The Classification of Permutation Groups with Maximum Orbits

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Abstract

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If no element of G moves any subset of Ω by more than m points (that is, if $|\Gamma^g \setminus \Gamma| \leq m$ for every $\Gamma \subseteq \Omega$ and $g \in G$), and the lengths two of orbits is p, and the rest of orbits have lengths equal to 3. Then the number t of G-orbits in Ω is at most $\lfloor \frac{1}{2}(3m-2) + \frac{5}{2p} \rfloor$. Moreover, we classifiy all groups for $t = \lfloor \frac{1}{2}(3m-2) + \frac{5}{2p} \rfloor$ is hold.(For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

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Introduction

1 Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g \setminus \Gamma|$ is bounded, for $g \in G$, we define the movement of Γ as move(Γ) = $\max_{g \in G} |\Gamma^g \setminus \Gamma|$. If move($\Gamma$) $\leq m$ for all $\Gamma \subseteq \Omega$,then G is said to have *bounded movement* and the *movement* of G is define as the maximum of move(Γ) over all subsets Γ, that is,

$$m := move(G) := sup\{|\Gamma^g \setminus \Gamma|| \Gamma \subseteq \Omega, g \in G\}.$$

This notion was introduced in [3]. By [3,Theorem 1],if G has bounded movement m,then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m.In particular it was shown that the number of G-orbits is at most $2m$-1.In this paper we will improve this to $\frac{1}{2}(3m-2) + \frac{5}{2p}$, if the lengths two of orbits is p, and the rest of orbits have lengths equal to 3. If m=1, then $t = \frac{1}{p}$, $|\Omega| = 2$ and G is Z_2 or S_2. So in this paper we suppose that m greater than 1. We present here a classification of all groups for which the bound $\frac{1}{2}(3m-2) + \frac{5}{2p}$ is attained. We shall say that an orbit of permutation group is nontrivial if its length is greater than 1.The main result is the following theorem.

Theorem 1.1. Let m be a positive integer and suppose that G is a permutation group on a set Ω such that G has no fixed points in Ω, and G has bounded movement equal to m . If the lengths two of orbits is p, and the rest of orbits have lengths

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equal to 3. Then the number t of G-orbits in Ω is at most $\frac{1}{2}(3m-2)+\frac{5}{2p}$. And also if $t=\frac{1}{2}(3m-1)+\frac{1}{p}$, then m is product of p in power of 3, and G is order pm, all G-orbits have length 3, and the pointwise stabilizers of the G-orbits are precisely the $\frac{1}{2}(3m-2)+\frac{5}{2p}$ distinct subgroups of G of index 3 .

Note that an orbit of a permutation group is non trivial if its length is greater than 1. The groups described below are examples of permutation groups with bounded movement equal to m which have exactly $\frac{1}{2}(3m-2)+\frac{5}{2p}$ nontrivial orbits.

2 Examples and Preliminaries

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle
representations has t nontrivial cycles of lengths $l_1, ..., l_t, say$. We might represent g as
$g=(a_1a_2...a_{l_1})(b_1b_2...b_{l_2})...(z_1z_2...z_{l_t})$. Let $\Gamma(g)$ denote a subset of Ω consisting $\lfloor l_i/2\rfloor$ points from the ith cycle , for each i, chosen in such a way that $\Gamma(g)^g \bigcap \Gamma(g)=\emptyset$. For example ,we could choose
$\Gamma(g)=\{a_2, a_4, ..., a_{k_1}, b_2, b_4, ..., b_{k_2}, ..., z_2, z_4, ..., z_{k_t}\}$, where $k_i=l_i-1$ if l_i is odd and $k_i=l_i$ if l_i is even . Note that $\Gamma(g)$is not uniquency determined as it depends on the way each cycle is written . For any set $\Gamma(g)$ consists of every point of very cycle of g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g \setminus \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t}\lfloor l_i/2\rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \setminus \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. [5, Lemma 2.1]. Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \setminus \Gamma| \leq \sum_{i=1}^{t}\lfloor l_i/2\rfloor$, where l_i is the length of the ith cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation . This upper bound is attained for $\Gamma=\Gamma(g)$ defined above .

Now we will show that there certainly is an infinite family of 3-groups for which the maximum bound obtained in Theorem 1.1 holds .

Example 2.2 . Let r be a positive integer , let G:=$Z_p^2Z_3^{r-2}$, let $t:=\frac{1}{2}(3m-2)+\frac{5}{2p}$, and let the lengths two of orbits is p, and the rest of orbits have lengths equal to 3, and $H_1, ..., H_t$ be an enumeration of the subgroups of index 3 in G. Define Ω_i to be the coset space of H_i in G and $\Omega := \Omega_1 \bigcup ... \bigcup \Omega_t$. If $g \in G \setminus 1$ then g lies in $\frac{1}{2}(p^2.3^{r-1}-1)+\frac{5}{2p}$ of the groups H_i and therefore acts on Ω as a permutation with $\frac{1}{2}(p^2.3^{r-1}-2)+\frac{3}{2p}=m-1$ fixed points and 3^{r-3} disjoint 3-cycles . Taking one point from each of these 3-cycles to form a set Γ we see that $m(G) \geq 3^{r-3}$,and it is not hard to prove that in fact $m(G)=3^{r-3}$. Thus $n=2t=(p^2.3^{r-1}-2)+\frac{5}{p}=$. This proves bound of $G-orbits$ of Theorem 1.1 . It follows that G has bounded movement equal to m, and G has $\frac{1}{2}(3m-2)+\frac{5}{2p}$ nontrivial orbits in Ω .

When $m>1$ the classification in Theorem 1.1 follows immediately from the following theorem about subsets with movement m.

Definition Let G be a permutation group on a set Ω with orbits Ω_i, for $i \in I$. We shall say that a subset $\Gamma \subseteq \Omega$ cuts across each G-orbit if $\Gamma_i := \Gamma \cap \Omega_i \notin \{\Phi, \Omega_i\}$, for every $i \in I$.

Theorem 2.3. Let $GlSym(\Omega)$ be a permutation group with t orbits for positive integer t, such that the lengths two of orbits is p, and the rest of orbits have lengths equal to 3. Moreover suppose that $\Gamma \subseteq \Omega$ such that move $(\Gamma)=m>1$, and Γ cuts across each G-orbit. Then $t \leq \frac{1}{2}(3m-2)+\frac{5}{2p}$ and moreover, if $t=\frac{1}{2}(3m-2)+\frac{5}{2p}$, then:

(1) G is an 3-group and all G-orbits of G has size 3 ;

(2) If the rank of the group G is r then $r \geq 2, t=(\frac{1}{2}(p^2.3^{r-1}-2)+\frac{5}{2p})$ and $m=p(3^{r-3})$;

(3)If one of the G-orbits is 3, then The t different G-orbits are (isomorphic to) the coset spaces of the $(\frac{1}{2}(p^2.3^{r-1}-2)+\frac{5}{2p})$ different subgroups of index 3 in G.

3 Proof of Theorem 2.3.

Proof: Let $\Omega_1, \ldots, \Omega_t$ be t orbits of G of lengths $n_1, \ldots, n_t$. Choose $\alpha_i \in \Omega$ and let $H_i := G_{\alpha_i}$, so that $|G : H_i| = n_i$. For $g \in G$, let $\Gamma(g) = \{\alpha_i | \alpha_i^g \neq \alpha_i\}$ be every second point of every cycle of g and let $\gamma(g) := |\Gamma(g)|$. Since $\Gamma(g) \cap \Gamma(g)^g = \emptyset$ it follows that $\gamma(g) \leq m$ for all $g \in G$. Let $\underline{\Omega} := \Omega_1 \cup \ldots \cup \Omega_t$, and let $\underline{G}$ and $\underline{H}_1, \ldots, \underline{H}_t$ denote the finite permutation groups on $\underline{\Omega}$ induced by G and $H_1, \ldots, H_t$ respectively. Then $n_i = |\underline{G}_1 : \underline{H}_i|$.

For $g \in G$, let $\underline{g} \in \underline{G}$ denote the permutation of $\underline{\Omega}$ induced by g. Then as $\gamma(1_G) = 0$, we have $\sum_{\underline{g} \in \underline{G}} \gamma(g) < m|\underline{G}|$. Now, Counting the pairs $(\underline{g}, i)$ such that $\underline{g} \in \underline{G}$ and $\alpha_i^g \neq \alpha_i$ gives

$$\sum_{\underline{g} \in \underline{G}} \gamma(g) = \sum_i |\{\underline{g} \in \underline{G} | \alpha_i^g \neq \alpha_i\}| = \sum_i |\{\underline{g} \in \underline{G} | g \notin H_i\}| = \sum_i (|\underline{G}| - |\underline{H}_i|) = |\underline{G}| \sum_i (1 - \frac{1}{n_i}).$$

It follows that $\sum_i (1 - \frac{1}{n_i}) < m$. Since $n_i \geq 3, p^2$ for each i, it follows that $\sum_i (1 - \frac{1}{n_i}) \geq \frac{2(p-1)}{p} + \frac{2}{3}(t-2)$ and hence $\frac{2(p-1)}{p} + \frac{2}{3}(t-2) < m$, that is, $tl\frac{1}{2}(3m-2) + \frac{5}{2p}$.

Consequently G has at most $\frac{1}{2}(3m-2) + \frac{5}{2p}$ orbits in Ω. Now Let m be a positive integer greater than 1. Suppose that $G \leq Sym(\Omega)$ with orbits$\Omega_1, \Omega_2, ..., \Omega_t$, where t=$\frac{1}{2}(3m-1) + \frac{1}{p}$. Suppose further that $\Gamma \subseteq \Omega$ has move $(\Gamma) = m$ and that cuts across each of the G-orbits Ω_i. For each i set $n_i = |\Omega_i|$ and $\Gamma_i = \Gamma \cap \Omega_i$. Note that $0 < |\Gamma_i| < n_i$.

Claim 3.1 If Theorem 2.3 holds for the special case in which $|\Gamma_i| = 1$ for $i = 1, ..., (\frac{1}{2}(3m-2) + \frac{5}{2p})$, then it holds in general .

Proof : Suppose that Theorem 2.3 holds for the case where each $|\Gamma_i| = 1$. For $i = 1, ..., t$, define $\sum_i := \{\Gamma_i^g | g \in G\}$, and note that $|\sum_i| \geq 3$ since Γ cuts across Ω_i. Set $\Sigma = \cup_{i \geq 1} \sum_i$. Then G induces a natural action on Σ for which the G-orbits are $\Sigma_1, ..., \Sigma_t$. Let G^Σ denote the permutation group induced by G on Σ , and let K denote the kernel of this action.

We claim that the t-element subset $\Gamma_\Sigma = \{\Gamma_1, ..., \Gamma_t\} \subseteq \Sigma$ has movement equal to m relative to G^Σ, and that Γ_Σ cuts across each Γ^Σ-orbit Σ_i. For each $g \in G$, $|\Gamma^g - \Gamma| lm$ and hence $|\Gamma_\Sigma^g - \Gamma_\Sigma| lm$. Thus move $(\Gamma_\Sigma) lm$. Also, Since $|\Sigma_i| \geq 3$ and $\Gamma_\Sigma \cap \Sigma_i$ Consists of the single element Γ_i of Σ_i , the set Γ_Σ cuts across each of the $\frac{1}{2}(3m-2) + \frac{5}{2p}$ orbits Σ_i. However, it follows that the number of G^Σ- orbits is at most $\frac{1}{2}(3.move(\Gamma_\Sigma) - 2) + \frac{5}{2p}$, and hence move $(\Gamma_\Sigma) = m$.

Thus the hypotheses of theorem 2.3 hold for the subset $\Gamma_\Sigma \subseteq \Sigma$ relative to G^Σ, and Γ_Σ meets each G^Σ-orbit in exactly one point. By our assumption it follows that $t = \frac{1}{2}(p^2 3^{r-1} - 2)\frac{5}{2p} = \frac{1}{2}(3m-2) + \frac{5}{2p}$ for some $r > 1$, and that $G^\Sigma = Z_3^r$ and each $|\Sigma_i| = 3$. Further, the subgroups H_i of G fixing Γ_i setwise range over the $\frac{1}{2}(p^2 3^{r-1} - 2) + \frac{5}{2p}$ distinct subgroups which have index 3 in G and which contain K. In particular, for each i, H_i is normal in G and hence the H_i-orbits in Ω_i are blocks of imprimitivity for G, and their number is at most $|G : H| = 3$. Since H_i fixes Γ_i setwise it follows that Γ_i is an H_i -orbit and $n_i = 3|\Gamma_i|$.

Let $g \in G \setminus K$. Then in its action on Σ , g moves exactly m of the Γ_i. Since the Γ_i are blocks of imprimitivity for G, each Γ_i^g is equal to either Γ_i or $\Omega_i - \Gamma_i$. It follows that $|\Gamma^g \setminus G|$ is equal to the sum of the sizes of the m subsets Γ_i moved by g. However, since move $(\Gamma) = m$, each of these m subsets Γ_i must have size 1. Since for each i we may choose an element g which moves Γ_i , we deduce that each of the Γ_i has size 1, and that K is the identify subgroup. It follows that theorem 2.3 hold for G . Thus the claim is proved .

From now on we may and shall assume that each $|\Gamma_i| = 1$. Let $\Gamma_i = \{\Omega_i\}$. Further we may assume that $n_1 l n_2 l ... l n_t$. For $g \in G$ let $c(g)$ denote the number of integers I such that $\omega_i^g = \omega_i$. Note that since move $(\Gamma) = m$, we have $c(g) > t - m = \frac{1}{2}(3m-2) + \frac{5}{2p} - m = \frac{m-2}{2} + \frac{5}{2p}$ and also $c(1_G) = t > \frac{m-2}{2} + \frac{5}{2p}$.

Lemma 3.2. If two of the orbits of G has length equal to p, then the rest orbits of G has size 3.

Proof : Let X denote the number of pairs (g,i) such that $g \in G$, $1 \le i \le t$, and $\omega_i^g = \omega_i$. Then $X = \sum_{g\in G} c(g)$, and by our observations, $X > |G|.(\frac{m-2}{2} + \frac{5}{2p})$. On the other hand, for each i, the number of elements of G which fix ω_i is $|G_{\omega_i}| = \frac{|G|}{n_i}$, and hence $X = |G| \sum_{i=1}^{t} n_i^{-1}$ If all the $n_i \ge 3$, and one of n_i is equal to p, then $X \le |G|.(\frac{2}{p} + \frac{t-1}{3}) = |G|(\frac{2}{p} + \frac{3m-2}{6} + \frac{2}{6p} + \frac{2}{3}) \le |G|.(\frac{m-2}{2} + \frac{17}{6p})$ (since $m \ge 3$) which is a contradiction. Hence n=3.

A similar argument to this enables us to show that except one of n_i the rest of n_i is $n_i = 3$, and hence that G is an $3 - group$.

Lemma 3.3. The group $G = Z_p^2.Z_3^r$ for some $r \ge 2$. Moreover for each $n_i = 3$, except one , the stabilizers $G_{\omega_i}(2lilt)$ are pair wise distinct subgroups of index 3 in G, and for each $g \ne 1, c(g) = (\frac{m-2}{2} + \frac{17}{6p})$.

Proof: By Lemma 3.2, except one of n_i the rest of n_i is $n_i = 3$. Thus $H := G_{\omega_i}$ is a subgroup of index 3. This time we compute the number Y of pairs (g, i) such that $g \in G \setminus H, 2lilt$, and $\omega_i^g = \omega_i$. For each such g, $\omega_1^g \ne \omega_1$ and hence there are $c(g)$ of these pairs with first entry g. Thus $Y = \sum_{g\in G\setminus H} c(g) \ge |G\backslash H|(\frac{m-1}{2} + \frac{5}{2p}) = |G|(\frac{m-1}{6} + \frac{5}{6p})$.

On the other hand, for each $i \ge 2$, the number of elements of G, which fix ω_i is $|G_{\omega_i}\backslash H|$. If $H = G_{\omega_i}$ then $|G_{\omega_i}\backslash H| = 0$, while if $G_{\omega_i} \ne H$, then $|G_{\omega_i}\backslash H| = \frac{|G_{\omega_i}|}{3} = \frac{|G|}{3n_i} l \frac{|G|}{9}$. Hence

$$\begin{aligned} Y &= \textstyle\sum_{i=2}^{t} |G_{\omega_i}\backslash H|l] \frac{|G|}{3} \sum_{i=2}^{t} \frac{1}{n_i} l \frac{|G|}{3}(\frac{2}{p} + \frac{t-2}{3}) \\ &= \frac{|G|}{3}(\frac{6+p(t-2)}{3p}) < |G|(\frac{m-2}{2} + \frac{17}{6p}) \end{aligned}$$

It follows that equality holds in both of the displayed approximations for Y. This means in particular that each $n_i = 2$, Whence $G = Z_p.Z_3^r$ for some r. Further, for each $i \ge 3, G_{\omega_i} \ne H$ and so $r \ge 2$. Arguing in the same way with H replaced by G_{ω_i}, for some $i \ge 2$, we see that $G_{\omega_i} \ne G_{\omega_j}$ if $j \ne i$, and also if $g \in G_{\omega_i}$ then $c(g) = (\frac{m-2}{2} + \frac{17}{6p})$. Thus the stabilizers $G_{\omega_i}(1lilt)$ are pairwise distinct , and if $gl1$ then $c(g) = (\frac{m-2}{2} + \frac{17}{6p})$. Finally we determine m.

Lemma 3.4.. $m = p(3^{r-2})$

Proof: We use the information in lemma3.3 to determine precise the quantity $X = \sum_{g\in G} c(g)$: $X = t + (|G| - 1).(\frac{1}{2}(m-2) + \frac{5}{2p}) = \frac{1}{2}(3m-2) + \frac{5}{2p} + (p^2.3^{r-2} - 1)(\frac{1}{2}(m-2)+$

$$X = |G| \sum_{i=1}^{t} n_i^{-1} = |G|.(\frac{2}{p} + \frac{t-2)}{3}) = p^2.3^{r-2}.(\frac{2}{p} + \frac{3m-2}{6} + \frac{5}{6p} - \frac{2}{3}).$$

Thus implies that $m = p(3^{r-3})$.

The proof of theorem 2.3 now follows from lemmas 3.2-3.4.

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