Insertion of Factors Property in Boolean Like Semi Rings

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Abstract

In this paper, Insertion of factors property is introduced in a Boolean like semi ring (shortly BLSR) and also study some of its properties. Denote IFP Boolean like semi ring by shortly IFPBLSR. Also prove that the collection of nilpotent elements forms a sub ring in IFPBLSR. Further prove that annihilator of x, denotes A(x) is an ideal in IFPBLSR R and also any sub sets of R, A(S) is an ideal of R.

Keywords: IFPBLSR, Annihilator, IFP Near Ring.

Mathematics Subject Classification.16Y30, 16Y60

Introduction

The concept of BLSR is due to Venkateswarlu and Murthy in [2]. Present article is a continuous work on BLSR by introducing the concept of Insertion of Factors Property. The current article is divided into two sections. In section1, the preliminary concepts and results connecting to BLSR [2,3] are given. In section 2, IFP is introduced in BLSR and furnish the examples of near rings with IFP & IFPBLSRs and conclude that these two near rings are different. Also give conditions for IFP near rings to be IFPBLSRs in theorem2.2. and also show that every weak commutative BLSR is IFPBLSR. After that some certain properties of IFPBLSR are studied and prove a typical theorem 2.4, the collection of all nil potent elements forms a sub ring in IFPBLSR. Finally, prove that A(x) is an ideal of IFPBLSR, which is extended to any subset S of R, A(S) is also ideal of R.

Preliminaries:.

Recollect definitions and results connecting with BLSRs from [2].

Definition 1.1. An algebraic structure (Ɍ, +, .) is said to be BLSR if the following holds

Ɍ1. (Ɍ,+) is a group

Ɍ2. (Ɍ , .) is a semi-group

Ɍ3. $\check{r}_1.(\check{r}_2+\check{r}_3) = \check{r}_1\check{r}_2 + \check{r}_1\check{r}_3\ \forall\ \check{r}_1,\check{r}_2,\check{r}_3 \in R$

Ɍ4. $\check{r}_1+\check{r}_1= 0\ \forall\ \check{r}_1 \in R$

Ɍ5. $\check{r}_1\check{r}_2(\check{r}_1+\check{r}_2+\check{r}_1\check{r}_2) = \check{r}_1\check{r}_2\ \forall\ \check{r}_1,\check{r}_2 \in R$

Definition 1.2.A BLSR R is called Weak commutative (for short W.C) if $\check{r}\check{s}\check{u} = \check{r}\check{u}\check{s}\ \forall\ \check{r},\check{u},\check{s} \in R$.

Remark 1.1.1

1. Condition Ɍ3 is called left distributive law and hence a system *(*Ɍ*,+,.)* satisfying Conditions Ɍ*1,* Ɍ*2* andɌ*3* is called a (left) near ring

2. From R4 of the definition 1.1, clearly the near ring R is of characteristic 2 and (R,+)is commutative.

Example 1. Let $R_1 = \{0,\check{r}_1,\check{r}_2,\check{r}_3\}$ The binary operations + and . are defined as follows

+	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
0	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
$\check{r}_1$	$\check{r}_1$	0	$\check{r}_3$	$\check{r}_2$
$\check{r}_2$	$\check{r}_2$	$\check{r}_3$	0	$\check{r}_1$
$\check{r}_3$	$\check{r}_3$	$\check{r}_2$	$\check{r}_1$	0

.	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
0	0	0	0	0
$\check{r}_1$	0	0	$\check{r}_1$	$\check{r}_1$
$\check{r}_2$	0	0	$\check{r}_2$	$\check{r}_2$
$\check{r}_3$	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$

Then (R_1, +, .) is a BLSR. We observe that R_1 is not W.C., as $\check{r}_3\check{r}_1\check{r}_2 \neq \check{r}_3\check{r}_2\check{r}_1$.

Example 2. Let $R_2 = \{0,\check{r}_1,\check{r}_2,\check{r}_3\}$. The binary operations + and .are defined as follows

+	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
0	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
$\check{r}_1$	$\check{r}_1$	0	$\check{r}_3$	$\check{r}_2$
$\check{r}_2$	$\check{r}_2$	$\check{r}_3$	0	$\check{r}_1$
$\check{r}_3$	$\check{r}_3$	$\check{r}_2$	$\check{r}_1$	0

.	0	$\check{r}_1$	$\check{r}_2$	$\check{r}_3$
0	0	0	0	0
$\check{r}_1$	0	$\check{r}_1$	0	$\check{r}_1$
$\check{r}_2$	0	0	0	0
$\check{r}_3$	0	$\check{r}_3$	0	$\check{r}_3$

Then (R_2, +, .) is a W.C BLSR.

Example 3. Let $R_3 = \{0, \alpha, \beta, \gamma, \delta, \sigma, \tau, 1\}$. The binary operations + and . are defined as follows.

+	0	α	β	γ	δ	σ	τ	1
0	0	α	β	γ	δ	σ	τ	1
α	α	0	γ	β	σ	δ	1	τ
β	β	γ	0	α	τ	1	δ	σ
γ	γ	β	α	0	1	τ	σ	δ
δ	δ	σ	τ	1	0	α	β	γ
σ	σ	δ	1	τ	α	0	γ	β
τ	τ	1	δ	σ	β	γ	0	α
1	1	τ	σ	δ	γ	β	α	0

.	0	α	β	γ	δ	σ	τ	1
0	0	0	0	0	0	0	0	0
α	0	0	0	0	α	α	α	α
β	0	0	β	β	0	0	β	β
γ	0	0	β	β	α	α	γ	γ
δ	0	0	0	0	δ	δ	δ	δ
σ	0	α	0	α	δ	σ	δ	σ
τ	0	0	β	β	δ	δ	τ	τ
1	0	α	β	γ	δ	σ	τ	1

Then $Ř_3$ is a BLSR. We notice that $Ř_3$ is not W.C. ,as $\sigma\tau\gamma \neq \sigma\gamma\tau$

Definition 1.3. A non-void subset Ï of Ř is said to be an ideal if

1. (Ï,+) is a sub group of (Ř,+) , i.e , $\forall$ m,n $\in$ Ï $\Rightarrow$ m+n $\in$ Ï
2. řm̧ $\in$ Ï, $\forall$ m̧ $\in$ Ï , ř $\in$ Ř, i.e Ř Ï $\subseteq$ Ï
3. ($ř_1$ + m̧)$ř_2$ + $ř_1 ř_2$ $\in$ Ï , $\forall$ $ř_1$,$ř_2$ $\in$ Ř, , m̧ $\in$ Ï

Lemma 1.4. ä.0 =0=0.ä, $\forall$ ä $\in$ Ř.

Lemma 1.5. $ä^4 = ä^2$ $\forall$ ä $\in$ Ř.

Lemma 1.6. If R is a Boolean like semi ring then, $ä^n$ = ä or $ä^2$ or $ä^3$, for any integer n > 0

Section-2:

IFPBLSR.

A Boolean like semi ring Ř (shortly BLSR) is called an IFP Boolean like semi ring (shortly IFPBLSR) if it fulfills

Insertion of Factors Property in BLSR. i.e. for s, t $\in$ Ř such that s t = 0, then for any n$\in$Ř, s n t = 0

Here some examples are given, in which near rings with Insertion of Factors Property are not necessarily BLSRs.

Example 2.1. LetT_1 = { 0,1,2,3 }. The binary operations + and .are defined as follows. It is IFP near ring.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	1	2	3

Then (T_1,+, .) is a (left) near ring, satisfies $ř_1ř_2$ ($ř_1$+$ř_2$+ $ř_1ř_2$) = $ř_1ř_2$, $\forall$ $ř_1$,$ř_2$ $\in$ T_1,

and ChrT_1 $\neq$ 2, since 1+1 = 2 $\neq$ 0. This example shows that condition Ř5 holds but condition Ř4 fails of the definition 1.1. and hence not all near rings to be BLSRs.

Example 2.2. Let T_2 = {0,1,2,3} , "+" denotes addition modulo 4 and "." Defined by

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	0	0	0

Then (T_2,+, .) is a (left) IFP near ring , but not W.C BLSR and Chr $T_2 \neq 2$ and also $ř_1ř_2 (ř_1+ř_2+ ř_1ř_2) = ř_1ř_2 \,\forall\, ř_1,ř_2 \in T_2$, is not true, as 1.2 (1+2+1.2) =1 $\neq$ 1.2 and also 123$\neq$ 132.

Example 2.3. Let T_3 = {0,1,2,3} , "+" denotes addition modulo 4 and "." Defined by

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	1	2	3

Clearly (T_3,+, .) is a (left) IFP near ring in which $ř_1ř_2 (ř_1+ř_2+ ř_1ř_2) = ř_1ř_2, \,\forall\, ř_1,ř_2 \in T_3$ is true and Chr$T_3 \neq 2$

Example 2.4 Let $T_4 = \{0, \delta_1, \delta_2\, \delta_3\}$. The binary operations + and .are defined as follows

+	0	δ_1	δ_2	δ_3
0	0	δ_1	δ_2	δ_3
δ_1	δ_1	0	δ_3	δ_2
δ_2	δ_2	δ_3	0	δ_1
δ_3	δ_3	δ_2	δ_1	0

.	0	δ_1	δ_2	δ_3
0	0	0	0	0
δ_1	0	δ_1	δ_2	δ_3
δ_2	0	0	δ_2	δ_2
δ_3	0	δ_1	δ_2	δ_3

Then (T_4, +,.) is IFP near ring and ChrT_4 = 2 also $ř_1ř_2 (ř_1+ř_2+ ř_1ř_2) = ř_1ř_2, \,\forall\, ř_1,ř_2 \in T_4$ is not true.

Example 2.5. Let $T_5 = \{0,ř_1,ř_2,ř_3\}$. The binary operations + and .are defined as follows

+	0	δ_1	δ_2	δ_3
0	0	δ_1	δ_2	δ_3
δ_1	δ_1	0	δ_3	δ_2
δ_2	δ_2	δ_3	0	δ_1
δ_3	δ_3	δ_2	δ_1	0

.	0	δ_1	δ_2	δ_3
0	0	0	0	0
δ_1	0	δ_1	0	δ_1
δ_2	0	0	0	0
δ_3	0	δ_3	0	δ_3

Then (T_5 +, .) is a BLSR with satisfies IFP and hence It is IFPBLSR.

Remark 2.1. From the above examples, It is concluded that an IFP near rings not necessarily a BLSRs.

Theorem 2.1. An IFP near ring R is IFPBLSR if Chr R =2 and $ř_1ř_2 (ř_1+ř_2+ ř_1ř_2) = ř_1ř_2 , \forall\, ř_1,ř_2 \in R$.

Proof is straight forward from the Definition 1.1

Theorem 2.2. If R is weak commutative BLSR then R is IFPBLSR.

Proof. For s, t $\in$R such that st = 0, then for any n$\in$R, s t n = 0n $\Rightarrow$ s n t = 0 (by W.C), hence proved

Lemma 2.3.Let R be a IFPBLSR. The following are true, $\forall$ σ,τ ϵ R,

(i) $(\sigma+\tau)\sigma = (\sigma+\tau)\sigma\tau = (\sigma+\tau)\sigma\tau^2 = (\sigma+\tau)\sigma\tau^3$ if $\sigma^2 = 0$

(ii) $(\sigma+\tau)\tau = (\sigma+\tau)\tau\sigma = (\sigma+\tau)\tau\sigma^2 = (\sigma+\tau)\tau\sigma^3$ if $\tau^2 = 0$

(iii) $(\sigma+\tau)\sigma\tau = (\sigma+\tau)\sigma\tau\sigma$ if $\tau^2 = 0$

(iv) $(\sigma+\tau)\tau\sigma = (\sigma+\tau)\tau\sigma\tau$ if $\sigma^2 = 0$

(v) $(\sigma+\tau)^2 = 0$ if $\sigma^2 = 0 = \tau^2$

(vi) If $\sigma^3 = 0$ then necessarily $\sigma^2 = 0$.

Proof.

(i) $(\sigma+\tau)\sigma = (\sigma+\tau)\sigma((\sigma+\tau) + \sigma + (\sigma+\tau)\sigma) = (\sigma+\tau)\sigma(\tau+(\sigma+\tau)\sigma) = (\sigma+\tau)\sigma\tau + (\sigma+\tau)\sigma(\sigma+\tau)\sigma$

$= (\sigma+\tau)\sigma\tau + (\sigma+\tau)(\sigma^2+\sigma\tau)\sigma = (\sigma+\tau)\sigma\tau + (\sigma+\tau)\sigma\tau\sigma = (\sigma+\tau)\sigma\tau$, hence proved.

(Since, if $\sigma^2 = 0 \Rightarrow \sigma\, r\, \sigma = 0$, $\forall\, r \in R$ (IFP))

Now $(\sigma+\tau)\sigma = (\sigma+\tau)\sigma\tau \Rightarrow (\sigma+\tau)\sigma\tau = (\sigma+\tau)\sigma\tau\tau = (\sigma+\tau)\sigma\tau^2$ and

$(\sigma+\tau)\sigma\tau = (\sigma+\tau)\sigma\tau^2 \Rightarrow (\sigma+\tau)\sigma\tau\tau = (\sigma+\tau)\sigma\tau^2\,\tau \Rightarrow (\sigma+\tau)\sigma\tau^2 = (\sigma+\tau)\sigma\tau^3$

(ii) Follows from (i), interchanging σ and τ

(iii) $(\sigma+\tau)\sigma\tau = (\sigma+\tau)\sigma\tau((\sigma+\tau) + \sigma\tau + (\sigma+\tau)\sigma\tau) = (\sigma+\tau)\sigma\tau(\sigma+\tau) + (\sigma+\tau)\sigma\tau\sigma\tau + (\sigma+\tau)\sigma\tau(\sigma+\tau)\sigma\tau$

$= (\sigma+\tau)\sigma\tau\sigma + (\sigma+\tau)\sigma\tau^2$ ($\tau^2 = 0 \Rightarrow \tau\, r\, \tau = 0$, $\forall\, r \in R$ (IFP))

$= (\sigma+\tau)\sigma\tau\sigma$

(iv) Follows from (iii), interchanging σ and τ

$(\sigma+\tau)\tau\sigma = (\sigma+\tau)\tau\sigma\tau$

(v) $(\sigma+\tau)^2 = (\sigma+\tau)(\sigma+\tau) = (\sigma+\tau)\sigma + (\sigma+\tau)\tau = 0$, follows from (i) &(ii)

(vi) $\sigma^3 = 0 \Rightarrow \sigma^3\sigma = 0\sigma \Rightarrow \sigma^4 = 0 \Rightarrow \sigma^2 = 0$ (Lemma 2.3 & 2.4)

Theorem 2.4. The collection of all nilpotent elements of IFPBLSR R forms a sub ring of R.

Proof.Let $S = \{ \upsilon \in R \,/\, \upsilon^n = 0$,for some +ve integer n $\}$,Clearly $S \neq \phi$, $S \subseteq R$

Let σ, τ ϵ S, then $\sigma^m = 0$, $\tau^n = 0$ for some +ve integers m, n

By lemma , $\sigma^{m} = 0$ either $\sigma = 0$ or $\sigma^{2} = 0$ or $\sigma^{3} = 0$ and $\tau^{n} = 0$ either $\tau = 0$ or $\tau^{2} = 0$ or $\tau^{3} = 0$

The following cases are discussed

(i) $\sigma = 0$, $\tau = 0$ $\sigma + \tau = 0$

(ii) $\sigma = 0$, $\tau^{2} = 0$ implies $(\sigma + \tau)^{2} = 0$ $(\sigma + \tau)$

(iii) $\sigma = 0$, $\tau^{3} = 0$, by (v) & (vi) $(\sigma + \tau)^{3} = 0$

(iv) $\sigma^{2} = 0$, $\tau = 0$ implies $(\sigma + \tau)^{2} = 0$

(v) $\sigma^{2} = 0$, $\tau^{2} = 0$, by (v), $(\sigma + \tau)^{2} = 0$

(vi) $\sigma^{2} = 0$, $\tau^{3} = 0$ by (v) & (vi) $(\sigma + \tau)^{2} = 0$

(vii) $\sigma^{3} = 0$, $\tau^{2} = 0$ and $\sigma^{3} = 0$, $\tau^{3} = 0$, by (v) & (vi) $(\sigma + \tau)^{2} = 0$

Hence in any case, $(\sigma + \tau) \in S$

Consider, $(\sigma\tau)^{m} = \sigma\tau\sigma\tau\sigma\tau \ldots \sigma\tau$ (m-times) ,

Discussed from above cases and IFP, $(\sigma\tau)^{m} = 0$ implies that $\sigma\tau \in S$

Hence S is sub ring of R.

Theorem 2.5. If R is IFPBLSR and Ï is ideal of R then R/Ï is IFPBLSR.

Proof is straight forward by

Definition 2.6. If R is near ring and I is an ideal of R such that R/Ï is IFP near ring then Ï is called IFP ideal.

Theorem 2.7. If R is IFPBLSR and $x \in R$ then $A(x)=\{a \in R / xa = 0\}$ is an ideal of R.

Proof. If $k \in A(x)$, $s \in R$, then $xk=0$ and hence by IFP, $xsk = 0 \Rightarrow x(sk)=0 \Rightarrow sk \in A(x)$

If $u,v \in R$, $k \in A(x)$ then $xk=0$.

Consider $x[(u+k)v+uv)] = x(u+k)v + xuv = (xu+xk)v + xuv = xuv + xuv = 0$ (ChrR = 2)

Corollary 2.8. If R is IFPBLSR and S is any subset of R then A(S) is an ideal of R

Theorem 2.9. If R_1 and R_2 are IFPBLSRs then $R_1 \times R_2=\{ (r_1, r_2)/ r_1, r_2 \in R\}$ is IFPBLSR with point wise addition and multiplication.

Proof is straight forward.

Corollary 2.10. If $R_1, R_2, R_3, \ldots, R_n$ are IFPBLSRs then $R_1 \times R_2 \times R_3 \times \ldots \times R_n$ is IFPBLSR with point wise addition and multiplication.

Theorem 2.11. If R and S are IFPBLSRs then the homorphic image of R is IFPBLSR.

Proof. Let ϯ: $R \rightarrow S$ be a homomorphism.

If $ř_1, ř_2 \in ƫ(Ꞧ)$ then $\exists\ ä_1, ä_2 \in Ꞧ$, $ƫ(ä_1) = ř_1$, $ƫ(ä_2) = ř_2$

Then $\forall\ š \in ƫ(R)$, $\exists\ ř \in Ꞧ$ such that $ƫ(ř) = š$

Consider, $ř_1 š ř_2 = ƫ(ä_1)\, ƫ(ř)\, ƫ(ä_2) = ƫ(ä_1 ř ä_2) = ƫ(0) = 0'$

Corollary 2.12. If $ƫ : R \to Ꞧ/Ï$ is homomorphism of Ꞧ onto Ꞧ/Ï then Ꞧ/ Ï is IFPBLSR.

Theorem 2.13. If Ꞧ is IFPBLSR with unity element and if $e^2=e$ then for all $r \in Ꞧ$, $er(e+1)$ is nilpotent.

Proof. $e+e =0 \Rightarrow e^2 +e =0 \Rightarrow e(e+1) =0 \Rightarrow er(e+1)=0\ \forall\ r \in Ꞧ$ by IFP and hence $er(e+1)^n = 0$.

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