2(t,x) &= \frac{1}{\ln(5)}\ln(\frac{20+\sqrt{x(t)}}{1+t}). \end{aligned}$$

Then easily we can deduce that:

$$\begin{aligned} |f_1(t,z)-f_1(s,y)| &\leq |\frac{1}{t+1}-\frac{1}{s+1}|+|\sin(\frac{z(t)}{4})-\sin(\frac{y(t)}{4})| \\ &\leq \frac{1}{4}\,|t-s|+\frac{1}{4}\,|z-y| \end{aligned}$$

and

$$\begin{aligned} |f_2(t,z)-f_2(s,x)| &\leq \frac{1}{ln(5)}|\ln(1+t)-\ln(1+s)|+\frac{1}{ln(5)}|\ln(20+\sqrt{z(t)})-\ln(20+\sqrt{x(s)})| \\ &\leq \frac{1}{ln(5)}|t-s)|+\frac{1}{2\sqrt{\xi}.ln(5)(20+\sqrt{\xi})}|\sqrt{z(t)}-\sqrt{x(s)}| \\ &\leq \frac{1}{ln(5)}|t-s)|+\frac{1}{4\xi.ln(5)(20+\sqrt{\xi})}|\sqrt{z(t)}-\sqrt{x(s)}|.|\sqrt{z(t)}+\sqrt{x(s)}| \\ &\leq \frac{1}{ln(5)}|t-s)|+\frac{1}{4\xi.ln(5)(20+\sqrt{\xi})}|z(t)-x(s)| \\ &\leq \frac{1}{ln(5)}|t-s)|+\frac{100}{804.ln(5)}|z-x| \end{aligned}$$

2 Coupled system of Two-points boundary value problems

Now, let $z(t)=\frac{dx(t)}{dt}$ and $w(t)=\frac{dy(t)}{dt}$, using the boundary conditions then we get

$$\begin{aligned} x(t) &= x(0)+Iz(t) \\ x(\eta) &= x(0)+Iz(\eta) \\ x(\eta) &= bx(\eta)+Iz(\eta) \\ x(\eta)(1-b) &= Iz(\eta) \\ x(\eta) &= \frac{1}{1-b}Iz(\eta) \\ x(0) &= bx(\eta)=\frac{b}{1-b}Iz(\eta) \\ x(t) &= \frac{b}{1-b}Iz(\eta)+Iz(t). \end{aligned}$$

By a similar way, we have

$$y(t)=\frac{a}{1-a}Iw(\eta)+Iw(t).$$

Therefore, the coupled system (3) has the form:

$$\begin{aligned} z(t) &= f_1(t,\frac{a}{1-a}Iw(\eta)+Iw(t)), \quad t\in J \\ w(t) &= f_2(t,\frac{b}{1-b}Iz(\eta)+Iz(t)), \quad t\in J, \end{aligned}$$

Definition 4. *A pair of functions (x,y) is a solution of (3), if the functions x and y are absolutely continuous on J and satisfy the coupled system (3).*

Then we can deduce the following theorem.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied, then the coupled system (3) has at least one solution $(x,y)\in Y$.*

3 Coupled system of differential equations of fractional order

Now, let $z(t) = \frac{dx(t)}{dt}$ and $w(t) = \frac{dy(t)}{dt}$, then we get

$$I^{1-\beta}\frac{dx(t)}{dt} = I^{1-\beta}z(t) = D^{\beta}x(t)$$

and similarly

$$I^{1-\alpha}\frac{dy(t)}{dt} = I^{1-\alpha}w(t) = D^{\alpha}y(t).$$

Then the coupled system of differential equations of fractional order (4) has the form:

$$z(t) = f_1(t, I^{1-\alpha}w(t)), \quad t \in J$$

$$w(t) = f_2(t, I^{1-\beta}z(t)), \quad t \in J.$$

Definition 5. *A pair of functions (x, y) is a solution of (4), if the functions x and y are absolutely continuous on J and satisfy the coupled system (4).*

Then we can deduce the following theorem.

Theorem 4. *Let the assumptions of Theorem 2 be satisfied, then the coupled system (4) has at least one solution $(x, y) \in Y$.*

Now, letting $\alpha, \beta \to 1$, then as a particular case of Theorem 4 we can obtain existence result of the following coupled system of functional equations

$$\frac{dx(t)}{dt} = f_1(t, \frac{dy(t)}{dt}), \quad t \in J$$

$$\frac{dy(t)}{dt} = f_2(t, \frac{dx(t)}{dt}), \quad t \in J.$$

Letting $z(t) = \frac{dx(t)}{dt}$ and $w(t) = \frac{dy(t)}{dt}$, then we get

$$z(t) = f_1(t, w(t)), \quad t \in J$$

$$w(t) = f_2(t, z(t)), \quad t \in J.$$

4 Coupled system of Cauchy problems of fractional orders

Now, let $z(t) = {}_RD^{\alpha}x(t)$ and $w(t) = {}_RD^{\beta}y(t)$, then we get

$$\begin{aligned} \frac{d}{dt}I^{1-\alpha}x(t) &= z(t) \\ I^{1-\alpha}x(t) - I^{1-\alpha}x(t)|_{t=0} &= Iz(t); \\ I^{1-\alpha}x(t) &= Iz(t); \\ I^{1}x(t) &= I^{\alpha+1}z(t); \\ x(t) &= I^{\alpha}z(t). \end{aligned}$$

Similarly

$$y(t) = I^{\beta}w(t).$$

Then the coupled system of differential equations of fractional order (5) has the form:

$$z(t) = f_1(t, I^{\beta}w(t));$$

$$w(t) = f_2(t, I^{\alpha}z(t)).$$

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