Generalized Weinstein Transform in Quantum Calculus

Youssef Bettaibi, Hassen Ben Mohamedl

Department of Mathematics Faculty of Sciences of Gabes. Tunisia

e-mail: youssef.bettaibi@yahoo.com, hassenbenmohamed@yahoo.fr

Abstract

In this paper, we introduce a q-analogue of the Weinstein operator and we investigate its eigenfunction. Next, we define and study its associated Fourier transform which is a q-analogue of the Weinstein transform. In addition to several properties, we establish an inversion formula and prove a Plancheral theorem for this q-Weinstein transform.

Key words: q-theory, Weinstein transform, q-integral transform
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1 Introduction

The classical Weinstein operator defined by the use of the Laplacian and Bessel operators as

$$\Delta_\alpha = \Delta + L_\alpha, \quad \alpha \geq -\frac{1}{2}. \tag{0.1}$$

It is now known as an important operator in analysis, due to its applications in pure and applied Mathematics, especially in Fluid Mechanics ([10]).
The relevant harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem ([7, 8, 9]). In particular, the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator, called as Weinstein transform.

During the few last decades many authors were interested in q-analogues of different integral transforms and their applications.
In [2, 1], W. H. Abdi studied a q-analogue of the Laplace transform, in [17], T. H. Koornwinder and R. F. Swarttouw studied a q-analogue of the Hankel transform, in [13], A. Fitouhi et al. studied a q-analogue of the Bessel transform, in [11], the authors studied a q-analogue of the Mellin transform, in [18, 19], Rubin studied a q-analogue of the Fourier transform and in [5], the authors studied a q-analogue of the Dunkl transform .... But in literature there is no manuscript on the subject of Weinstein transform. So, it is in this context that this paper is built around the construction of a q-analogue of the Weinstein transform and study its properties.

This paper is organized as follows: In Section 2, we present some preliminaries results and notations that will be useful in the sequel. Section 3 is devoted to recall some result about the q-Rubin's transform, about the normalized third Jackson q-Bessel function and about the q-Bessel transform. Is Section 4, we introduce the q-Weinstein operator and its eigenfunction. Finally, in Section 5, we introduce and study the q-Weinstein transform. We provide for this transform an inversion formula and a Plancheral theorem.

1 Notations and preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer the reader to the general references [14] and [16], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions. Throughout this paper, we assume $q \in]0,1[$ and we denote $\mathbb{R}_q = \{\pm q^n \quad : \quad n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n \quad : \quad n \in \mathbb{Z}\}$.

1.1 Basic symbols

For $x \in \mathbb{C}$, the q-shifted factorials are defined by

$$(x;q)_0 = 1; \quad (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k), \quad n = 1,2,...; \quad (x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k). \tag{1.1}$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}. \tag{1.2}$$

1.2 Operators and elementary special functions

The q-Gamma function is given by ([15])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0,-1,-2,...$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q\longrightarrow 1^-}\Gamma_q(x) = \Gamma(x), \Re(x) > 0. \tag{1.3}$$

The q-trigonometric functions q-cosine and q-sine are defined by ([18, 19])

$$\cos(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_q!} \quad , \quad \sin(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_q!}. \tag{1.4}$$

The q-analogue exponential function is given by ([18, 19])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{1.5}$$

These three functions are absolutely convergent for all z in the plane and when q tends to 1 they tend to the corresponding classical ones pointwise and uniformly on compacts.
Note that we have for all $x \in \mathbb{R}_q$ ([18])

$$|\cos(x;q^2)| \leq \frac{1}{(q;q)_\infty}, \quad |\sin(x;q^2)| \leq \frac{1}{(q;q)_\infty},$$

and

$$| e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}. \tag{1.6}$$

The q^2-analogue differential operator is ([18, 19])

$$\partial_q(f)(z) = \begin{cases} \dfrac{f\left(q^{-1}z\right)+f\left(-q^{-1}z\right)-f\left(qz\right)+f\left(-qz\right)-2f(-z)}{2(1-q)z} & if \quad z\neq 0 \\ \lim\limits_{x\to 0}\partial_q(f)(x) \qquad (\text{in} \ \ \mathbb{R}_q) & if \quad z=0. \end{cases} \tag{1.7}$$

Remark that if f is differentiable at z, then $\lim\limits_{q\to 1}\partial_q(f)(z) = f'(z)$.
A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

For $\beta = (\beta_1, \beta_2) \in \mathbb{N}\times\mathbb{N}$, we use the notation

$$D_q^\beta = \partial_{x,q}^{\beta_1}\partial_{y,q}^{\beta_2}.$$

The q^2-analogue Laplace operator or q-Laplacian is given by

$$\Delta_q = \partial_{x,q}^2 + \partial_{y,q}^2.$$

The following lemma lists some useful computational properties of ∂_q, and reflects the sensitivity of this operator to parity of its argument. The proof is straightforward.

Lemma 1.1
1) $\partial_q \sin(x;q^2) = \cos(x;q^2)$, $\partial_q \cos(x;q^2) = -\sin(x;q^2)$ and $\partial_q e(x;q^2) = e(x;q^2)$.
2) For all function f on $\mathbb{R}_q$, $\partial_q f(z) = \dfrac{f_e(q^{-1}z)-f_e(z)}{(1-q)z} + \dfrac{f_o(z)-f_o(qz)}{(1-q)z}$.
3) For two functions f and g on $\mathbb{R}_q$, we have

- *if f even and g odd*

$$\partial_q(fg)(z) = q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) = \partial_q(g)(z))f(z) + qg(qz)\partial_q(f)(qz);$$

- *if f and g are even*

$$\partial_q(fg)(z) = \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z).$$

Here, for a function f defined on $\mathbb{R}_q$, f_e and f_o are its even and odd parts respectively.
The q-Jackson integrals are defined by ([15])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \tag{1.8}$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n), \tag{1.9}$$

and

$$\int_{-\infty}^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n), \tag{1.10}$$

provided the sums converge absolutely.
The following simple result, giving q-analogues of the integration by parts theorem, can be verified by direct calculation.

Lemma 1.2
1) For $a>0$, if $\int_{-a}^{a}(\partial_q f)(x)g(x)d_qx$ exists, then

$$\int_{-a}^{a}(\partial_q f)(x)g(x)d_qx = 2\left[f_e(q^{-1}a)g_o(a)+f_o(a)g_e(q^{-1}a)\right]-\int_{-a}^{a}f(x)(\partial_q g)(x)d_qx. \tag{1.11}$$

2) If $\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx$ exists,

$$\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx = -\int_{-\infty}^{\infty}f(x)(\partial_q g)(x)d_qx. \tag{1.12}$$

In the end of this subsection, let δ^α, $\alpha\geq-\frac{1}{2}$, denotes the Dirac-measure at $y\in\mathbb{R}_q$ defined on $\mathbb{R}_q$ by

$$\delta_y^\alpha(x)=\begin{cases}\left[(1-q)|y|^{2\alpha+2}\right]^{-1} & if \quad x=y\\ \\ 0 & if \quad x\neq y.\end{cases} \tag{1.13}$$

We recall that the q-Rubin's exponential function satisfies the following orthogonality relation ([6]).

$$\int_{-\infty}^{\infty}e(ixt;q^2)e(-iyt;q^2)d_qt=\left[\frac{2\Gamma_{q^2}(\frac{1}{2})}{(1+q)^{\frac{1}{2}}}\right]^2\delta_y^{-\frac{1}{2}}(x),\quad x,y\in\mathbb{R}_q. \tag{1.14}$$

1.3 Sets and spaces

By the use of the q^2-analogue differential operator ∂_q, we note:
• $\mathcal{E}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$, satisfying

$$\forall n\in\mathbb{N},\quad a\geq 0,\qquad P_{n,a}(f)=\sup\left\{|\partial_q^k f(x)|; 0\leq k\leq n; x\in[-a,a]\cap\mathbb{R}_q\right\}<\infty$$

and

$$\lim_{x\to 0}\partial_q^n f(x)\quad (\text{in}\quad \mathbb{R}_q)\qquad \text{exists.}$$

We provide it with the topology defined by the semi norms $P_{n,a}$.
• $\mathcal{E}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{E}_q(\mathbb{R}_q)$ constituted of even functions.
• $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n,m\in\mathbb{N},\qquad P_{n,m,q}(f)=\sup_{x\in\mathbb{R}_q}\mid x^m\partial_q^n f(x)\mid<+\infty$$

and

$$\lim_{x\to 0}\partial_q^n f(x)\quad (\text{in}\quad \mathbb{R}_q)\qquad \text{exists.}$$

• $\mathcal{S}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ constituted of even functions.
• $\mathcal{D}_q(\mathbb{R}_q)$ the space of functions defined on $\mathbb{R}_q$ with compact supports.
• $\mathcal{D}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{D}_q(\mathbb{R}_q)$ constituted of even functions.
• $\mathcal{E}_q(\mathbb{R}_q\times\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q\times\mathbb{R}_q$, satisfying forall $n\in\mathbb{N}$ and all $a\geq 0$,

$$P_{n,a}(f)=\sup\left\{|D_q^\beta f(x,y)|,\mid\beta\mid\leq n;(x,y)\in\mathbb{R}_q\times\mathbb{R}_q:\|(x,y)\|=\sqrt{x^2+y^2}\leq a\right\}<\infty$$

and

$$\lim_{(x,y)\to(0,0)} D_q^{\beta} f(x,y) \quad (\text{ in } \quad \mathbb{R}_q \times \mathbb{R}_q) \qquad \text{exists,}$$

where $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ and $\mid \beta \mid = \beta_1 + \beta_2$.
We provide it with the topology defined by the semi norms $P_{n,a}$.

- $\mathcal{E}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ the space of functions in $\mathcal{E}_q(\mathbb{R}_q \times \mathbb{R}_q)$, even with respect to the last variable.
- $\mathcal{S}_q(\mathbb{R}_q \times \mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q \times \mathbb{R}_q$ satisfying

$$\forall n \in \mathbb{N}, \qquad P_{n,q}(f) = \sup_{x,y \in \mathbb{R}_q} \sup_{|\beta| \le n} \left| D_q^{\beta} \left(\| (x,y) \|^{2n} f(x,y) \right) \right| < +\infty$$

and

$$\lim_{(x,y)\to(0,0)} D_q^{\beta} f(x,y) \quad (\text{in } \quad \mathbb{R}_q \times \mathbb{R}_q) \qquad \text{exists.}$$

- $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ the space of functions in $\mathcal{S}_q(\mathbb{R}_q \times \mathbb{R}_q)$, even with respect to the last variable.

Using the q-Jackson integrals, we note for $p > 0$ and $\alpha \ge -\frac{1}{2}$,

- $L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}$,
- $L_{\alpha,q}^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}$,
- $L_{\alpha,q}^p(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{0}^{\infty} |f(x)|^p x^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}$,
- $L_{\alpha,q}^p(\mathbb{R}_q \times \mathbb{R}_{q,+}) = \left\{ f : \|f\|_{L_{\alpha,q}^p(\mathbb{R}_q \times \mathbb{R}_{q,+})} = \left(\int_{0}^{\infty} \int_{-\infty}^{+\infty} |f(x,y)|^p y^{2\alpha+1} d_q x d_q y \right)^{\frac{1}{p}} < \infty \right\}$,
- $L_q^{\infty}(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < \infty \right\}$,
- $L_q^{\infty}(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_{q,+}} |f(x)| < \infty \right\}$,
- $L_q^{\infty}(\mathbb{R}_q \times \mathbb{R}_{q,+}) = \left\{ f : \|f\|_{L_q^{\infty}(\mathbb{R}_q \times \mathbb{R}_{q,+})} = \sup_{(x,y) \in \mathbb{R}_q \times \mathbb{R}_{q,+}} |f(x)| < \infty \right\}$.

2 q-Bessel Fourier and q-Rubin's Fourier transforms

2.1 q-Rubin's Fourier transform

R. L. Rubin defined in [19] the q^2-analogue Fourier transform as

$$\widehat{f}(x;q^2) = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} f(t) e(-itx;q^2) d_q t. \tag{2.1}$$

Letting $q \uparrow 1$ subject to the condition

$$\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}, \tag{2.2}$$

gives, at least formally, the classical Fourier transform. In the remainder of this paper, we assume that the condition (2.2) holds.

It was shown in [19] that $\widehat{f}(.;q^2)$ verifies the following properties:
1) If $f(u),\quad uf(u) \in L_q^1(\mathbb{R}_q)$, then $\partial_q \left(\widehat{f} \right)(x;q^2) = (-iuf(u))\widehat{\ }(x;q^2)$.
2) If $f,\quad \partial_q f \in L_q^1(\mathbb{R}_q)$, then $(\partial_q f)\widehat{\ }(x;q^2) = ix\widehat{f}\,(x;q^2)$.
3) $\widehat{f}\,(.;q^2)$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ onto itself. For $f \in L_q^2(\mathbb{R}_q)$, we have

$$\forall x \in \mathbb{R}_q,\quad \left(\widehat{f}\right)^{-1}(x;q^2) = \widehat{f}(-x;q^2) \text{ and } \|\widehat{f}\,(.;q^2)\|_{2,q} = \|f\|_{2,q}.$$

2.2 q-Bessel Fourier Transform

The normalized q-Bessel function is defined by

$$j_\alpha(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}. \tag{2.3}$$

Note that we have

$$j_\alpha(x;q^2) = (1-q^2)^\alpha \Gamma_{q^2}(\alpha+1)\left((1-q)x\right)^{-\alpha} J_\alpha((1-q)x;q^2), \tag{2.4}$$

where

$$J_\alpha(x;q^2) = \frac{x^\alpha (q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} . {}_1\varphi_1(0;q^{2\alpha+2};q^2,q^2x^2) \tag{2.5}$$

is the Jackson's third q-Bessel function.
Using the relations (2.3) and (1.4), we obtain

$$j_{-\frac{1}{2}}(x;q^2) = \cos(x;q^2), \tag{2.6}$$

$$j_{\frac{1}{2}}(x;q^2) = \frac{\sin(x;q^2)}{x} \tag{2.7}$$

and

$$\partial_q j_\alpha(x;q^2) = -\frac{x}{[2\alpha+2]_q} j_{\alpha+1}(x;q^2). \tag{2.8}$$

In [12], the authors proved the following estimation.

Lemma 2.1 *For $\alpha \geq -\frac{1}{2}$ and $x \in \mathbb{R}_q$, we have*

- $$|j_\alpha(x;q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2\alpha+1};q^2)_\infty}{(q^{2\alpha+1};q^2)_\infty} \begin{cases} 1, & \text{if } |x| \leq \frac{1}{1-q} \\ q^{\left(\frac{Log(1-q)|x|}{Log q}\right)^2}, & \text{if } |x| \geq \frac{1}{1-q} \end{cases}$$
- *for all* $v \in \mathbb{R}$, $j_\alpha(x;q^2) = o(x^{-v})$ *as* $|x| \longrightarrow +\infty$ (in $\mathbb{R}_q$).

As a consequence of the previous lemma and the relation (2.8), we have for $\alpha \geq -\frac{1}{2}$,

$$j_\alpha(.;q^2) \in \mathcal{S}_{*,q}(\mathbb{R}_q).$$

Then, from the relations (2.6), (2.7) and (1.5), we deduce that the two q-trigonometric functions and the Rubin's q-exponential function are in $\mathcal{S}_q(\mathbb{R}_q)$.
Furthermore, for $\alpha > -\frac{1}{2}$, $j_\alpha(.;q^2)$ has the following q-integral representations of Mehler type

$$\begin{aligned} j_\alpha(x;q^2) &= C(\alpha;q^2)\int_0^1 W_\alpha(t;q^2)\cos(xt;q^2)d_qt \\ &= \frac{1}{2}C(\alpha;q^2)\int_{-1}^1 W_\alpha(t;q^2)e(-ixt;q^2)d_qt, \end{aligned} \tag{2.9}$$

where

$$C(\alpha;q^2) = (1+q)\frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \tag{2.10}$$

and

$$W_\alpha(t;q^2) = \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}. \tag{2.11}$$

In particular, using the inequality (1.6), we obtain

$$|j_\alpha(x;q^2)| \le \frac{2}{(q;q)_\infty}, \forall x \in \mathbb{R}_q. \tag{2.12}$$

The orthogonality relation of the Jackson's third q-Bessel function $J_\alpha(.;q^2)$ proved in [17] gives the following orthogonality relation for the normalized q-Bessel function:

$$\int_0^{+\infty} j_\alpha(xt;q^2)j_\alpha(yt;q^2)t^{2\alpha+1}d_qt = \left[(1+q)^\alpha\Gamma_{q^2}(\alpha+1)\right]^2\delta_x^\alpha(y), \qquad x,y\in\mathbb{R}_{q,+}. \tag{2.13}$$

Using the same technique as in [13], one can prove the following result.

Proposition 2.1 *For $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda x;q^2)$ is the unique analytic even solution of the problem*

$$\left\{\begin{array}{rcl} L_q^\alpha f(x) &=& -\lambda^2 f(x), \\ f(0) &=& 1, \end{array}\right. \tag{2.14}$$

where $L_q^\alpha f(x) = \frac{1}{|x|^{2\alpha+1}}\partial_q[|x|^{2\alpha+1}\partial_q f(x)]$ is the q-Bessel operator.

Definition 2.1 *The q-Bessel Fourier transform is defined for $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, by*

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q}\int_0^\infty f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx \tag{2.15}$$

where

$$c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}. \tag{2.16}$$

Letting $q \uparrow 1$ subject to the condition (2.2), gives, at least formally, the classical Bessel-Fourier transform.
Some properties of the q-Bessel Fourier transform are given in the following results. For their proofs, we refer to [5].

Proposition 2.2

1. *For* $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *we have* $\mathcal{F}_{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_{q,+})$ *and*

$$\|\mathcal{F}_{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,q}.$$

2. *For* $f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *we have*

$$\int_0^\infty f(x)\mathcal{F}_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx = \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q\lambda. \tag{2.17}$$

3. *If* f *and* $L^\alpha_q f$ *are in* $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *then*

$$\mathcal{F}_{\alpha,q}(L^\alpha_q f)(\lambda) = -\lambda^2\mathcal{F}_{\alpha,q}(f)(\lambda).$$

4) If f *and* $x^2 f$ *are in* $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *then*

$$L^\alpha_q(\mathcal{F}_{\alpha,q}(f)) = -\mathcal{F}_{\alpha,q}(x^2 f).$$

Proposition 2.3 *If* $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, *then*

$$\forall x \in \mathbb{R}_{q,+}, \qquad f(x) = c_{\alpha,q}\int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)j_\alpha(\lambda x; q^2)\lambda^{2\alpha+1}d_q\lambda.$$

Theorem 2.1
1) Plancherel formula
For all $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, *we have*

$$\|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{2.18}$$

2) Plancherel theorem
The q*-Bessel transform can be uniquely extended to an isometric isomorphism on* $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *with* $\mathcal{F}^{-1}_{\alpha,q} = \mathcal{F}_{\alpha,q}$.

3 The q-Weinstein operator and its eigenfunctions

Let us now introduce the generalized Weinstein operator $\triangle^\alpha_q$ defined on $\mathbb{R}_q \times \mathbb{R}_{q,+}$ by:

$$\triangle^\alpha_q = \partial^2_{q,x} + \frac{1}{|y|^{2\alpha+1}}\partial_{q,y}(|y|^{2\alpha+1}\partial_{q,y}) = \partial^2_{q,x} + L^\alpha_q, \qquad \alpha \geq -\frac{1}{2}, \tag{3.1}$$

where L^α_q is the q-Bessel operator.
In the case $\alpha = -\frac{1}{2}$, $\triangle^\alpha_q$ reduces to the q^2-analogue Laplace operator $\triangle_q$.
The q-Weinstein operator $\triangle^\alpha_q$ tends, as q tends to 1, to the classical Weinstein operator $\triangle_\alpha$ given by:

$$\triangle_\alpha = \frac{\partial^2}{\partial x^2} + \frac{1}{|\, y\, |^{2\alpha+1}}\frac{\partial^2}{\partial y^2}\left(|\, y\, |^{2\alpha+1}\frac{\partial^2}{\partial y^2}\right).$$

Remark 3.1 *According to Lemma 1.1, we have*

$$L_q^\alpha f(x,y) = \frac{1-q^{2\alpha+1}}{(1-q)y}\partial_{q,y} f(x,qy) + \partial_{q,y}^2 f(x,y)$$

Proposition 3.1 *For all f and g such that*

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \triangle_q^\alpha f(x,y) g(x,y) y^{2\alpha+1} d_q x d_q y \quad \text{exists} \tag{3.2}$$

we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \triangle_q^\alpha f(x,y) g(x,y) y^{2\alpha+1} d_q x d_q y = \int_0^{+\infty}\int_{-\infty}^{+\infty} \triangle_q^\alpha g(x,y) f(x,y) y^{2\alpha+1} d_q x d_q y. \tag{3.3}$$

That is $\triangle_q^\alpha$ is self-adjoint.

Let f and g verify the hypothesis of the proposition. Then, according to the integration by parts theorem and the Fubini's theorem, we have

$$\begin{aligned}\int_0^{+\infty}\int_{-\infty}^{+\infty} \triangle_q^\alpha f(x,y) g(x,y) y^{2\alpha+1} d_q x d_q y &= \int_0^{+\infty}\left(\int_{-\infty}^{+\infty} \partial_{q,x}^2 f(x,y) g(x,y) d_q x\right) y^{2\alpha+1} d_q y \\ &+ \int_{-\infty}^{+\infty}\left(\int_0^{+\infty} L_q^\alpha f(x,y) g(x,y) y^{2\alpha+1} d_q y\right) d_q x \\ &= \int_0^{+\infty}\left(\int_{-\infty}^{+\infty} f(x,y)\partial_{q,x}^2 g(x,y) d_q x\right) y^{2\alpha+1} d_q y \\ &+ \int_{-\infty}^{+\infty}\left(\int_0^{+\infty} f(x,y) L_q^\alpha g(x,y) y^{2\alpha+1} d_q y\right) d_q x \\ &= \int_0^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\triangle_q^\alpha g(x,y) y^{2\alpha+1} d_q x d_q y.\end{aligned}$$

■

Proposition 3.2 *For all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ the system*

$$\begin{cases} L_q^\alpha u(x,y) &= -\lambda_2^2 u(x,y), \\ \partial_{q,x}^2 u(x,y) &= -\lambda_1^2 u(x,y) \\ u(0,0) &= 1, \\ \partial_{q,y} u(0,0) &= 0, \\ \partial_{q,x} u(0,0) &= -i\lambda_1 \end{cases} \tag{3.4}$$

has as analytic solution Λ_q^α even with respect to the second variable the function given by

$$\Lambda_{q,\lambda}^\alpha(x,y) = e(-i\lambda_1 x; q^2) j_\alpha(\lambda_2 y; q^2). \tag{3.5}$$

Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and impose $u(x,y) = e(-i\lambda_1 x; q^2) j_\alpha(\lambda_2 y; q^2)$. Then, according to the expression (2.14), we have

$$\begin{aligned} L_q^\alpha u(x,y) &= e(-i\lambda_1 x; q^2) L_q^\alpha \left(j_\alpha(\lambda_2 .; q^2)\right)(y) \\ &= -\lambda_2^2 u(x,y)\end{aligned}$$

and according to Lemma 1.1, we have

$$\partial_{q,x}^2 u(x,y) = -\lambda_1^2 u(x,y).$$

Furthermore, from the definition of the q-Rubin's exponential function and of the normalized q-Bessel function, we get $u(0,0)=1$, $\partial_{q,y}u\,(0,0)=0$ and
$\partial_{q,x}u(0,0) = -i\lambda_1$. ■

The function $\Lambda_{q,\lambda}^{\alpha}(x,y)$ called q-Weinstein kernel has a unique extension to $\mathbb{C}\times\mathbb{C}$. In the following result, we summarise some of its properties:

Proposition 3.3 *Let $\lambda = (\lambda_1,\lambda_2), z \in \mathbb{R}^2$ and $a \in \mathbb{C}$. Then,*

1. *The q-Weinstein fonction $\Lambda_{q,\lambda}^{\alpha}$ satisfies the following q-differential equation*

$$\triangle_q^{\alpha}(\Lambda_{q,\lambda}^{\alpha}) = - \parallel \lambda \parallel^2 \Lambda_{q,\lambda}^{\alpha}. \tag{3.6}$$

2. $\Lambda_{q,\lambda}^{\alpha}(z) = \Lambda_{q,z}^{\alpha}(\lambda), \qquad \Lambda_{q,a\lambda}^{\alpha}(z) = \Lambda_{q,\lambda}^{\alpha}(az)$ *and*

$$\overline{\Lambda_{q,\lambda}^{\alpha}(z)} = \Lambda_{q,-\lambda}^{\alpha}(z) = \Lambda_{q,(-\lambda_1,\lambda_2)}^{\alpha}(z).$$

3. *If $\alpha = -\frac{1}{2}$ then*

$$\Lambda_{q,\lambda}^{\alpha}(x,y) = e(-i\lambda_1 x; q^2)\cos(\lambda_2 y; q^2).$$

4. *For $\alpha > -\frac{1}{2}$, the function $\Lambda_{q,\lambda}^{\alpha}$ has the following q-integral representation*

$$\Lambda_{q,\lambda}^{\alpha}(x,y) = a_{\alpha,q} e(-i\lambda_1 x; q^2) \int_0^1 W_{\alpha}(t,q^2)\cos(\lambda_2 y t; q^2) d_q t. \tag{3.7}$$

5. *For all $\lambda \in \mathbb{R}_q \times \mathbb{R}_q$ the function $\Lambda_{q,\lambda}^{\alpha}$ is bounded on $\mathbb{R}_q \times \mathbb{R}_q$ and we have*

$$\mid \Lambda_{q,\lambda}^{\alpha}(x,y) \mid \leq \frac{4}{(q;q)_\infty^2}, \qquad (x,y) \in \mathbb{R}_q \times \mathbb{R}_q. \tag{3.8}$$

1. Let $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}\triangle_q^{\alpha}(\Lambda_{q,\lambda}^{\alpha}) &= \partial_{q,x}^2(\Lambda_{q,\lambda}^{\alpha}) + L_q^{\alpha}(\Lambda_{q,\lambda}^{\alpha}) = -(\lambda_1^2+\lambda_2^2)\Lambda_{q,\lambda}^{\alpha} \\ &= - \parallel \lambda \parallel^2 \Lambda_q^{\alpha}.\end{aligned}$$

2. follows from the definition of $\Lambda_{q,\lambda}^{\alpha}$.

3. is a direct consequence of the relation (2.6).

4. The q-integral representation of Mehler type of the normalized q-Bessel function (2.9) gives the relation (3.7).

5. follows from the two relations (1.6) and (2.12).

■

Remark 3.2 *For $\alpha > -\frac{1}{2}$, the function $\Lambda^{\alpha}_{q,\lambda}$ has also the following q-integral representation*

$$\Lambda^{\alpha}_{q,\lambda}(x,y) = \frac{a_{\alpha,q}}{2} e(-i\lambda_1 x; q^2) \int_{-1}^{1} W_{\alpha}(t, q^2) e(i\lambda_2 yt; q^2) d_q t. \tag{3.9}$$

Now, let $\delta^{\alpha}_{(x,y)}$, $\alpha \geq -\frac{1}{2}$, denotes the weighted Dirac-measure at $(x,y) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$ defined by

$$\delta^{\alpha}_{(x,y)}(z,t) = \begin{cases} \left[(1-q)^2 |z| t^{2\alpha+2}\right]^{-1} & if \quad (x,y) = (z,t), \\ 0 & ifnot. \end{cases} \tag{3.10}$$

Note that for all (x,y), $(z,t) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$ we have

$$\delta^{\alpha}_{(x,y)}(z,t) = \delta^{-\frac{1}{2}}_{x}(z) \delta^{\alpha}_{y}(t), \tag{3.11}$$

with δ^{α}, $\alpha \geq -\frac{1}{2}$, is given by (1.13).

Proposition 3.4 *For all $(t,z) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$, we have*

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \delta^{\alpha}_{(t,z)}(x,y) y^{2\alpha+1} d_q x d_q y = f(t,z).$$

For $(t,z) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$, we have

$$\begin{aligned} \int_0^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \delta^{\alpha}_{(t,z)}(x,y) y^{2\alpha+1} d_q x d_q y &= \int_0^{+\infty} \int_{-\infty}^{+\infty} f(x,y) \delta^{-\frac{1}{2}}_{z}(x) \delta^{\alpha}_{t}(y) y^{2\alpha+1} d_q x d_q y \\ &= \int_0^{+\infty} \delta^{\alpha}_{t}(y) y^{2\alpha+1} \left[\int_{-\infty}^{+\infty} f(x,y) \delta^{-\frac{1}{2}}_{z}(x) d_q x\right] d_q y. \end{aligned}$$

Then from the definition of the q-Jackson's integral, we obtain the result. ■

Proposition 3.5 *For all (x,y), $(t,z) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$, we have*

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \Lambda^{\alpha}_{q,\lambda}(x,y) \overline{\Lambda^{\alpha}_{q,\lambda}(z,t)} \lambda_2^{2\alpha+1} d_q\lambda_1 d_q\lambda_2 = \left[2(1+q)^{\alpha-\frac{1}{2}} \Gamma_{q^2}(\frac{1}{2}) \Gamma_{q^2}(\alpha+1)\right]^2 \delta^{\alpha}_{(x,y)}(z,t). \tag{3.12}$$

For all (x,y), $(t,z) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$, we have

$$\begin{aligned} \int_0^{+\infty} \int_{-\infty}^{+\infty} \Lambda^{\alpha}_{q,\lambda}(x,y) \overline{\Lambda^{\alpha}_{q,\lambda}(z,t)} \lambda_2^{2\alpha+1} d_q\lambda_1 d_q\lambda_2 &= \left(\int_0^{+\infty} j_{\alpha}(\lambda_2 y; q^2) j_{\alpha}(\lambda_2 t; q^2) \lambda_2^{2\alpha+1} d_q\lambda_2\right) \\ &\times \left(\int_{-\infty}^{+\infty} e(-i\lambda_1 x; q^2) e(i\lambda_1 z; q^2) d_q\lambda_1\right). \end{aligned}$$

The relations (1.14), (2.13) and (3.11) finish the proof. ■

Proposition 3.6

1. *The operator $\triangle^{\alpha}_{q}$ lives $\mathcal{E}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ and $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ invariant.*

2. *For all $\beta=(\beta_1,\beta_2)\in\mathbb{N}^2$, $\lambda,(x,y)\in\mathbb{R}_q\times\mathbb{R}_q$, we have*

$$\left|D_q^{\beta}\Lambda_{q,\lambda}^{\alpha}(x,y)\right|\leq\frac{4|\lambda_1|^{\beta_1}|\lambda_2|^{\beta_2}}{(q;q)_\infty}. \tag{3.13}$$

3. *For all $\lambda\in\mathbb{R}_q\times\mathbb{R}_q$, $\Lambda_{q,\lambda}^{\alpha}\in\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$.*

1. The result follows from the fact that

$$\triangle_q^{\alpha}f(x,y)=\partial_{q,x}^2f(x,y)+\frac{1-q^{2\alpha+1}}{(1-q)y}\partial_{q,y}f(x,qy)+\partial_{q,y}^2f(x,y).$$

2. The q-integral representation (3.9) of the q-Weinstein function, Lemma 1.1 and the relation (1.6) give the result.

3. Since the q-Rubin exponential function is in $S_q(\mathbb{R}_q)$ and the normalized third Jackson q-Bessel function is in $S_{*,q}(\mathbb{R}_q)$, then for all $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}_q\times\mathbb{R}_q$, the function $\Lambda_{q,\lambda}^{\alpha}(x,y)=e(i\lambda_1x;q^2)j_\alpha(\lambda_2y;q^2)$ is in $S_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$.

■

Remark 3.3 *A repeated application of the operator $\triangle_q^{\alpha}$ is defined by induction as*

$$\left(\triangle_q^{\alpha}\right)^0f=f,\qquad\left(\triangle_q^{\alpha}\right)^{n+1}f=\triangle_q^{\alpha}\left(\left(\triangle_q^{\alpha}\right)^nf\right).$$

From (1), we have for all $n\in\mathbb{N}$ and all $f\in\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$, $(\triangle_q^{\alpha})^nf\in\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$.

4 q-Weinstein transform

Definition 4.1 *The q-Weinstein Fourier transform is defined for $f\in L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})$, by*

$$\mathcal{F}_W^{\alpha,q}(f)(\lambda)=K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty}f(x,y)\Lambda_{q,\lambda}^{\alpha}(x,y)y^{2\alpha+1}d_qxd_qy \tag{4.1}$$

where

$$K_{\alpha,q}=\frac{(1+q)^{\frac{1}{2}-\alpha}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)\Gamma_{q^2}(\alpha+1)}. \tag{4.2}$$

Remark 4.1 *Letting $q\uparrow 1$ subject to the condition (2.2), gives, at least formally, the classical Weinstein transform.*

Some properties of the q-Weinstein transform are summarised in the following proposition.

Proposition 4.1

1. *For $f\in L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})$, we have $\mathcal{F}_W^{\alpha,q}(f)\in L_q^{\infty}(\mathbb{R}_q\times\mathbb{R}_{q,+})$,*

$$\|\mathcal{F}_W^{\alpha,q}(f)\|_{L_q^{\infty}(\mathbb{R}_q\times\mathbb{R}_{q,+})}\leq\frac{4K_{\alpha,q}}{(q;q)^2_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})} \tag{4.3}$$

and

$$\lim_{\|\lambda\|\to\infty}\mathcal{F}_W^{\alpha,q}(f)(\lambda)=0.$$

2. *For $f \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$ such that $\triangle^\alpha_q f \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$, we have*

$$\mathcal{F}^{\alpha,q}_W(\triangle^\alpha_q f)(\lambda) = - \parallel \lambda \parallel^2 \mathcal{F}^{\alpha,q}_W(f)(\lambda). \tag{4.4}$$

3. *For $f, g \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$, we have*

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \mathcal{F}^{\alpha,q}_W(f)(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 = \int_0^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathcal{F}^{\alpha,q}_W(g)(x,y)y^{2\alpha+1}d_qxd_qy.$$

1. Let $f \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$. From (3.8), we have

$$\forall (x,y),\lambda \in \mathbb{R}_q \times \mathbb{R}_{q,+}, \quad \left|f(x,y)\Lambda^\alpha_{q,\lambda}(x,y)\right| \leq \frac{4}{(q;q)^2_\infty}\left|f(x,y)\right|.$$

Then, from the definition of the q-generalized Weinstein transform $\mathcal{F}^{\alpha,q}_W$, we have

$$\begin{aligned} \mid \mathcal{F}^{\alpha,q}_W(f)(\lambda) \mid &\leq K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} \mid f(x,y) \mid\mid \Lambda^\alpha_{q,\lambda}(x,y) \mid y^{2\alpha+1}d_qxd_qy \\ &\leq \frac{4K_{\alpha,q}}{(q;q)^2_\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty} \mid f(x,y) \mid y^{2\alpha+1}d_qxd_qy \\ &= \frac{4K_{\alpha,q}}{(q;q)^2_\infty}\|f\|_{1,\alpha,q} \end{aligned}$$

and according to the Riemann-Lebesgue theorem, we get

$$\lim_{\|\lambda\|\to\infty} \mathcal{F}^{\alpha,q}_W(f)(\lambda) = \int_0^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\lim_{\|\lambda\|\to\infty}\Lambda^\alpha_{q,\lambda}(x,y)y^{2\alpha+1}d_qxd_qy = 0.$$

2. The result follows from the relation (3.3) and Proposition 3.3.

3. Let $f, g \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$. Then, from the relation (3.8), we have

$$\begin{aligned} &\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty} \left|f(x,y)g(\lambda_1,\lambda_2)\Lambda^\alpha_{q,\lambda}(x,y)\right| y^{2\alpha+1}\lambda_2^{2\alpha+1}d_qxd_qyd_q\lambda_1, d_q\lambda_2 \\ \leq\ & \frac{4}{(q;q)_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})}\|g\|_{L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})}. \end{aligned}$$

So, by the Fubini-Tonelli theorem, we can exchange the order of the q-integrals and obtain the desired result.

■

Theorem 4.1 *For all $f \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$, we have for all $(x,y) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$,*

$$\begin{aligned} f(x,y) &= K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} F^{\alpha,q}_W(f)(\lambda_1,\lambda_2)\Lambda^\alpha_{q,(-\lambda_1,\lambda_2)}(x,y)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 \\ &= \overline{F^{\alpha,q}_W(\overline{F^{\alpha,q}_W(f)})}(x,y). \end{aligned} \tag{4.5}$$

Let $f \in L^1_{\alpha,q}(\mathbb{R}_q \times \mathbb{R}_{q,+})$ and $(x,y) \in \mathbb{R}_q \times \mathbb{R}_{q,+}$.

$$\begin{aligned} & K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} F_W^{\alpha,q}(f)(\lambda_1,\lambda_2)\Lambda^\alpha_{q,(-\lambda_1,\lambda_2)}(x,y)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 \\ = \; & K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} F_W^{\alpha,q}(f)(\lambda_1,\lambda_2)\Lambda^\alpha_{q,(x,y)}(-\lambda_1,\lambda_2)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 \\ = \; & K^2_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}\int_{-\infty}^{+\infty} f(t_1,t_2)\Lambda^\alpha_{q,\lambda}(t_1,t_2)t_2^{2\alpha+1}d_qt_1d_qt_2\right)\Lambda^\alpha_{q,(x,y)}(-\lambda_1,\lambda_2)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2. \end{aligned}$$

But, from the relation (3.8) and the fact that $\Lambda^\alpha_{q,(x,y)} \in \mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$, we have

$$\begin{aligned} & \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left|f(t_1,t_2)\Lambda^\alpha_{q,\lambda}(t_1,t_2)\Lambda^\alpha_{q,(x,y)}(-\lambda_1,\lambda_2)\right| t_2^{2\alpha+1}\lambda_2^{2\alpha+1}d_qt_1d_qt_2d_q\lambda_1d_q\lambda_2 \\ \leq \; & \frac{4}{(q;q)^2_\infty}\|f\|_{L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})}\|\Lambda^\alpha_{q,(x,y)}(\cdot)\|_{L^1_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})} < \infty. \end{aligned}$$

Hence, by the Fubini-Tonelli theorem, we can exchange the order of the q-integrals, and by Propositions 3.5 and 3.4, we obtain

$$\begin{aligned} & K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} F_W^{\alpha,q}(f)(\lambda_1,\lambda_2)\Lambda^\alpha_{q,(-\lambda_1,\lambda_2)}(x,y)\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 \\ = \; & K^2_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(t_1,t_2)\left(\int_0^{+\infty}\int_{-\infty}^{+\infty}\Lambda^\alpha_{q,\lambda}(t_1,t_2)\Lambda^\alpha_{q,(-\lambda_1,\lambda_2)}(x,y)\lambda_2^{2\alpha+1}d_q\lambda_1d_q\lambda_2\right)t_2^{2\alpha+1}d_qt_1d_qt_2 \\ = \; & K^2_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(t_1,t_2)\delta^\alpha_{(x,y)}(t_1,t_2)t_2^{2\alpha+1}d_qt_1d_qt_2 \\ = \; & f(x,y). \end{aligned}$$

The second equality is a follows directly from the definition of the q-Weinstein transform, the definition of the q-Jackson integral and Proposition 3.3. ■

Theorem 4.2 Plancherel formula
For $\alpha \geq -1/2$, the q-Weinstein transform $F_W^{\alpha,q}$ is an isomorphism from $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ onto itself. Moreover, for all $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$, we have

$$\|F_W^{\alpha,q}(f)\|_{L^2_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})} = \|f\|_{L^2_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})}. \quad (4.6)$$

From Theorem 4.1, to prove the first part of theorem it suffices to prove that $F_W^{\alpha,q}$ lives $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ invariant. Moreover, from the definition of $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ and the properties of the operator ∂_q (Lemma 1.1), one can easily see that $\mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$ is also the set of all function defined on $\mathbb{R}_q \times \mathbb{R}_q$, such that for all $l \in \mathbb{N}$ and $\beta = (\beta_1,\beta_2) \in \mathbb{N}^2$, we have

$$\sup_{x,y\in\mathbb{R}_q}\left|D_q^\beta\left(\|\,(x,y)\,\|^{2l}\,f(x,y)\right)\right| < \infty \quad \text{and} \quad \lim_{(x,y)\to(0,0)}\partial_q^\beta f(x,y) \quad \text{exists.}$$

Let $f \in \mathcal{S}_{*,q}(\mathbb{R}_q \times \mathbb{R}_q)$, $l \in \mathbb{N}$ and $\beta = (\beta_1,\beta_2) \in \mathbb{N}^2$, from the relation (4.4), we have

$$\begin{aligned} \|\,\lambda\,\|^{2l}\,F_W^{\alpha,q}(f)(\lambda) &= (-1)^l F_w^{\alpha,q}((\Delta_q^\alpha)^l f)(\lambda) \\ &= (-1)^l K_{\alpha,q}\int_0^\infty\int_{-\infty}^\infty(\Delta_q^\alpha)^l f(x,y)\Lambda^{\alpha,}_{q,\lambda}(x,y)y^{2\alpha+1}d_qxd_qy. \end{aligned}$$

So, using the relation (3.13), we obtain

$$\begin{aligned}\left|D_q^\beta(\|\lambda\|^{2l} F_W^{\alpha,q}(f)(\lambda))\right| &= \left|(-1)^l K_{\alpha,q}\int_0^\infty\int_{-\infty}^\infty (\Delta_q^\alpha)^l f(x,y) D_q^\beta \Lambda_{q(x,y)}^\alpha(\lambda)|y|^{2\alpha+1}d_qxd_qy\right| \\ &\leq \frac{4K_{\alpha,q}}{(q;q)_\infty^2}\int_0^\infty\int_{-\infty}^\infty |(\Delta_q^\alpha)^l f(x,y)|x^{\beta_1}|y|^{2\alpha+\beta_2+1}d_qx.\end{aligned}$$

This together with the Remark 3.3 and the Lebesgue theorem prove that $F_W^{\alpha,q}(f)$ belongs to $\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$.
By Theorem 4.1, we deduce that $F_W^{\alpha,q}$ is an isomorphism of $\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$ onto itself and for $f\in\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$, we have $(F_W^{\alpha,q})^{-1}(f)(x,y) = F_W^{\alpha,q}(f)(-(x,y)),\quad x,y\in\mathbb{R}_q$.
Finally, the Plancheral formula (4.6) is a direct consequence of the second equality in Theorem 4.1 and the relation (4.5). ■

Theorem 4.3 Plancheral theorem *The q-Weinstein transform can be uniquely extended to an isometric isomorphism on* $L^2_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})$. *Its inverse transform* $(F_W^{\alpha,q})^{-1}$ *is given by :*

$$(F_W^{\alpha,q})^{-1}(f)(x,y) = K_{\alpha,q}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\lambda)\Lambda_{q,\lambda}^\alpha(x,y).\lambda_2^{2\alpha+1}d_q\lambda_1 d_q\lambda_2 = F_W^{\alpha,q}(f)(-(x,y)). \quad (4.7)$$

The result follows from Plancherel formula, Theorem 4.1 and the density of $\mathcal{S}_{*,q}(\mathbb{R}_q\times\mathbb{R}_q)$ in $L^2_{\alpha,q}(\mathbb{R}_q\times\mathbb{R}_{q,+})$. ■

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