New Results on Extension of Linear Operators and Markov Moment Problem

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(Received: 2020-03-25, Accepted: 2020-04-02)

Abstract

One recalls earlier applications of extension of linear operators with two constraints to the abstract Markov moment problem and Mazur-Orlicz theorem. Next we generalize one of our previous results on the characterization for the existence of a linear extension T preserving the sandwich condition $T_1 \leq T \leq T_2$ on the positive cone of the domain (where T_1, T_2 are given linear operators). Precisely, a similar characterization is obtained, when the sandwich condition on the extension T is $Q \leq T \leq P$ on C, where $P, -Q$ are sublinear operators, and C is an arbitrary convex cone (that might be the entire domain space). In the end, solutions of moment and Mazur-Orlicz problems are discussed, pointing out evaluation of their norms. All these solutions are obtained from the theorems previously stated or proved in this work. Some of the solutions are Markov operators.

Keywords: Markov moment problem; Mazur-Orlicz theorem; characterizing the existence of a solution; Markov operator

2010 MSC: 47A57; 46A22; 46B42; 41A10

1. Introduction

We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894-1895 (see the basic book of N.I. Akhiezer [1] for details): find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders k ($k = 0,1,2,\ldots$) are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(s_k)_{k\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ ($t \geq 0$), which verifies the moment conditions:

$$\int_0^{\infty} t^k d\sigma = s_k \qquad (k = 0,1,2,\ldots)$$

This is a one dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The numbers $s_k, k \in \mathbb{N}$ are called the moments of the measure $d\sigma$. Existence, uniqueness and construction of the solution σ are studied. The present work concerns firstly the existence problem. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one denotes by $\varphi_j, \varphi_j(t) := t^j, j \in \mathbb{N}, t \in [0,\infty)$, $\mathcal{P}$ the vector space of polynomials with real coefficients and $T_0: \mathcal{P} \to \mathbb{R}, T_0\left(\sum_{j\in J_0} \alpha_j \varphi_j\right) := \sum_{j\in J_0} \alpha_j s_j$, where $J_0 \subset \mathbb{N}$ is a finite subset, then the moment conditions $T_0(\varphi_j) = s_j, j \in \mathbb{N}$ are obviously verified. It remains to check whether the linear form T_0 has nonnegative values at nonnegative polynomials. If the latter condition is also accomplished, then one looks for the existence of a linear positive extension T of T_0 to a larger ordered function space X which contains both $\mathcal{P}$ and the space of continuous compactly supported functions, then representing T by means of a positive regular Borel measure μ on $[0,\infty)$, via Riesz representation theorem. Alternately one can apply directly Haviland theorem. If an interval (for example $[a,b]$, $\mathbb{R}$, or $[0,\infty)$) is replaced by a closed subset of $\mathbb{R}^n, n \geq 2$, we have a multidimensional moment problem. The case of multidimensional moment problem on compact semi-algebraic subsets in $\mathbb{R}^n$ was intensively studied (see [4], [13], [29], [30], [32] and many other articles). Clearly, the classical moment problem is related to the form of positive polynomials on the involved closed subset of $\mathbb{R}^n$. As it is well-known, there exists nonnegative polynomials on the entire space $\mathbb{R}^n, n \geq 2$, which are not sums of squares of polynomials (contrary to the case $n = 1$). The analytic form of positive polynomials on special closed unbounded finite dimensional subsets is crucial in solving classical moment problems on such subsets (see [11] for the expression of nonnegative polynomials on a strip, in terms of sums of squares). Such results are useful in characterizing the existence of a positive solution by means of signatures of quadratic forms. In case of Markov moment problem, approximation of nonnegative compactly supported continuous functions (with their support contained in a closed subset) by special nonnegative polynomials on that subset, having known their analytic form is very important. Details and other aspects of the moment problem can be found in [13], [21], [22], [23], [26], [28]. The basic result on polynomial approximation over closed unbounded subsets of $\mathbb{R}^n, n \geq 1$ was first completely proved in [21], Lemma 7. It was republished and applied in [22] as well as in the review papers [23], [26], [28]. Besides known results, the latter three review works contain a few applications of such approximation type results to new theorems. In most of the cases, the uniqueness of the solution of the Markov moment problem on spaces $L_{1,\mu}(A)$ follows too, thanks to the density of polynomials in such spaces; here A is a closed unbounded subset of $\mathbb{R}^n$ and μ is a positive regular M-determinate Borel measure on A. Recall that a measure is $M-$determinate (moment determinate), if it is uniquely determinate by its classical moments (or, equivalently, by its values on polynomials). For determinacy, uniqueness and non-uniqueness of solutions of some moment problems see [5], [7], [10], [31]. For the construction of some solutions see [2], [12], [19]. Connection of the moment problem to operator theory is partially revealed in [5], [9], [29], [30], [32]. An interesting connection of some moment sequences to fixed point theory is pointed out in [3]. To conclude, for characterizing the existence of a solution for a classical moment problem, extension Hahn-Banach results and their generalizations accompanied by knowing the analytic form of positive polynomial on the set under discussion are the basic tools. Sometimes, especially in Markov moment problem, the uniqueness of the solution follows too, thanks to the density of polynomials in some function spaces (even over an unbounded closed subset). Otherwise, the uniqueness problem requires specific methods. Basic monographs on the moment problem are [1], [8] as well as the recent book [30]. In [2], the connection of the moment problem to Hahn-Banach type results is realized, when the sandwich condition is defined by a concave upper constraint and a convex lower constraint (conversely with respect to the classical Hahn-Banach type result). Sandwich results of this type occur naturally when working over simplexes. A sketch of an algorithm in approximating solutions of systems with infinite many equations and unknowns was pointed out in [19]. Inverse problems solved started form the moments are also sketched in more recent papers, such as [24], [26]. Finally, connections of Markov moment problem with optimization are studied in [27] (see also the references therein). The monograph [15] contains (among other information) many interesting inequalities involving classical convex functions and their generalizations. All the papers and monographs [1]-[33] are more or less related to the present work. The rest of this paper is organized as follows. Section 2 is devoted to recalling known results and methods on vector-valued Markov moment problem and Mazur-Orlicz theorem. In section 3, the first result of Section 2 is applied to obtain a characterization for the existence of a linear extension preserving two nonlinear constraints, which have to be accomplished on an arbitrary convex cone (which might be the entire domain space). Section 4 deals with applications of the results in Sections 2 and 3. In some cases, the linear solutions are Markov operators. Generally, the norms of the solutions can be determined by means of the norms of the bounded sublinear upper constraints. Section 5 concludes the paper.

To prove the above mentioned results, we use the following methods.

Methods

The basic methods used along this paper are:

1) Extension of linear operators with two nonlinear constarints.

2) Solving abstract and classical Markov moment problems and Mazur-Orlicz type theorems.

3) Pointing out Markov operators as linear solutions.

2. Results and Discussion

Almost all the results of this section will be applied in the sequel. Three of them have been published first in [18]. Their proofs are based on previous theorems on constrained extension of linear operators published in [16], [17]. The next result was published first in [16], where its proof was sketched as well. The detailed proof can be found in [17]. In the following statement, E will be a real vector space, F an order-complete vector lattice, $A, B \subset E$ convex subsets, $Q: A \to F$ a concave operator, $P: B \to F$ a convex operator, $H \subset E$ a vector subspace, $T_0: H \to F$ a linear operator. All vector spaces and linear operators are considered over the real field.

Theorem 2.1. ([16]). *Assume that*

$$T_0(x) \ge Q(x)\ \forall x \in H \cap A, \quad T_0(x) \le P(x)\ \forall x \in H \cap B$$

The following statements are equivalent:

(a) *there exists a linear extension* $T: E \to F$ *of the operator* T_0 *such that* $T|_A \ge Q, T|_B \le P$;

(b) *there exists* $P_1 : A \to F$ *convex and* $Q_1 : B \to F$ *concave operator such that for all*

$$(\rho, t, \lambda', a_1, a', b_1, b', v) \in [0,1]^2 \times (0,\infty) \times A^2 \times B^2 \times H,$$

one has

$$\begin{aligned}&(1-t)a_1 - tb_1 = v + \lambda'[(1-\rho)a' - \rho b'] \Rightarrow \\ &(1-t)P_1(a_1) - tQ_1(b_1) \ge T_0(v) + \lambda'[(1-\rho)Q(a') - \rho P(b')]\end{aligned} \tag{2.1}$$

Thus in the last relation we have a convex operator on the left hand side, and a concave operator on the right hand side.

Theorem 2.2. ([18]). *Let E be a preordered vector space,* F *an order complete vector lattice,* $P: E \to F$ *a convex operator,* $\{x_j\}_{j\in J} \subset E$, $\{y_j\}_{j\in J} \subset F$ *given families. The following statements are equivalent*

(a) *there exists a linear positive operator* $T: E \to F$ *such that*

$$T(x_j) = y_j\ \forall j \in J, \quad T(x) \le P(x)\ \forall x \in E;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *we have*

$$\sum_{j\in J_0} \lambda_j x_j \le x \in E \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le P(x)$$

If in addition we assume that P *is isotone* $(u \le v \Rightarrow P(u) \le P(v))$, *the assertions (a) and (b) are equivalent to (c), where*

(c) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *the following inequality holds*

$$\sum_{j\in J_0}\lambda_j y_j \le P\left(\sum_{j\in J_0}\lambda_j x_j\right)$$

Theorem 2.3. ([18]). Let E, F, $\{x_j\}_{j\in J}$, $\{y_j\}_{j\in J}$ *be as in Theorem 2.2,* $T_1, T_2 \in L(E,F)$ *two linear operators. The following statements are equivalent*

(a) *there is a linear operator* $T \in L(E,F)$ *such that*

$$T_1(x) \le T(x) \le T_2(x)\ \forall x \in E_+, T(x_j) = y_j\ \forall j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *the following implication holds true*

$$\left(\sum_{j\in J_0}\lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in E_+\right) \Rightarrow \sum_{j\in J_0}\lambda_j y_j \le T_2(\psi_2) - T_1(\psi_1).$$

The next result is a variant of Mazur-Orlicz Theorem, where the interpolation conditions $T(x_j) = y_j\ \forall j \in J$ from Theorem 2.2 are replaced by the weaker requirements $T(x_j) \ge y_j\ \forall j \in J$. Consequently, the corresponding weaker conditions appear in (b), (c) (see below), where it is sufficient that the coefficients $\lambda_j, j \in J_0$ to be nonnegative.

Theorem 2.4. ([18]). *Let* E *be a preordered vector space,* F *an order complete vector space* $\{x_j\}_{j\in J}$, $\{y_j\}_{j\in J}$ *be as in Theorem 2.2,* $P: E \to F$ *a sublinear operator. The following statements are equivalent*

(a) *there exists a linear positive operator* $T: E \to F$ *such that*

$$T(x_j) \ge y_j\ \forall j \in J,\ T(x) \le P(x)\ \forall x \in E;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R_+ = [0,\infty)$, *the following implication holds true*

$$\sum_{j\in J_0}\lambda_j x_j \le x \in E \Rightarrow \sum_{j\in J_0}\lambda_j y_j \le P(x)$$

If in addition we assume that P is isotone, the assertions (a) and (b) are equivalent to (c), where

(c) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R_+$, *the following inequality holds*

$$\sum_{j\in J_0}\lambda_j y_j \le P\left(\sum_{j\in J_0}\lambda_j x_j\right)$$

All the three results recalled above can be proved using theorems stated and proved in [24], [25], the latter paper completing and giving detailed proofs for the results published in [24]. For the sake of completeness, we illustrate how the polynomial results recalled in the Introduction can be applied to known results, in order to furnish new characterizations for the existence of a/the solution. We start by recalling basic result from [18].

Theorem 2.5. ([11]). *Suppose that* $p(t_1,t_2) \in \mathbb{R}[t_1,t_2]$ *is non – negative on the strip* $A = [0,1] \times \mathbb{R}$. *Then* $p(t_1,t_2)$ *is expressible as*

$$p(t_1,t_2)=\sigma(t_1,t_2)+\tau(t_1,t_2)t_1(1-t_1),$$

where $\sigma(t_1,t_2),\tau(t_1,t_2)$ are sums of squares in $\mathbb{R}[t_1,t_2]$.

Let $A=[0,1]\times\mathbb{R}$, ν a positive $M-$ determinate regular Borel measure on A, with finite moments of all orders, $E:=L_{1,\nu}(A), \varphi_j(t_1,t_2):=t_1^{j_1}t_2^{j_2}, j=(j_1,j_2)\in\mathbb{N}^2, (t_1,t_2)\in A$. Let F be on order complete Banach lattice, $(y_j)_{j\in\mathbb{N}^2}$ a sequence of given elements in F. Combining Theorem 2.5 with a polynomial approximation result deduced from Lemma 7 [21], we infer the following theorem (see [28], Theorem 3.1.7, p. 99).

Theorem 2.6. *Let $F_2\in B_+(E,F)$ be a linear bounded positive operator from E to F. The following statements are equivalent:*

(a) *there exists a unique bounded linear operator $T:E\to F$, such that*

$$T(\varphi_j)=y_j, \forall j\in \mathrm{N}^2,$$

T is between zero and T_2 on the positive cone of E, $\|T\|\le\|T_2\|$;

(b) *for any finite subset $J_0\subset\mathbb{N}^2$, and any $\{\lambda_j; j\in J_0\}\subset\mathbb{R}$, we have*

$$0\le\sum_{i,j\in J_0}\lambda_i\lambda_j\, y_{i+j}\le\sum_{i,j\in J_0}\lambda_i\lambda_j\, T_2(\varphi_{i+j});$$

$$0\le\sum_{i,j\in J_0}\lambda_i\lambda_j\left(y_{i_1+j_1+1,i_2+j_2}-y_{i_1+j_1+2,i_2+j_2}\right)\le$$

$$\sum_{i,j\in J_0}\lambda_i\lambda_j\left(T_2\left(\varphi_{i_1+j_1+1,i_2+j_2}-\varphi_{i_1+j_1+2,i_2+j_2}\right)\right), i=(i_1,i_2), j=(j_1,j_2)\in J_0$$

3. Existence of a linear extension preserving two nonlinear constraints

Our next goal is to find a variant of Theorem 2.3, when the linear operators involved in the constraints on the solution T are more general. Namely, T_2 will be replaced by an arbitrary sublinear operator, T_1 will be a supralinear operator, and the positive cone E_+ will be an arbitrary convex cone C (which might be equal to the whole domain space E). Consequently, the linear subspace $C\cap(-C)$ is not generally reduced to the origin.

Theorem 3.1. *Let E be a vector space, F an order complete vector lattice, $C\subset E$ an arbitrary convex cone, $\Phi: C\to F$ a sublinear operator, $Q: C\to F$ a supralinear operator. Let $H\subset E$ be a vector subspace and $T_0: H\to F$ a linear operator. Assume that $Q|_{H\cap C}\le T_0|_{H\cap C}\le\Phi|_{H\cap C}$. The following statements are equivalent*

(a) *there exists a linear extension $T: E\to F$ of T_0 such that $Q\le T|_C\le\Phi$;*

(b) *for all $(f,h_1,h_1',h_2,h_2')\in H\times C^4$, the following implication holds true*

$$f=h_1+h_1'-(h_2+h_2')\Rightarrow T_0(f)\le\Phi(h_1)+\Phi(h_1')-\left(Q(h_2)+Q(h_2')\right) \tag{3.1}$$

(c) *for all $(f,\tilde h_1,\tilde h_2)\in H\times C^2$, one has*

$$f=\tilde h_1-\tilde h_2\Rightarrow T_0(f)\le\Phi(\tilde h_1)-Q(\tilde h_2) \tag{3.2}$$

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows directly from Theorem 2.1, applied to $A = B = C, P = P_1 := \Phi, Q_1 := Q, v = f$. The implication (2.1) can be written as

$$f = (1-t)a_1 + \lambda'\rho b' - (tb_1 + \lambda'(1-\rho)b') \Rightarrow$$

$$T_0(f) \le (1-t)\Phi(a_1) + \lambda'\rho\Phi(b') - \big(tQ(b_1) + \lambda'(1-\rho)Q(a')\big) =$$

$$\Phi\big((1-t)a_1\big) + \Phi(\lambda'\rho b') - (Q(tb_1) + Q(\lambda'(1-\rho)a') =$$

$$\Phi(h_1) + \Phi(h_1') - \big(Q(h_2) + Q(h_2')\big),$$

where $h_1 := (1-t)a_1 \in C, h_1' := \lambda'\rho b' \in C, h_2 := tb_1 \in C, h_2' := \lambda'(1-\rho)a' \in C$. In other words, for Φ sublinear and Q supralinear, (2.1) is equivalent to (3.1). According to Theorem 2.1, (2.1) (hence (3.1)) is equivalent to the existence of a linear extension T of T_0, from E to F, with the properties stated at point (a) of the present theorem. Namely, one has $Q \le T|_C \le \Phi$. Thus $(a) \Leftrightarrow (b)$ of the present theorem is proved. To prove $(a) \Leftrightarrow (c)$, observe that $(a) \Rightarrow (c)$ is obvious, thanks to the properties of T:

$$f = \tilde{h}_1 - \tilde{h}_2, \tilde{h}_1, \tilde{h}_2 \in C \Rightarrow T_0(f) = T(f) = T(\tilde{h}_1) - T(\tilde{h}_2) \le \Phi(\tilde{h}_1) - Q(\tilde{h}_2)$$

For the implication $(c) \Rightarrow (a)$, it is sufficient to prove that $(c) \Rightarrow (b)$, which is obvious. Indeed, under the hypothesis and using the notations from (b), we have

$$f = \tilde{h}_1 - \tilde{h}_2,$$

where $\tilde{h}_1 := h_1 + h_1', \tilde{h}_2 := h_2 + h_2'$. The hypothesis (3.2) from (c) leads to

$$T_0(f) \le \Phi(h_1 + h_1') - Q(h_2 + h_2') \le \Phi(h_1) + \Phi(h_1') - \big(Q(h_2) + Q(h_2')\big)$$

Thus (b) is verified and (a) follows according to implication proved previously. The proof is done. □

Remark 3.1. The basic implication of Theorem 3.1 is $(c) \Rightarrow (a)$.

4. Solutions of Markov moment problems and evaluating their norms

The aim of this Section is to point out some cases when the results of Section 2 and Section 3 can be applied. A special care is accorded to the control of the norm of the solutions.

Theorem 4.1. *Let E be a normed vector lattice, F an order complete normed vector lattice, $\Phi: E \to F$ a bounded sublinear operator, $\{e_j; j \in J\}$ an arbitrary family of linearly independent elements of E, $\{y_j; j \in J\}$ a family of given elements of F. The following statements are equivalent*

(a) *there exists a bounded linear operator T from E to F such that*

$$T(e_j) = y_j, j \in J, \qquad |T(h)| \le \Phi(h)\ \ \forall h \in E_+, \|T\| \le 2\|\Phi\|;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, $h_1, h_2 \in E_+$, the following implication holds*

$$\sum_{j \in J_0} \alpha_j e_j = h_1 - h_2 \Rightarrow \sum_{j \in J_0} \alpha_j y_j \le \Phi(h_1) + \Phi(h_2)$$

Proof. Condition (b) of the present statement, also using the notations of Theorem 3.1, where $H := Span\{e_j; j \in J\}$, $T_0: H \to F, T_0\big(\sum_{j \in J_0} \alpha_j e_j\big) := \sum_{j \in J_0} \alpha_j y_j, Q(h) := -\Phi(h), h \in E_+ := C$ lead to the conclusion that condition (c) of Theorem 3.1 is accomplished. Application of (c)⇒(a) of the latter theorem, yields the existence of a linear extension T of T_0 with the following properties

$$T(e_j) = T_0(e_j) := y_j, j \in J, -\Phi(h) \le T(h) \le \Phi(h), \forall h \in E_+ \Rightarrow |T(h)| \le \Phi(h) \ \ \forall h \in E_+$$

$$\Rightarrow \|T(h)\| \le \|\Phi(h)\| \le \|\Phi\|\|h\| \ \forall h \in E_+$$

If $x \in E$, then

$$\|T(x)\| = \|T(x^+) - T(x^-)\| \le \|T(x^+)\| + \|T(x^-)\| \le$$

$$\|\Phi\|(\|x^+\| + \|x^-\|) \le 2\|\Phi\|\||x|\| = 2\|\Phi\|\|x\|$$

We have obtained $\|T\| \le 2\|\Phi\| < \infty$. In particular, T is bounded. On the other side, observe that the implication (a)⇒(b) is obvious. This concludes the proof. □

Theorem 4.2. *Under assumptions and using the notations of Theorem 4.1, additionally assume that Φ is isotone. The following statements are equivalent*

(a) *there exists a positive linear operator $T: E \to F$ such that*

$$T(e_j) = y_j, j \in J, \qquad T(h) \le \Phi(h) \ \ \forall h \in E, \|T\| \le \|\Phi\|$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, the following inequality holds*

$$\sum_{j \in J_0} \alpha_j y_j \le \Phi\left(\sum_{j \in J_0} \alpha_j e_j\right)$$

Proof. The implication (a)⇒(b) is obvious, because of the following relations:

$$\sum_{j \in J_0} \alpha_j y_j = \sum_{j \in J_0} \alpha_j T(e_j) = T\left(\sum_{j \in J_0} \alpha_j e_j\right) \le \Phi\left(\sum_{j \in J_0} \alpha_j e_j\right)$$

To prove the converse, we apply Theorem 2.2, where x_j stands for $e_j, j \in J$ and P stands for Φ. Since Φ is isotone and condition (c) of Theorem 2.2 is clearly accomplished, application of the latter theorem yields the existence of a linear positive operator T that verifies

$$T(e_j) = y_j, j \in J, \qquad T(h) \le \Phi(h) \ \ \forall h \in E, \tag{4.1}$$

To obtain the last assertion of point (a), observe that the isotonicity of Φ and (4.1) lead to

$$\pm T(h) = T(\pm h) \le \Phi(\pm h) \le \Phi(|h|) \Rightarrow$$

$$|T(h)| \le \Phi(|h|) \Rightarrow \|T(h)\| \le \|\Phi(|h|)\| \le$$

$$\|\Phi\|\|h\|, h \in E \Rightarrow \|T\| \le \|\Phi\|$$

and the proof is done. □

The next results refer to the Markov moment problem on the space $C(K)$, where K is a compact Hausdorff topological space. All the linear solutions T appearing in the sequel are Markov operators.

Theorem 4.3. *Let K be a compact Hausdorff topological space, μ a positive regular Borel measure defined on the class of Borel subsets of $K, C(K)$ the Banach lattice of all real valued continuous functions on $K, \{\varphi_j\}_{j \in J}$ a family* *of linearly independent elements in* $C(K), \{y_j\}_{j\in J}$ *a given family of elements in* $L_{\infty,\mu}(K)$*. The following statements are equivalent*

(a) *there exists a linear positive (bounded) operator* $T: C(K) \to L_{\infty,\mu}(K)$ *such that*

$$T(\varphi_j) = y_j, j \in J, T(\varphi) \le \left(\sup_{t\in K} \varphi(t)\right) ¶ \ \forall \varphi \in C(K) \tag{4.2}$$

In particular, the following equalities hold

$$T(¶) = ¶, \|T\| = 1;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, *the following relation holds true*

$$\sum_{j\in J_0} \alpha_j y_j \le \sup_{t\in K}\left(\sum_{j\in J_0} \alpha_j \varphi_j(t)\right) ¶ \tag{4.3}$$

Proof. The implication $(\boldsymbol{a})$**⇒(**b) is obvious, thanks to the properties of T. To prove $(b) \Rightarrow (a)$, one applies Theorem 2.2, implication $(c) \Rightarrow (a)$, for

$$E = C(K), F = L_{\infty,\mu}(K), \quad \Phi(\psi) = \left(\sup_{t\in K} \psi(t)\right) ¶, \psi \in E \tag{4.4}$$

Observe that Φ defined by (4.4) is a scalar valued sublinear nondecreasing functional multiplied by the class of the constant function ¶ in $L_{\infty,\mu}(K)$, hence is an isotone sublinear operator. The inequality (4.3) is equivalent to the fact that condition written at point (c) of Theorem 2.2 is accomplished. Since $L_{\infty,\mu}(K)$ is an order complete vector lattice, according to Theorem 2.2, there exists a positive linear operator $T: E \to F$ with the properties mentioned at point (a) of the latter theorem. In particular, it results $T(\varphi_j) = T_0(\varphi_j) \coloneqq y_j, j \in J$. Moreover the following implications hold

$$\varphi \in E \Rightarrow T(\varphi) \le \left(\sup_{t\in K} \varphi(t)\right) ¶, -T(\varphi) \le \left(\sup_{t\in K} -\varphi(t)\right) ¶ =$$

$$= -\left(\inf_{t\in K} \varphi(t)\right) ¶ \Rightarrow \left(\inf_{t\in K} \varphi(t)\right) ¶ \le T(\varphi) \le \left(\sup_{t\in K} \varphi(t)\right) ¶$$

Thus (4.2) are proved. In particular, for $\varphi = ¶$, it results $T(¶) = ¶$. Since any $\varphi \in E$ with $\|\varphi\|_E \le 1$ is situated in the order interval $[-¶, ¶]$, the positivity of T leads to $T(\varphi) \in [T(-¶), T(¶)] = [-¶, ¶] \Rightarrow \|T(\varphi)\|_F \le 1 \Rightarrow \|T\| \le 1$. But we have already seen that $\|T(¶)\|_F = \|¶\|_F = 1$. Hence $\|T\| = 1$ and the proof is done. □

In the next theorem, K will be a compact subset of $\mathbb{R}^n$ ($n \ge 1$ is a natural number), $E = C(K)$, F an order complete Banach lattice with a strong order unit $¶_F$ such that the order interval $[-¶_F, ¶_F]$ is equal to the closed unit ball of F. $j = (j_1, \dots, j_n) \in \mathbb{N}^n$. $t = (t_1, \dots, t_n) \in K, |j| = \sum_{k=1}^n j_k, \varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}$.

Theorem 4.4. *Let* $\{y_j; |j| \le m\} \subset F$, $m \in \mathbb{N}, m \ge 1$. *The following statements are equivalent*

(a) *there exists a positive linear operator* $T: C(K) \to F$ *such that*

$$T(\varphi_j) = y_j, |j| \le m, \left(\inf_{t\in K} \varphi(t)\right) ¶_F \le T(\varphi) \le \left(\sup_{t\in K} \varphi(t)\right) ¶_F \ \forall \varphi \in C(K), \|T\| = 1; \tag{4.5}$$

(b) *for any* $\{\beta_j; |j| \le m\} \subset \mathbb{R}$, *the following relation holds*

$$\sum_{\substack{j\in\mathbb{N}^n\\|j|\le m}}\beta_j y_j \le \left(\sup_{t\in K}\left(\sum_{|j|\le m}\beta_j t^j\right)\right)\P_F \tag{4.6}$$

Proof. One repeats the proof of Theorem 4.3, where we replace $L_{\infty,\mu}(K)$ by F, $\varphi_j(t)=t^j, t\in K, j\in J:=\mathbb{N}^n$, $|j|\le m, \Phi(\psi)=\left(\sup_{t\in K}\psi(t)\right)\P_A, \psi\in E=C(K)$. Some of the notations have been defined before the statement. Clearly, from (4.5) with positive and unital T, the relation (4.6) follows. For the converse, repeating the arguments from the proof of Theorem 4.3, the existence of a positive linear operator verifying (4.5) follows (via Theorem 2.2., (c)⇒(a)). Here the subspace $H=Sp\{\varphi_j; j\in\mathbb{N}^n, |j|\le m\}$ is the vector subspace of all polynomial functions on K, of degree $\le m$. From (4.5), in particular, $T(\P_E)=\P_F$ follows as well and the proof is done. □

When applying Mazur-Orlicz theorem (Theorem 2.4), one can work with the subspace of all polynomial functions on $K\subset\mathbb{R}^n$, without any restriction on the their degree (one proves a full Mazur-Orlicz theorem). Such a result is not a direct consequence of the density of polynomials in $C(K)$ (that could be the case of the full moment problem for $C(K), K\subset\mathbb{R}^n$). In theorem 4.4, a solution for a truncated moment problem is proposed. A linear operator T from $C(K)$ to F is called a Markov operator if T is positive and $T(\P)=\P_F$ (the definition is valid for any Hausdorff compact K). It is easy to observe that a linear operator $T\in L(C(K),F)$ is a Markov operator if and only if $T(\varphi)\le\left(\sup_{t\in K}\varphi(t)\right)\P_F\ \forall\varphi\in C(K)$. In particular, solutions T from Theorems 4.3, 4.4 and 4.5 (the latter being proved below) are Markov operators. Let F be an order complete Banach space, having a strong order unit $\P_F$, $(y_j)_{j\in\mathbb{N}^n}$ a sequence in F. We prove the following theorem.

Theorem 4.5. *With the notations from Theorem 4.4, let $K=K_1\times\cdots\times K_n\subset\mathbb{R}_+^n$ be such that $K_l\subset\mathbb{R}_+$ is compact and denote $r_l=supK_l, l=1,\dots,n,\ r^j=r_1^{j_1}\cdots r_n^{j_n}, j=(j_1,\dots,j_n)\in\mathbb{N}^n$. The following statements are equivalent:*

(a) *there exists a (positive) linear operator $T: C(K)\to F$ such that*

$$T(\varphi_j)\ge y_j, j\in\mathbb{N}^n, \left(\inf_{t\in K}\varphi(t)\right)\P_F\le T(\varphi)\le\left(\sup_{t\in K}\varphi(t)\right)\P_F\ \ \forall\varphi\in C(K), \|T\|=1;$$

(b) $y_j\le r^j\P_F\ \ \forall j\in\mathbb{N}^n$.

Proof. The implication (a)⇒(b) is obvious, thanks to the properties of T. Namely, the following relations hold true:

$$y_j\le T(\varphi_j)\le\left(\sup_{t\in K}\varphi_j(t)\right)\P_A=\left(\sup_{t\in K}(t_1^{j_1}\cdots t_n^{j_n})\right)\P_A=r^j\P_A,\qquad j\in\mathbb{N}^n$$

To prove (b)⇒(a), we use the implication (c)⇒(a) of Theorem 2.4. The conditions mentioned at (c) of the latter theorem is accomplished, since for any finite subset $J_0\subset\mathbb{N}^n$ the following inequalities hold

$$y_j\le r^j\P_F, \lambda_j\ge 0, \forall j\in J_0\Rightarrow$$

$$\sum_{j\in J_0}\lambda_j y_j\le\left(\sum_{j\in J_0}\lambda_j r^j\right)\P_F=\left((\sum_{j\in J_0}\lambda_j t^j)|_{t=(r_1,\dots,r_n)}\right)\P_F=$$

$$\sup_{t\in K}\left(\sum_{j\in J_0}\lambda_j t^j\right)\P_F=\Phi\left(\sum_{j\in J_0}\lambda_j\varphi_j\right), \Phi(\psi):=\left(\sup_{t\in K}\psi(t)\right)\P_F, \psi\in C(K)$$

According to Theorem 2.4, (c)⇒(a), there exists a positive linear operator $T: C(K) \to F$ with the properties mentioned at point (a) of the present theorem. The proof is done. □

Observe that for Mazur-Orlicz theorem, it is not necessary that F be a lattice (F is an order complete Banach space, which is sufficient for applying Theorem 2.4). On the other side, in [12], Theorem 2.1, one gives a sufficient condition ensuring the existence of signed real valued measure-solution of a Markov moment problem. The proof is based on Theorem 2.3 of the present work.

Conclusions

Sections 3 and 4 contain our main new results. Such results could be proved for convex dominating operators as upper constraint. All dominating operators Φ appearing in Section 4 are sublinear. However, some of our results (such as Theorem 2.1, Theorem 2.2) hold true for convex operators. A class of examples of convex operators P which are not sublinear can be constructed in the following way. Let E be a normed vector space, F a normed vector lattice (the norm on F is solid: $u, v \in F, |u| \leq |v| \Rightarrow \|u\| \leq \|v\|$). Recall that most of the usual spaces have a natural structure of Banach lattice. Let $u_0 \in F_+ \setminus \{0_F\}, p \in (1, \infty)$. Define $P: E \to F, P(x) = \|x\|^p u_0$. Then P is convex, symmetric, non-sublinear, $P(\mathbf{0}) = \mathbf{0}$. If $T: E \to F$ is linear, such that $T \leq P$ on E, then it is easy to see that $\|T\| \leq \|u_0\|$. This example could be a motivation for a future work on this subject. Paper [14] points out many examples of interesting sublinear operators.

Conflicts of Interest

The author declares that there is no conflict of interest in publishing this paper.

Funding Statement

This research work was not funded in any way.

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