A Rigorous Geometrical Proof on Constructability of Magnitudes (A Classical Geometric Solution for the Factors: $\sqrt{2}$ and $\sqrt[3]{2}$)

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Abstract

In the absence of a comprehensive geometrical perceptive on the nature of rational (commensurate) and irrational (incommensurate) geometric magnitudes, a solution to the ages-old problem on the constructability of magnitudes of the forms $\sqrt{2}$ and $\sqrt[3]{2}$ (the square root of two and the cube root of two as used in modern mathematics and sciences) would remain a mystery. The primary goal of this paper is to reveal a pure geometrical proof for solving the construction of rational and irrational geometric magnitudes (those based on straight lines) and refute the established notion that magnitudes of form $\sqrt[3]{2}$ are not geometrically constructible. The work also establishes a rigorous relationship between geometrical methods of proof as applied in Euclidean geometry, and the non-Euclidean methods of proof, to correct a misconception governing the geometrical understanding of irrational magnitudes (expressions). The established proof is based on a philosophical certainty that "the algebraic notion of irrationality" is not a geometrical concept but rather, a misrepresentative language used as a means of proof that a certain problem is geometrically impossible.

KeyWords: Delian Constant, Pythagorean Factor, Constructible Magnitudes, Constructible Points, Rational Numbers, Irrational Numbers, Euclidean Geometry, Doubling A Square, Constructability

Abbreviations

2 D Two dimensional

CAD Computer-Aided Design

Notations

$\overline{AB}$ Denotes a straight line segment (length)

$\widehat{AB}$ Denotes circular arc or curve

1. Introduction

One of the most famous theorems used in mathematics is the Pythagorean Theorem. The theorem relates the shorter sides of a right triangle to its hypotenuse (the longest side) as found in (Book I of Euclid Elements, Proposition 47), which asserts "the squares on the sides of a right triangle, taken together, have the same content as the square on the hypotenuse (Longest side)" [1, 2]. The theorem is very often expressed in terms of algebra as $a^2 + b^2 = c^2$, with a, b and c representing magnitudes of the right triangle. An application of the Pythagorean theorem leads to a definition of the magnitude "length" in the Euclidean plane which is based on coordinates as $x_{1,}y_1$, and $x_{2,}y_2$. For instance, if we consider the right triangle depicted in figure (1), it is pretty easy to apply Pythagoras's theorem based on coordinates ($x_{2,}y_1$, $x_{1,}y_1$, $x_{2,}y_2$), to determine the magnitudes of the three edges.

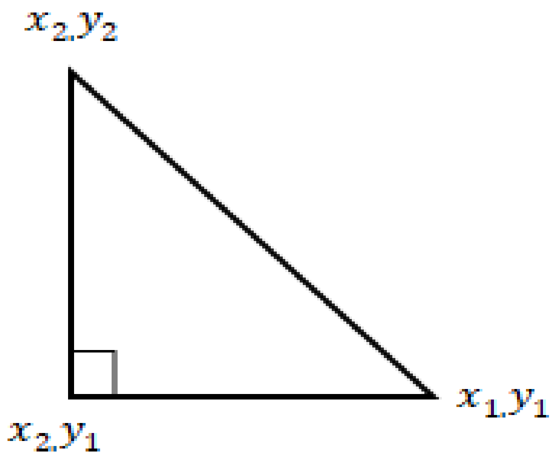

Figure 1: Definition of a line segment using Pythagorean Theorem

From figure (1), the magnitude of the hypotenuse (longest side) can be defined as $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where l is the length. In respect to geometric rigor, this work would treat the equation $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ as a definition, and not a theorem as it might be perceived in other contexts. Suppose we introduce some numbers as they are very often applied in the description of a physical system, based on coordinate pairs: $(1,0)$ for $x_{1,}y_1$ and $(0,1)$ for $x_{2,}y_2$ respectively, for a plane. Applying the equation $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ one can easily find the magnitude l as: $l = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$. Thus $l = \sqrt{2}$. The primary goal of this work is to illustrate the possibility of constructability of the two factors $\sqrt{2}$ and $\sqrt[3]{2}$. The constructability of $\sqrt[3]{2}$ is defined in the famous problem of doubling the volume of a cube, often referred to as a solution to the Delian constant. The problem of cube duplication asks that: "given compass and straightedge, construct a cube whose volume is double the volume of the given cube". Letting x and y to represent the lengths (magnitudes) of the original and new cube respectively the problem can be established as $y = x\sqrt[3]{2}$. To this effect, the goal has been to deduce the two factors $\sqrt{2}$ and $\sqrt[3]{2}$ from a simple Euclidean geometric plane (2D plane), which would be dealt with throughout this work. The discovery for factor $\sqrt{2}$ was an immediate and difficult problem for the Pythagoreans. For centuries, the factor $\sqrt{2}$ has not found clear and elementary geometrical interpretation and it is only alleged in regard to algebra, to be an irrational factor. The statement of $\sqrt{2}$ as irrational magnitude exhibits one serious geometric problem as it shall be exposed in later parts of this work, that the statement is itself not geometrically valid to define a geometric concept as wholly irrational. The Delian constant $\sqrt[3]{2}$ was equally a serious problem to the ancient Greek mathematicians, and its geometrical solution has remained an open challenge to this date [3, 4]. This paper is devoted to establishing the classical geometric interpretation of factors $\sqrt{2}$ and $\sqrt[3]{2}$ which in their explicit algebraic forms can be represented as $x^2 = 2$ and $x^3 = 2$ respectively. The equation $x^2 = 2$ is clearly a square of factor $\sqrt{2}$ and considering x^2 as sort of magnitude, then the expression implies doubling a square. We can establish the problem of doubling a square as given an arbitrary square of magnitude y^2, conceive geometrically another square of magnitude x^2 such that $2y^2 = x^2$, with y and x as sides of the given and the new squares respectively. Similarly, for the Delian constant $\sqrt[3]{2}$, one can state the cube duplication problem as; given an arbitrary cube of magnitude, y^3 construct another cube of magnitude x^3 such that $2y^3 = x^3$ where y and x are edges of the given and the new cubes respectively. In solving these geometric puzzles, this paper would employ two approaches, Euclidean and non-Euclidean methods, to establish the descriptive significance of the factors $\sqrt{2}$ and $\sqrt[3]{2}$ for both cases. The two forms of approach would be supplemented by a philosophical account, for clarity in redefining the concept of constructability. This mode of approach is employed with the goal of clearing a serious conceptual language misconception governing the relationship between geometric and algebraic concepts of irrationals. This work is based on the classical belief that geometrically defined concept of "irrationality for $\sqrt{2}$" is not proven in classical geometry and that all geometric problems can purely be sought by a sort of compass-straightedge constructions. For completeness, this paper would prove the claim that all geometric magnitudes are constructible for compass-straightedge methods.

1.1. Geometric Perspective on Constructability of Magnitudes of form $\sqrt{2}$ and $\sqrt[3]{2}$

This section provides some brief account on the nature of Euclidean geometry, very often referred to as the classical geometry. This is the ordinary form of geometry concerned with the use of only two classical geometric tools: a compass and straightedge. It's a form of geometry governed by some important rules, and one very conspicuous such rules is that geometric constructions must be solely performed with the aid of straightedge and compass. By its' genetic design, a straightedge has no markings on it, and as such, its nature forbids any form of measurements in solution to a problem. It is also required that any form of geometric reasoning aimed at a solution to some classical problem should not involve arithmetic. Euclidean geometry is a collection of geometric concepts presented in a treatise called the Elements developed approximately 2,300 years ago by Euclid himself [1, 2, 5] . The Elements is fractioned into thirteen books based on the nature of early mathematical developments: books $\mathrm{I}-\mathrm{VI}$ involves plane geometric concepts, books $\mathrm{VII}-\mathrm{X}$ deals with number theory including the known Euclidean algorithm, the infinitude of primes, and the irrationality of $\sqrt{2}$ which is a subject to this work. Books $\mathrm{XI}-\mathrm{XIII}$ mainly deal with the form of geometry called solid geometry and ends with the construction of the five regular (platonic solids) (which include cube as a regular polygon). The discussion throughout this paper is largely between books $\mathrm{I}-\mathrm{VI}$ and the contents of the other texts. Starting with books $\mathrm{I}-\mathrm{VI}$ which deal solely with compass-straightedge constructions, one can easily deduce that the use of the real numbers is simply form of analysis that obscures one of the most interesting aspects of the natural development of geometry: the concept of continuity. The concept of continuity is based on geometry and gradually, by correspondence it has been applied to numbers, leading eventually to Dedekind's edifice of the field of real numbers [6]. This deduction poses the need for proper examination of the relation between magnitudes and numbers in Greek's geometry, an approach aimed at establishing the inconsistency between algebra and geometry when algebra is used as means of proving geometric concepts. It is this form of reasoning which creates the broad scale of the problem being sought in this paper.

1.1.1. The Nature of Numbers in Greek's Geometry and their Implication

Perhaps, understanding the nature of the ancient Greek's geometry, and the perception of the early mathematicians on the use of numbers as geometric magnitudes is a significant matter in this work. In classical Greek geometry, the only allowed numbers were positive integers or numbers of the form: $2, 3, 4, \ldots$ and unity (1). At the time, any other number which did not lie in the prescribed set of numbers (negative numbers and zero) was not accepted. Geometrically conceived quantities such as; line segments, angles, areas, and volumes were referred to as magnitudes. It is found that magnitudes of the same kind (line segments to line segments or areas to areas and so on) could be compared for size: less, equal, or greater, and they could be added or subtracted (the lesser from the greater). These magnitudes could not be multiplied. The aspect of multiplication involved a multi-step operation such as forming a rectangle from two straight line segments, or a volume from a straight line segment or as well, an area from a straight line segment. These operations could be considered a form of multiplication of magnitudes, and the result was a magnitude of a different kind. This paper asserts and as it is proven in later parts of the work, Euclidean geometry is not complete either. The concept of equality (often called congruence) is not defined in Euclid's Elements, for line segments, which could be tested by placing one segment on the other as a test to whether they coincide exactly. This form of operation is the nature of classical geometry in which the equality (congruence) or inequality (incongruence) of a quantity and a magnitude is perceived directly from the geometry without being imposed with the assistance of real numbers to measure magnitudes. In Euclid's Elements [1, 2, 5], the concept of commensurability is also not defined. But, one can carefully deduce its geometrical implication as established ordinarily in books $\mathrm{I}-\mathrm{VI}$, that two magnitudes of the same kind are commensurable if there exists a third magnitude of the same kind such that the first two are (regarding whole numbers) multiples of the third. Otherwise, they are incommensurable. For instance, from book X proposition 21, Euclid says (and proves) that the diagonal (hypotenuse) of a square is incommensurable with its side. So any two line segments if one taken as standard (reference) line segment, can be commensurable or incommensurable. However, it must be noted that Euclid did not say the square root of two (a number) is irrational or in other terms, the square root of two is not a rational number. Such a claim is not proven, and this does not imply Euclid did not understand algebra. Spectacularly, in-plane geometry where these problems lie, Euclid completely avoided the use of algebra. In essence, as it will be demonstrated later, Euclid used the diagonal and side of a square as a perfect geometrical example of illustrating incommensurability in its simplest form (in the case of straight lines). To this point, the discussion has been based on a line segment. Looking at a different quantity such as area would aid in understanding the difference in language between classical geometric language and the modern mathematical language, and this is very significant in developing the justification that uses numbers as concepts of classical geometry obscure the ordinary nature of compass-straightedge constructions. Throughout the contents in the Elements, there is no real number measure of the area of a plane figure. Instead, the equality of planes is verified by cutting in elemental geometric pieces and adding and subtracting congruent triangles (the Pythagorean Theorem). The Pythagorean Theorem is proved as the culmination of a series of propositions demonstrating equal areas for various figures (for example, triangles with congruent bases and congruent altitudes have the same area). For the theory of similar triangles, one can assert; two triangles are similar if their sides are proportional, meaning the ratios of the lengths of corresponding sides are equal to a fixed real number. Euclid instead uses the theory of proportion, due to Eudoxus, that is developed in Book V of the Elements. Two magnitudes a and b of the same kind are said to have a ratio $a{:}\,b$. In Euclid's context, this ratio is not a number, nor is it a magnitude. Its main role is explained by the fifth definition of Book V as two ratios a:b and c : d are equal (in which case one can assert that there is a proportion a is to b as c is to d, expressed as $a{:}\,b :: c{:}\,d$) if, for every choice of whole numbers (m, n), the multiple ma is less than, equal to, or greater than the multiple nb if and only if mc is less than, equal to, or greater than nd, respectively. There are no arithmetic operations (addition, multiplication, subtraction, and division) defined for these ratios, but they can be ordered by size. The whole theory of similar triangles is developed in Book VI based on the definition that two triangles are similar if their corresponding sides are proportional in pairs. To this point, it is pellucid that Euclid developed his geometry without using real numbers to measure line segments, angles, areas or any other magnitude.

1.1.2. Algebraic Perspective on Constructability of Factors $\sqrt{2}$ and $\sqrt[3]{2}$

In terms of algebra, it is clear that factors $\sqrt{2}$ and $\sqrt[3]{2}$ explicitly represent numbers (square root of two and cube root of two respectively both with unknown numerical values) and the factor $\sqrt[3]{2}$ is not considered a constructible geometrical magnitude. The factors $\sqrt{2}$ and $\sqrt[3]{2}$ are considered irrationals and they are believed to have no geometric solution since irrational numbers are not constructible. However, as it will also be observed later, the factors $\sqrt{2}$ and $\sqrt[3]{2}$ have clear geometrical interpretations, a concept which was inconceivable to the Pythagoreans and not established on any other text to this day. Section (2) provides algebraic proofs that $\sqrt{2}$ and $\sqrt[3]{2}$ are irrational numbers.

1.2. Distinctive Observations between Euclidean and Non-Euclidean Geometry (Case of Algebra)

Based on the so far established discussions, one can construe the following three rigorous and essential observations.

1. One of the goals of this work is to establish the distinction between plane geometry and the other forms of geometry. This was essentially done at first by Euclid himself, by putting together a collection of early mathematics materials as developed by the ancient mathematicians in classical categorical forms, which include books $\mathrm{I}-\mathrm{V}$, books $\mathrm{VI}-\mathrm{X}$ and books $\mathrm{XI}-\mathrm{XIII}$. This classification preserves the tradition of the form of mathematics found in each of the three classes of books.

2. The modern form of mathematics involves largely the use of numbers to describe physical systems. Those systems which cannot be fully defined using real numbers are described using other types of numbers

(rational, irrational, or symbolic numbers). Taking the case of symbolic representation where a symbol such as $\sqrt{2}$ would significantly represent an algebraic number, it is pellucid from section (1.1) that the relationship between the geometric and the algebraic significance of factor $\sqrt{2}$ is different. The factor $\sqrt{2}$ is inherently a geometric constant whose significance is found in the proof of the Pythagorean theorem, while in terms of algebra, $\sqrt{2}$ can best lead to definition (a measure) of a line segment. First, Euclidean geometry is not based on measurements, and again, a definition is different from a theorem. In terms of geometry, the square of $\sqrt{2}$ would be treated as a magnitude, while in algebra, the square of $\sqrt{2}$ is defined as a number. It is now known that numbers had typically significant implications in geometry than they are applied from algebra. The same is true for the Delian constant $\sqrt[3]{2}$ which inherently stems from plane geometry (since the factor $\sqrt[3]{2}$ can represent straight line segment). Geometrically, the cube of the factor $\sqrt[3]{2}$ would imply the magnitude volume, which is based on geometrical objects, while in algebra, the cube of $\sqrt[3]{2}$ could mean simply a number 2, which has no geometric significance.

3. Euclidean geometry has some significant limitations, however. It was earlier established that the geometrically consistently used terminologies "rational" and "commensurable" or "irrational" and "incommensurable" are not defined in Euclid's Elements. Mathematicians and scientists have deduced the implications of the two terms based on how they are applied in Euclid's work. These terms are not defined in this paper, but they are consistently applied where suitable in respect to Euclid's notion of commensurability and incommensurability.

2. Limitations of the Irrationality Proofs on Constructability of Factors $\sqrt{2}$ and $\sqrt[3]{2}$

This section provides a provable argument aimed at disproving the belief that $\sqrt{2}$ and $\sqrt[3]{2}$ are irrational magnitudes, and in particular, the factor $\sqrt[3]{2}$ is not constructible magnitudes. It begins by establishing some important background based on Euclid's notion of plane geometry in relation to the factors $\sqrt{2}$ and $\sqrt[3]{2}$ and culminates by exposing the limitations with the use of the irrationality avowal in defining the geometric significance of $\sqrt{2}$ and $\sqrt[3]{2}$.

2.1. Important Definitions According to Euclid

So far, it is clear we lack two very crucial definitions of the terms "equal" and "commensurate" from Euclid's Elements as specified in section (1.1). However, this work would follow Euclid's formal application of the two terms, so as to operate in the confines of classical geometry. Other important terms to be considered in this discussion include "Magnitude", "Ratio", "Constructability" and "Measurable magnitude". Understanding these terms and their use in geometry would play a very significant role in this work as well.

a) Magnitude: According to Euclid (Euclid's Elements book $X\ definition$ 1) [1, 2], magnitude is a part of a magnitude, the less of the greater, when it measures the greater. As stated earlier, magnitudes do not represent a measurement in geometry. Rather, a magnitude could be some length, area, and volume, inherently not based on measurement. Strictly, only magnitudes of similar kinds are comparable in classical geometry. The classical comparison of similar magnitudes is highly used as a means of proof in Euclidean geometry.

b) Ratio: From the Elements [1, 2] book $V\ definition$ 3, we are provided with the definition of a ratio as a sort of relation in respect of size between two magnitudes of the same kind. Magnitudes which have the same ratio are said to be proportional. The aspect of proportions is largely applied in the development of similar triangles. This approach will soon be adopted in later parts of this work, to prove that the square root of two ($\sqrt{2}$) is both rational and irrational with respect to geometry.

c) Commensurable: According to Euclid's Elements (book $X\ definition$ 1) [1, 2], those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure. This is not a definition for commensurability. It is a concept of relating magnitudes.

d) Commensurability in Straight Lines: Euclid's provides illustrative definition of incommensurability in straight lines using their square as provided in (book X $definition$ 2) [1, 2] that; straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure.

e) Constructability: In regard to this work, the concept of constructability would be defined based on real numbers starting from $\{1, 2, 3, \ldots\}$ strictly, based on the Greeks notion of using numbers in geometric constructions. In this respect one can consider a very clear fact about numbers in geometry that, a real number n is said to be constructible if and only if, given a line segment of $unit\ length$, a line segment of magnitude $|n|$ is constructible with compass and straightedge in a finite number of steps. Equivalently in algebra, n is said constructible if and only if there is a closed-form expression for n using only the integers 0 and 1 and the operations for addition, subtraction, multiplication, division, and square roots.

f) Measurable Magnitudes: The concept of measure applies very much in mathematical analysis. A measure on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is treated as a generalization of the concepts of length, area, and volume. For instance, a common form of measure of the interval $(0, 1)$ in the real numbers is its length in the everyday sense of the word, specifically, 1.

2.2. Limitations of Algebraic Proofs on the Constructability of the Factors $\sqrt{2}$ and $\sqrt[3]{2}$

This section provides purely algebraic proofs that the factors $\sqrt{2}$ and $\sqrt[3]{2}$ are irrational, and that $\sqrt[3]{2}$ is not geometrically constructible. The section would be based on the previously established discussion that the only allowed numbers in Greek's geometry were the positive integers. These type of numbers would be treated in some cases, as geometric magnitudes. In any case, where the two factors $\sqrt{2}$ and $\sqrt[3]{2}$ are used to represent a geometric magnitude, it would imply such a magnitude is irrational. The goal of this section is to refute the assertion that constants of the form $\sqrt{n}$ and $\sqrt[3]{n}$ are geometric irrationals, where n is not a perfect square or a perfect cube.

2.2.1. The Problems with Irrationality Proof for Constructability of $\sqrt{2}$

Definitions a and b, and the deduced definition c from section (2.1) provide very significant information about the nature of the proof to be revealed in this section. Considerably, we discover that the deduced definition on "commensurable straight lines" only applies to straight lines, and from (d) one can easily put it as lines are ever said to be geometrically commensurable in the square. So certainly, commensurable lines are also commensurable in the square. However, lines can be commensurable in the square but not commensurable otherwise. Consider the square face $ABCD$ shown in figure (2).

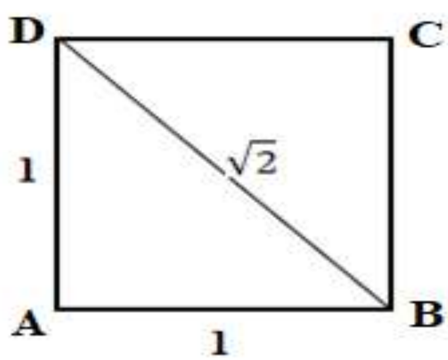

Figure 2: Reconsidering the proof that diagonal of a square is incommensurate with its sides

To test for commensurability between the diagonal $\overline{BD}$ and the side $\overline{DA}$ in figure (2), (in this case by using the ratio of the sides $\overline{BD}$ and $\overline{DA}$ respectively) it is pretty simple to show that the side $\overline{DA}$ is not finitely relational with $\overline{BD}$. This result is depicted in figure (3). This is one of the methods used by Euclid to test for commensurability between straight line segments, besides considering their squares. This method has an algebraic interpretation, exposed in section (2.2.1.1). The idea of using algebra as a means of proving geometric concepts is largely based on the work of David Hilbert (Hilbert's Axioms) [7]. However, Hebert's axioms (which in some way appear to be modernizing Euclidean geometry) are not sufficiently strong enough to decree the validity of using algebra in defining geometric concepts, and this avowal is addressed in later sections of this work. Euclid used the words rational and irrational differently than it is used by other mathematicians. The usual uses of these words corresponded to the use of the terms commensurable and incommensurable, respectively. But when applied to lines, Euclid makes them correspond to commensurable in square and incommensurable in the square. His workflow involves setting up a standard segment of line, then another line is called rational if the two lines are commensurable in their squares, and irrational if not. Thus, the diagonal on the square on the standard line is rational, even though it's incommensurable with the standard line since it's commensurable in the square with it. To this point, one can possibly make a reasonable conclusion that the idea of using naturally the diagonal and a side of the square to verify the irrationality of $\sqrt{2}$ is only limited to straight line segments and not to the squares of straight line segments. Thus the statement of $\sqrt{2}$ as a geometric irrational does not formally meet the threshold of testing for commensurability as established in Euclidean geometry (which concerns the comparison of magnitudes in two fashions: between line segments and between squares of lines segments). This observation makes the algebraic notion of considering the diagonal of the square (line of magnitude $\sqrt{2}$) as an irrational magnitude rationally invalid.

2.2.1.1. Algebraic Proof that the Factor $\sqrt{2}$ is Irrational

According to algebra, only rational numbers or magnitudes (those expressible in the form a/b where a and b are co-prime) are constructible. Some trivial cases like numbers of the form $\sqrt{n}$ are also constructible, and n is not a perfect square. The primary goal of this section is to illustrate that the algebraic definition of irrational numbers as geometric magnitudes is incomplete, and not geometrically effective. Consider claim 1.

Claim 1: The square root of the number two ($\sqrt{2}$) is irrational.

2.2.1.1.1. Proof

This proof will be based on figure (2), in regard to the product of numbers as established in [8]. To start out with, consider the follows three important algebraic statements:

1. The product between any two even numbers is always even. An even number has the form $2n$ where $n = 1, 2, 3, \ldots..$ then such a statement is true by definition, and the proof is always simple and clear (the proof for $2n$ is not to be dealt with in this context).

2. The product between an even number and an odd number would always give an even result. Odd numbers are classed as those numbers not completely divisible by 2, or they are not multiples of two. This is a true statement considering the definition of an even number in (1) since odd numbers form part of n, .

3. The product between two odd numbers is odd. This is equally true based on the definition of an odd number in (2), since an odd number is not completely divisible by two nor is it a multiple of two.

Now let us look at the proof, by contradiction approach. The proof is based on one condition that rational numbers are only those expressible in the form $\frac{x}{y}$, where x and y are relative primes.

2.2.1.1.1.1. Algebraic Interpretation of Euclid's Statement on Commensurability of $\sqrt{2}$

This interpretation can be illustrated from figure (2) in terms of ratios as follows:

The diagonal of square $ABCD$ is $\overline{DB}$ and one of the sides of the square is defined by edge $\overline{DA}$. The ratio $\overline{DB}/\overline{DA}$ does not result in finitely positive integer magnitudes between the two line segments as shown in figure (3). This observation has an algebraic interpretation in cases where measurements are imposed, to define length of the diagonal using the Pythagorean theorem. This definition results in Pythagorean factor $\sqrt{2}$, stated as the square root of two. The follows argument provides an algebraic interpretation of the ration $\overline{DB}/\overline{DA}$ using the square root of two ($\sqrt{2}$). By application of the Pythagorean theorem, $\overline{DB} = \sqrt{2}$, and $\overline{DA} = 1$. The goal here is to show that $\sqrt{2}$ cannot be written as x/y which then means $\sqrt{2}$ is irrational, and thus the diagonal of a square.

Suppose $\sqrt{2}$ was a rational number. Then $\sqrt{2}$ could be written as $\sqrt{2} = x/y$, with x and y being natural numbers having only 1 as a common factor. It then follows that $2 = \left(\frac{x}{y}\right)^2$, which imply:

$$2(y^2) = x^2 \qquad (1)$$

It was earlier stated that numbers of the form $2n$ are even, and thus $2(y^2)$ is an even term, which means x^2 must as well be even.

Suppose $x = 2b$. Using $2b$ in equation (1) we get, $2(y^2) = 4(b^2)$. This implies:

$y^2 = 2(b^2)$, and $2(b^2)$ is also an even term. (2)

In regard to equations (1) and (2), x and y are even numbers, a result which contradicts the formally established condition that the ratio between any two rational numbers should be a relative prime. Thus claim 1 is true that $\sqrt{2}$ is an irrational number.

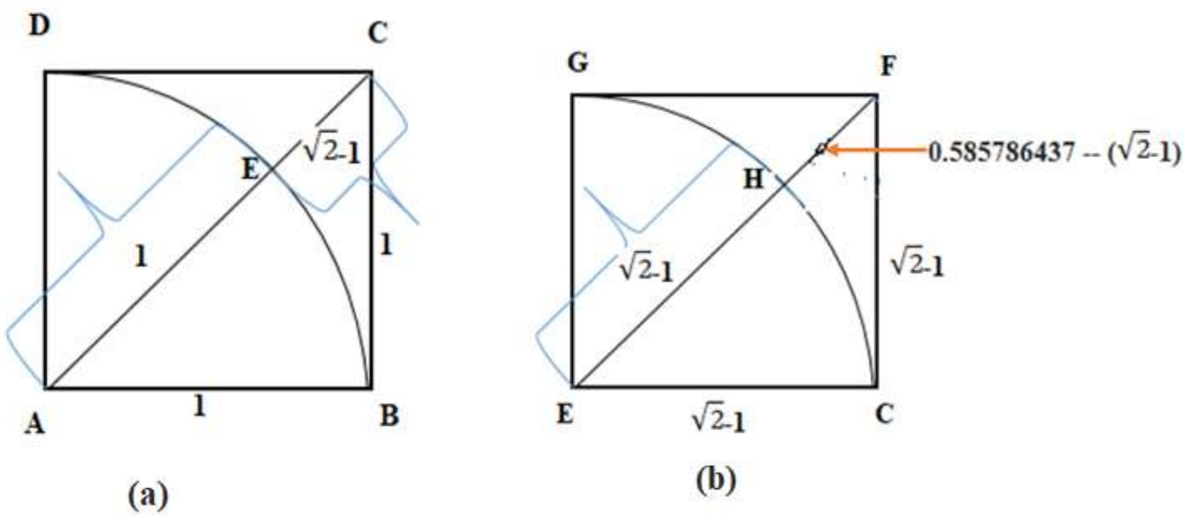

Figure 3: A geometrical proof that $\sqrt{2}$ (diagonal of a square) is not commensurate to *unity* (side of the square).

Figure (3): is essentially meant to prove that the diagonal of a square and its sides are not commensurate. Starting with figure (3, (a)), square $ABCD$ is constructed with sides one unit in magnitude. The reason for using a measured magnitude (or co-ordinates) is to relate the modern form of approach in geometry, and the Euclidean notion of numbers and magnitudes. The arc of the circle (drawn at center A) through points B, E and D is purposively meant to show that the diagonal of a square "is not commensurate" with its side(s) by comparisons. This can further be elaborated as; line segments $\overline{AB}$, $\overline{AE}$, and $\overline{AD}$ are congruent (radii of the arc $\widehat{BED}$ and sides of square $ABCD$). Therefore $\overline{AE} = 1$. The remaining fraction $\overline{EC}$ of the diagonal $\overline{AC}$ is equivalent to $\sqrt{2} - 1$, which constitutes of random and infinite decimals. The fraction $\overline{EC}$ is applied as the baseline in the construction of square $ECFG$ as shown in figure (3, (b)). In this case, $\overline{EC}$ is treated as sort of magnitude (length) and used to construct square $ECFG$ which yields yet another diagonal $\overline{EF}$. In figure (3, (b)), arc $\widehat{CHG}$ is constructed to aid in the subtraction of $\overline{EH}$ from $\overline{EF}$. Since the line segments $\overline{EC}$, $\overline{EH}$ and $\overline{EG}$ are congruent (radii of arc $\widehat{CHG}$) then $\overline{EH} = \sqrt{2} - 1$. A geometric subtraction of $\overline{EH}$ from $\overline{EF}$ results into yet another magnitude, and it is this nature of development that convincingly leads to the irrationality of $\sqrt{2}$ when considered as geometrical magnitude. However, this was an illustrative concept by Euclid to demonstrate incommensurability using straight lines. It is not the complete way to illustrate irrationalities (incommensurability) as Euclid also applies the use of squares of lines in his work.

2.2.1.1.1.2. Interpreted Fashion of Euclid's Proof that the Magnitude $\sqrt{2}$ is Rational

The previous argument shows the factor $\sqrt{2}$ as an irrational number, which when applied to geometry it would represent an irrational magnitude. But $\sqrt{2}$ is quite a strange factor which has both rational and irrational geometric characteristics. This section is aimed at showing that the diagonal of a square is commensurate (corresponding to rational) to its side. This proof will be based on information from figures (2) and (4). From figure (2) it has been established that $\overline{DB} = \sqrt{2}$ (diagonal of the square) and $\overline{DA} = 1$ an edge of the square. This section is concerned with the ratio of the square of the lines $(\overline{DB})^2$ and $(\overline{DA})^2$ as: ${(\overline{DB})^2}/{(\overline{DA})^2}$. Using the values for $\overline{DB}$ and $\overline{DA}$ in the ration we get ${(\sqrt{2})^2}/{(1)^2} = \frac{2}{1} = 2$, which is a rational result. Thus the diagonal of a square is geometrically commensurate to its sides and this statement is justifiable when proving the Pythagorean Theorem, as shown in figure (4). As stated earlier, the Pythagoras theorem can be expressed using the algebraic equation: $a^2 + b^2 = c^2$ in which case, each of the squares a^2, b^2 and c^2 represent an area as magnitude, and not a real number measure of line segments. Figure (4) is based on an isosceles right triangle in which, $A\_1 = A\_2$ (squares of corresponding equal sides). $A\_3 = 2$ (square of the diagonal). The ratio ${A\_3}/{A\_1} = {2}/{1} = 2$. Thus the diagonal of a square is a rational magnitude in square, as it results to the rational magnitude area.

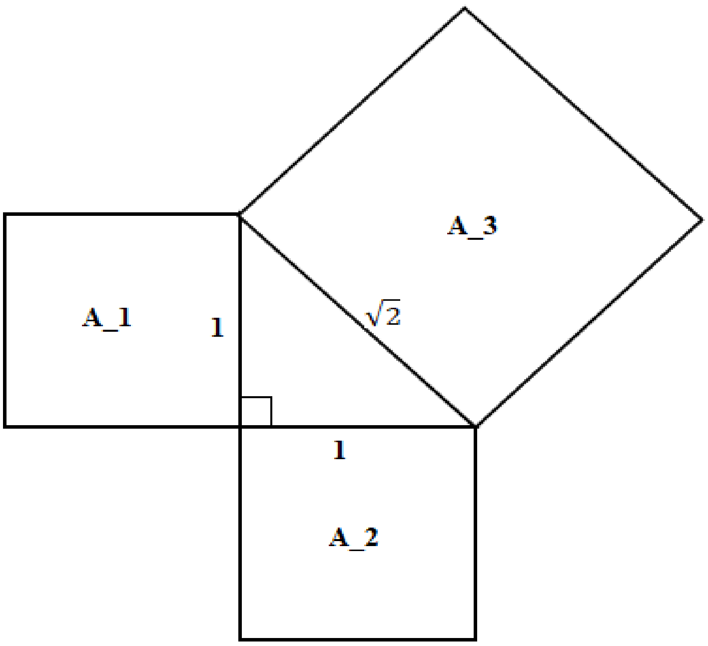

Figure 4: Geometrical Proof that the Diagonal of a Square is a Rational Magnitude

2.2.1.1.1.3. The Problems with Algebraic Proof that the $\sqrt{2}$ is Irrational

Through section (2.2) two problems with the algebraic argument that $\sqrt{2}$ is irrational are evident.

a) The proof (2.2.1.1.1.1) is based on the condition that rational numbers are only those expressible in the form $(^{a}/_{b})$, and a and b are relative primes. This condition is not inherently geometrical (as a and b can be positive/negative numbers and negative magnitudes do not exist in plane geometry), and it limits the irrationality argument into one of the fashions of Euclid's ways of testing for commensurability in case of straight lines, that the diagonal of square is incommensurate with the side of the square.

b) Assuming $\sqrt{2}$ to be used as a geometric magnitude (diagonal) of the square, the irrational statement fails when the commensurability test is applied based on the square of the diagonal and side of the square. So the statement of $\sqrt{2}$ as an irrational magnitude is not comprehensively defined with geometric rigor. Euclid did not prove the irrationality of the number $\sqrt{2}$ by using the diagonal, and side of the square to test for commensurability. The "irrational" statement is a misconstrued algebraic interpretation of Euclid's approach to the concepts of rational and irrational magnitudes because as it has been established earlier, $\sqrt{2}$ has both rational and irrational geometric physiognomies.

2.3. Algebraic Proof that the $\sqrt[3]{2}$ is Irrational

The goal of this section of proof is to demonstrate that $\sqrt[3]{2}$ is irrational, and the statement that $\sqrt[3]{2}$ is not geometrically constructible lack proper geometrical tests. The proof is based on the common notion of constructible number theorem which asserts: every number n that you can construct has the following properties:

(i) n is an algebraic number.

(ii) The degree of the characteristic polynomial of n is a power of 2.

These two properties are based on all numbers (not only the positive integers as used in Euclidean plane geometry) and so, as employed earlier, numbers would also be treated as forms of magnitudes for the purposes of this proof.

Claim 2: $\sqrt[3]{2}$ is not a constructible magnitude in Euclidean plane geometry.

2.3.1. Proof

2.3.1.1. Case 1: Proof based on the statement "All constructible Numbers are Algebraic"

In classical geometry, the factor $\sqrt[3]{2}$ often called the Delian constant is believed to have no geometrical solution [9]. The Delian constant $\sqrt[3]{2}$ defines an edge of a cube, whose volume is double the volume of a given cube. The Delian constant is expressed algebraically as $x^3 = 2$ where x represents a number. Based on the property (i) provided in section (2.3), $x^3 = 2$ has degree 3, which is not a power of 2, and hence not geometrically constructible number or magnitude.

2.3.1.2. Case 2: Proof based on Field of Constructible Numbers

The geometric formulation of constructible numbers is defined using the Cartesian coordinate system in which the point of origin (say point O) is associated to coordinate pair $(0, 0)$ and in which another point A can be associated with the coordinates $(1, 0)$. These points can be used to establish the relationship between geometry and algebra by defining a constructible number to be a coordinate of a constructible point [10]. The geometric equivalent definition is that a constructible number is the length of a constructible line segment [11]. Under this formulation, considering an initial cube be of a unit length and assume that one of the sides of the cube is the line between the coordinates $(0,0)$ and $(1,0)$. The volume of such a cube can be computed to be 1, so that constructing another cube of volume 2 would correspond to constructing some point $(\beta, 0)$, such that $\beta^3 = 2$. Letting F to be the smallest field containing 0 and 1, the minimum polynomial of β over F is $\beta^3 = 2$. This is a degree three polynomial and therefore, dimensionally we have:

$$[F(\beta):F]=3 \qquad (3)$$

Based on the application of algebra in plane geometry, the point $(\beta,0)$ is constructible from the configuration $\{(0,0)\ and\ (1,0)\}$, if $[F(\beta):F]$ is a power of two (as illustrated in [8]). However, as seen from equation (3), $[F(\beta):F]$ is a power of three and not two. In regard to these presumptions, one can, therefore, conclude that the relation $\beta^3=2$ has no rational roots for β^3, and thus the Delian constant $\sqrt[3]{2}$ is not constructible. This proof implies that if one could construct β, since $\beta=\alpha\sqrt[3]{2}$, then one could construct the quotient $\beta/_\alpha=\sqrt[3]{2}$. In a simpler viewpoint, the ratio $\beta/_\alpha$ corresponds to the ratio between two integers $a/_b$, which are described as relative primes. This shows (though not perfectly algebraic) as demonstrated earlier for $\sqrt{2}$, $\sqrt[3]{2}$ is similarly an irrational factor.

2.3.1.3 Problems with the Algebraic Proof that the Factor $\sqrt[3]{2}$ is not Constructible Magnitude (Number)

1. The property that for every number n that one can construct n is an algebraic number assumes that even negative magnitudes (represented by negative numbers) and zero are ordinary constructible geometric magnitudes, a case not established in plane geometry. Indeed, the number zero does not represent a valid magnitude or quantity in plane geometry.

2. The property also assumes that some algebraic magnitudes might not be constructible and that all geometrically constructible magnitudes are algebraic. This is not a geometric definition or property. It is an algebraic property used to define geometric concepts, while the claim is not inherently geometrical. The property simply translates geometry into algebra thus violating the conditions governing classical geometry. Figure (5) expose the limitation of such property in defining geometric concepts.

3. It has been concluded that as $\sqrt{2}$ cannot be expressed in the form $a/_b$, and so is true for $\sqrt[3]{2}$ as they are both irrational factors. But as established in the previous section about the rationality of $\sqrt{2}$, the irrationality proof is not complete to define commensurability or incommensurability of geometric magnitudes, as Euclid implied use of ratios in two forms; between lines, and between squares of the lines in test for commensurability and incommensurability.

4. From section (2.3.1.2), equation (3) conceives a notion in which a straight line (solution for $\sqrt[3]{2}$) is perceived as a three dimensional magnitude. This is a misconstruction which makes the entire proof obscure, since the cube duplication problem is defined in a two dimensional configuration.

2.4. Geometrical Proof that All Magnitudes are Constructible in Plane Geometry

The proof is ultimately illustrated based on Euclid's definition of a point, and Euclid's definition of a straight line in respect to Archimedes's postulate [12]. From Euclid's Elements Book I definition 1, "a point is that which has no part". On the other hand, the definition of straight line can be modified from Euclid's Elements based on Archimedes's postulate to read as "a straight line is (or a line) which, uniformly in respect to (all) its points, lies upright and stretched to the utmost towards the ends, such that, given two points, it is the shortest of the lines having them as ends". These two definitions are essentially important in understanding the physical geometric models, in which a point is defined based on coordinates, and these coordinates form constructible numbers, which in some way, can be applied to define some geometric magnitudes. Based on these two definitions, one can reasonably assume; two preceding points make the finitely shortest straight line segment. This definition is depicted in figure (5), and it was applied by Archimedes (287-212 BS) [12] in finding an exact formula for the area of a circle and a few other exceptional geometric figures (Apostol, p. 2) [13].

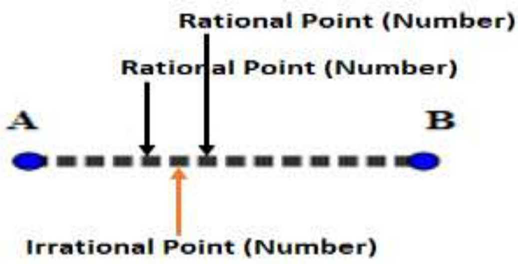

Figure 5: Geometric definition of a straight line

To prove that all magnitudes are geometrically constructible, one has to prove that all the magnitudes exist in some sort of a plane geometric configuration. For the purpose of this proof, figure (5) will be applied in the discussion. For clarity in illustrating the proof, this section would adopt two approaches; Algebraic and Geometric approaches, both applied systematically. In relating algebra to geometry, it will also be assumed that numbers represent both the constructible and non-constructible magnitudes. The straight-line segment would be preferably used as the simplest magnitude in this configuration. Considering that all magnitudes exist in a plane ($2D$ surface) and that since these magnitudes correspond to numbers in algebra, a handy way to prove that all magnitudes are geometrically constructible would be to show that all positive numbers are constructible. This proof is based on two types of numbers; positive rational numbers (those expressible in the form ${}^{x}/_{y}$ where x and y are co-primes) and positive irrational numbers (numbers of the form $\sqrt{n}$ and $\sqrt[3]{n}$ where n is an integer which is not a perfect square and perfect cube respectively). The section begins by an algebraic proof aimed to show that there is a form of relation between rational and irrational numbers, and later this proof is translated to geometry based on geometrical magnitudes. The following section provides an algebraic proof that there exists an irrational number between any two rational numbers.

Claim 3: Between any two real numbers there exists an irrational number.

2.4.1. Geometric Translation of the Algebraic Proof

Section (2.2.1.1.1.1) provided an algebraic proof that $\sqrt{2}$ is an irrational factor. This proof stems from the relation between $\sqrt{2}$ and natural numbers, cogitating only positive real numbers. Consider, suppose that x is a positive integer under the condition $x < y$. We can assert that there exists a natural number k such that

$$ {}^{\sqrt{2}}/_{k} < y - x. \qquad (4) $$

Now let $w = {}^{\sqrt{2}}/_{k}$. It is clear that w is an irrational number since k is a natural number and $\sqrt{2}$ is irrational. Starting at $x = 0$ on the positive dimension, one can stroll along the positive real numbers in intervals of w. The goal of this operation is to prove that one must step into the interval (x, y) since its width is greater than w.

Letting a configuration $S = \{j: jw \geq y\}$, and applying the Archimedean property shows that S is not an empty configuration.

Let m be the least element of S. Then, $mw \geq y$ and $(m-1)w < y$. (5)

Now it is necessary to illustrate that $x < (m-1)w$. Supposing $(m-1)w \leq x$, then $mw - w \leq x$ and $mw \leq x + w < x + y - x = y$. (6)

Equation (6) implies that $mw < y$, a contradiction. Thus, $x < (m-1)w < y$. Since w is irrational, $(m-1)w$ is also an irrational number.

This clearly proves that between any two positive rational numbers, there exists a positive irrational number. Geometrically, these numbers can be translated as geometric magnitudes in compass-straightedge construction. So the geometrical argument that all magnitudes (rational and irrational magnitudes) are constructible is based on demonstrating that all these points exist to one another, as part of any magnitude. This section of proof uses a straight line segment to illustrate the significance of these points in plane geometry. Considering figure (5), the straight line AB is made by a collection of points that are collinear between the two points. Any of these points can finitely be described using coordinate pairs, and the finitely shortest distance (for compass-straightedge construction) between them be defined with the aid of the Pythagorean theorem. It is known that the difference or the sum between an irrational number and a rational number is always an irrational result. This fact shows that the difference between any two points on the straight line AB is irrational, implying that, all irrational magnitudes are geometrically constructible (this approach completely assumes all other coordinate pairs in the case of using coordinates, and not the combination $(0, 0)$). This proof shows that the algebraic statement of constructible and non-constructible magnitudes has no allegiance to state that some significant geometric factors such as $\sqrt{2}$ and $\sqrt[3]{2}$ as geometrical irrationalities, and that $\sqrt[3]{2}$ is not a constructible geometric magnitude.

3. Geometrical Verification that all Magnitudes of Form $\sqrt{2}$ and $\sqrt[3]{2}$ are Constructible (Doubling a Square and Doubling Volume of a Cube)

This discussion would largely involve comparable magnitudes. To compare two magnitudes in classical geometry is quite simple with the aid of "the compass equivalent theorem", which is provided in proposition II of book I of Euclid's Elements. The compass equivalence theorem asserts that: "all constructions via "fixed" compass may be attained with a collapsing compass [14]. This implies it is possible to construct circles of equal radius centered at any point on a plane". This theorem would be employed in the subsequent constructions, as it plays a significant role in multi-step constructions, which may involve transfer of magnitudes.

3.1. Geometrical Duplication of a Square (A Proof on Rationality of $\sqrt{2}$)

The problem of doubling the area of a square has been an open problem in classical geometry. The objective of this section is to establish a provable geometric proof for doubling a square, using only two tools: compass and straightedge. The problem of doubling a square asks "given an arbitrary square, create geometrically another square of area double the given square" [1, 2]. The solution to this problem would imply resolving the geometrical irrationality of the Pythagoras factor $\sqrt{2}$. The problem can formally be formulated in algebraic form as: $({}^{1}/_{2})y^2 = x^2$ in which case, y define an edge of the newly constructed square, while x represents the side of the original square. One common and simplistic notion in doubling a square is to use the diagonal of the given square as an edge of the new square, whose area has to be double. Figure (6) depicts the geometric relationship between the newly constructed square $(EFGH)$ from the given square $(ABCD)$ of unity magnitude.

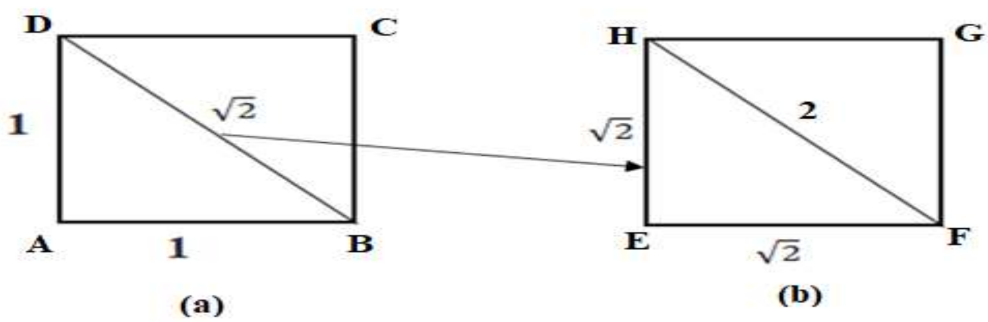

Figure 6: An Illustrative Result on Geometrical duplication of a square

In other terms, the area of a square is found using a very simple formula, L^2 where L is a side of the square. Geometrically, the concept of area is developed using straight lines, to construct the desired geometric object which has some content (area). This section employs both methods for comprehensiveness in the discussion. Beginning with figure (6, (a)), in some way the area of square $ABCD$ is a square unit, since the square has magnitude, unity. It is relatively easy to construct another square of area double the area of $ABCD$ by simply

translating the diagonal $\overline{BD}$ into a side of the new square. This is possible based on the application of the compass equivalence theorem discussed earlier, which allows for the transfer of distances using compass. So figure (6, (b)) is developed as a consequence of constructability of the diagonal $\overline{BD}$. If we allow numbers to represent geometric magnitudes, then from figure (6, (a)) and figure (6, (b)), the sides $\overline{AB}$ and $\overline{EF}$ have magnitudes 1 and $\sqrt{2}$ respectively. As done for $ABCD$ equally the area of $EFGH$ would be obtained by squaring

the factor $\sqrt{2}$ (side of the square) which gives 2. Now there are two magnitudes of areas; 1 and 2 for the squares $ABCD$ and $EFGH$ respectively. We observe that the diagonal of a square yields a square of area twice the initial square. Considering Euclid's concepts of commensurability and equality, one of the best results which we can obtain from figures (6, (a)) and (6, (b)) is that the two figures are commensurable in terms of their area (which equals the square of the sides). Figure (6, (b)) is twice in area (a multiple of) figure (6, (a)) and it is quite easy to deduce a reasonable geometric relation in terms of ratios. Figure (7) provides a geometrical sequence of constructions illustrating how a square can be doubled. Figure (7, (a)) has diagonal $\sqrt{2}$ and area 1 square unit. Figure (7, (b)) has diagonal 2 and area 2. Figure (7, (c)) has diagonal $2\sqrt{2}$ and area 4. And lastly, figure (7, (d)) has diagonal 4 and area 8. Sequentially, the areas are geometrically increasing by a geometric factor of 2, from 1 to 2 to 4 to 8.

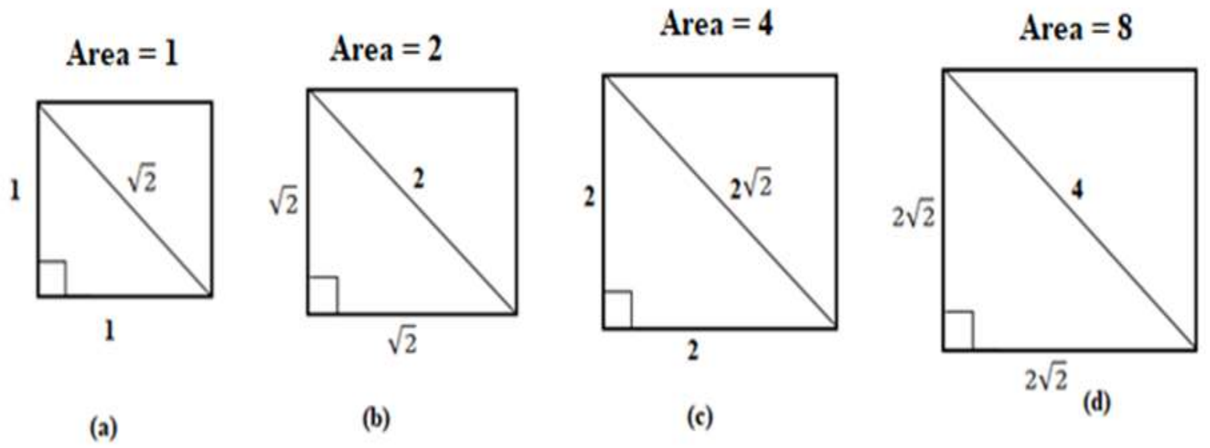

Figure 7: A decent proof of doubling a square and constructability of $\sqrt{2}$

The concepts of ratios in classical geometry is based on magnitudes of a similar kind. Euclid did not encourage the use of mixed ratios which could result in comparison of different magnitudes. In Euclidean geometry, ratios do not just represent ordinary numbers, but they convey a piece of important information on comparable magnitudes of the same kind. The concept of similar triangles would not be complete without the use of ratios. This work would employ similar triangles (taking case of an isosceles right triangle) to prove that the diagonal of a square does not represent an irrational magnitude. This approach is largely focused to illustrate how all magnitudes which cannot be expressed as $^a/_b$ do not represent a geometric irrationality (as it has been established), but rather represent a set of all constructible magnitudes in a plane.

3.1.1. Constructability of $\sqrt{2}$ in Doubling a Square

Consider the follows construction steps.

1. Construct square $ABCD$ of length $1unit$ defined by the edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.
2. Join vertices $\overline{AC}$ using a straight line segment (the diagonal). Since the square has an edge of length one unit, it becomes straight forward (by application of the Pythagoras theorem) that the diagonal $\overline{AC}$ is defined by the value $\sqrt{2}$.
3. Using $\overline{AB}$ as radius and center A, construct an arc through vertices B and D, which cuts $\overline{AC}$ at a point E.
4. Using radius $\overline{EC}$ and center A, make an arc that cuts the produced baseline $\overline{BA}$ externally at a point F.
5. Again using the same radius $\overline{EC}$ and center C, make an arc that cuts the edge $\overline{BC}$ externally at a point G.
6. Using either radius $\overline{FB}$ or $\overline{BG}$ and centers F and G, make two arcs that intersect at a point H as shown in figure (8).
7. Join points F and G using a straight line segment (new diagonal).

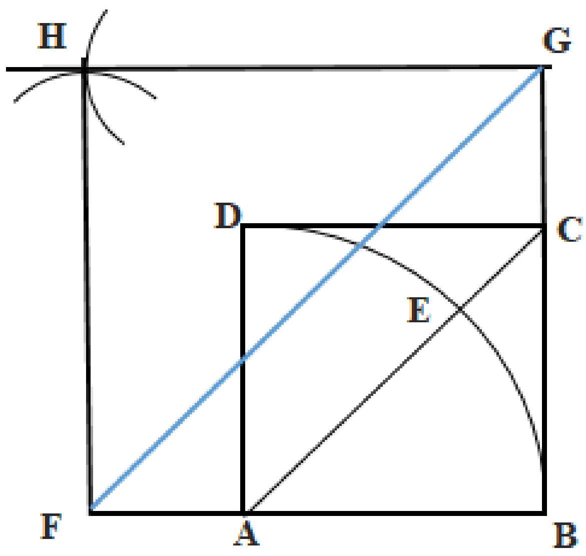

Figure 8: Geometric duplication of square

From figure (8), the edges $\overline{AB} \cong \overline{AE} \cong \overline{AD} = 1$, radii of the arc $\widehat{BED}$, and side of square $ABCD$. Edge $\overline{EC} = \sqrt{2} - 1$, since $\overline{AC} = \sqrt{2}$. $\overline{FB} \cong \overline{BG} \cong \overline{AC} = \sqrt{2}$, since $\overline{FA} \cong \overline{EC}$. Applying Pythagorean theorem to triangle FBG, $\overline{FG} = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$. Based on these deductions, it is easy to show that square $FBGH$ and square $ABCD$ are similar. From the construction, it is clear that the edges $\overline{FB}$ and $\overline{AB}$ have magnitudes $\sqrt{2}$ and 1 respectively. Using ratios and the similar sides between the two squares we have the relations: $\frac{\overline{FB}}{\overline{AB}} = \frac{\sqrt{2}}{1}, \frac{\overline{BG}}{\overline{BC}} = \frac{\sqrt{2}}{1}, \frac{\overline{GH}}{\overline{CD}} = \frac{\sqrt{2}}{1}, \frac{\overline{HF}}{\overline{DA}} = \frac{\sqrt{2}}{1}$ and $\frac{\overline{FG}}{\overline{AC}} = \frac{2}{\sqrt{2}}$. All these ratios result into a common solution of $\sqrt{2}$ and this is the linear geometrical factor relating the two squares. It implies that the two squares are similar. A simple and straightforward way to show that the factor $\sqrt{2}$ is a constructible geometric magnitude is illustrated in figure (9), which is a result of the following construction steps:

1. Construct square $ABCD$ defined by the edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.
2. Join vertices $\overline{AC}$ using a straight line segment (the diagonal).
3. Using $\overline{AB}$ as radius and center A, construct an arc through vertices B and D, which cuts $\overline{AC}$ at a point E.
4. Using radius $\overline{EC}$ and center A, make an arc that cuts the produced baseline $\overline{BA}$ externally at a point F.
5. Again using the same radius $\overline{EC}$ and center C, make an arc that cuts the edge $\overline{BC}$ externally at a point G.
6. Using either radius $\overline{FB}$ or $\overline{BG}$ and centers F and G, make two arcs that intersect at a point H as shown in figure (9).
7. Join points F and G using a straight line segment (new diagonal).
8. Join the vertices B and H using a straight line segment (also diagonal of the larger square).
9. Mark point I, the point of intersection of diagonals $\overline{FG}$ and $\overline{BH}$.
10. Using radius $\overline{HI}$ and center D (vertex of the initial square), construct a circle which passes through vertices A and C.

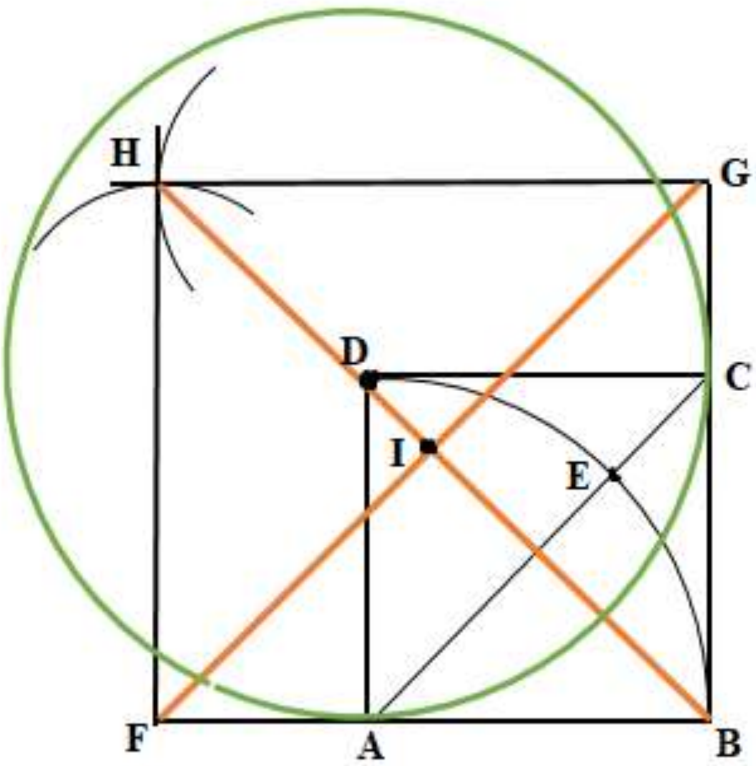

Figure 9: Geometrical proof that the factor $\sqrt{2}$ is not irrational geometric magnitude

Figure (9) represents a construction aimed at proving that $\sqrt{2}$ is a constructible rational magnitude. Square $FBGH$ is double square $ABCD$ for the following geometrical reasons: from the construction step (9), the circle in green with radius $\overline{HI}$ passes through vertices A and C. $\overline{AC}$ is diagonal of square $ABCD$ which implies $\overline{HI} \cong \overline{AB}$, and $\overline{AB}$ is an edge of square $ABCD$. The diagonals $\overline{FG}$ and $\overline{BH}$ intersect at I (they make right angles), implying $\overline{HI} \cong \overline{IG} \cong \overline{IF} \cong \overline{IG} \cong \overline{AB}$. Based on these decoctions and letting A_1 be the area to square $ABCD$ and A_2 be the area to square $FBGH$, then it is possible to show that $A_2 = A\_1$. We apply the common formula of a right triangle in computing A_2; $Area = \frac{1}{2}bh$, with b and h as sides of the triangles FIH, FIB, BIG and GIH. But since triangles FIH, FIB, BIG and GIH are all isosceles, then $b = h$. So the expression for calculating A_2 becomes $Area = \frac{1}{2}b^2$ applied as follows:

$A\_2 = 4\left(\frac{1}{2}b^2\right) = 2b^2$.

Thus $A\_2 = 2b^2$ (7)

The area of a square is simply the square of its side.

Thus $A\_1 = b^2$ (8)

Equations (7) and (8) shows that $A\_2 = 2A\_1$, implying square $FBGH$ is double square $ABCD$. Hence justifying that it is possible to double a square typically using Euclidean geometric constructions. If we introduce measurements and let a number b to represent the length of a square, applying the equation $A\_2 = 2b^2$ would generate the sequence of constructions shown in figure (7). In figure (7), the diagonal of the square follows a simple sequence $\sqrt{2}$, 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, and so on. The sides of the squares form the sequence 1, $\sqrt{2}$, 2, $2\sqrt{2}$, 4, and so on. So the relationship is, starting with a square of side a unit, its diagonal becomes the side of the succeeding square. As discussed earlier that there is always an irrational constructible magnitude between any two rational magnitudes, duplication of a square provides an elegant illustration to justify this statement. From figure (7), one can infer that, there is an irrational constructible magnitude (due to $\sqrt[2]{2}$) between any two rational magnitudes, and the opposite is true, there is a rational magnitude between any two irrational constructible magnitudes.

3.1.2. Doubling a Square Using CAD (An Application of GeoGebra) for Results Visualization

The use of computer-aided design methods in Euclidean geometric constructions provides a very rigorous environment for results visualization. GeoGebra is one such geometric dynamic package designed pretty well to apply the concepts of classical geometry. Despite the fact that GeoGebra might have some limitations (some established in [8]) such as high error margins at higher accuracies of results visualization, this software provides one of the most friendly basic geometric environments. This section would perform a multistep construction aimed at showing that the Pythagorean factor $\sqrt{2}$ is geometrically a constructible and not an irrational magnitude. Consider figure (10) which is a result of the following construction steps.

1. Construct square $ABCD$ defined by the edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.
2. Join vertices $\overline{AC}$ using a straight line segment (the diagonal).
3. Using $\overline{AB}$ as radius and center A, construct an arc through vertices B and D, which cuts $\overline{AC}$ at a point E.
4. Using radius $\overline{EC}$ and center A, make an arc that cuts the produced baseline $\overline{BA}$ externally at a point F.
5. Again using the same radius $\overline{EC}$ and center C, make an arc that cuts the edge $\overline{BC}$ externally at a point G.
6. Using either radius $\overline{FB}$ or $\overline{BG}$ and centers F and G, make two arcs that intersect at a point H as shown in figure (10).

7. Join points F and G using a straight line segment (new diagonal).

8. Join the vertices B and H using a straight line segment (also diagonal of the larger square).

9. Mark point I, the point of intersection of diagonals $\overline{FG}$ and $\overline{BH}$.

10. Using radius $\overline{HI}$ and center D (vertex of the initial square), construct a circle which passes through vertices A and C.

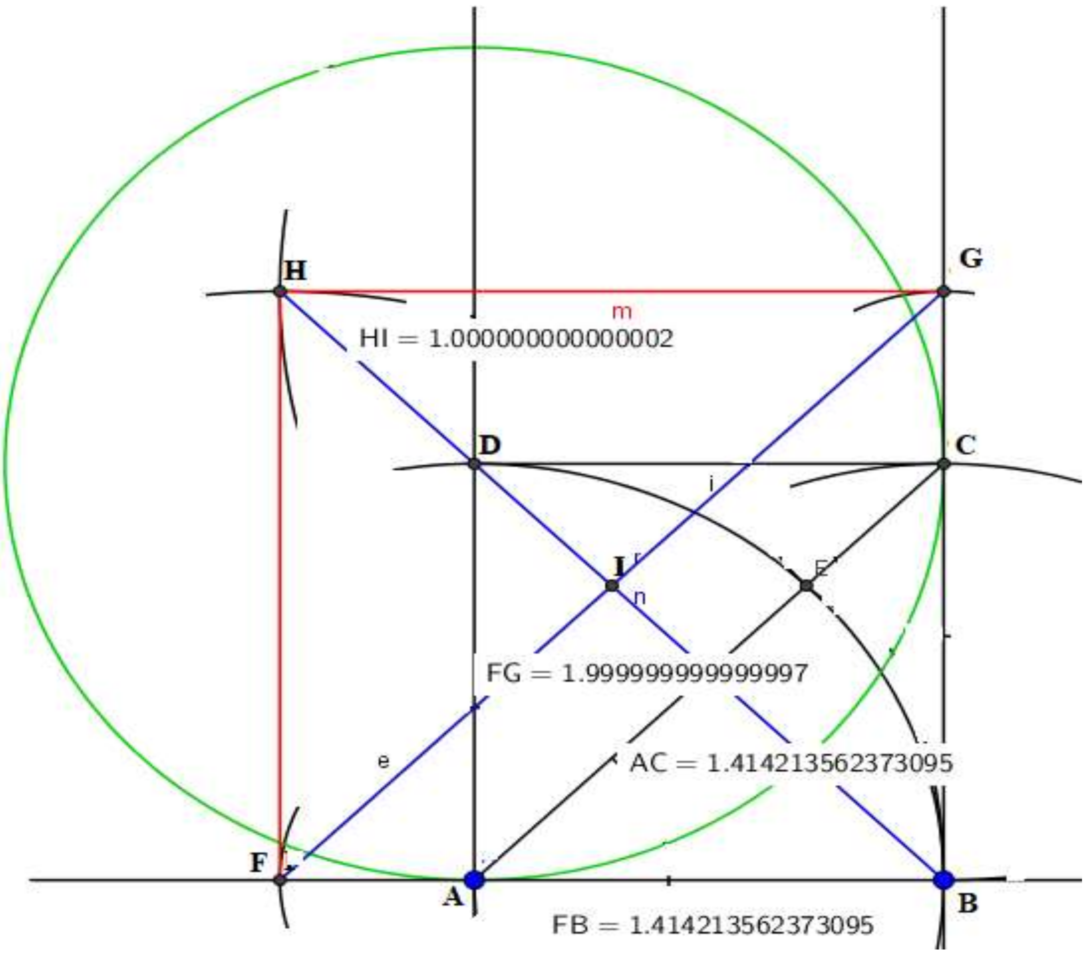

Figure 10: Decent proof that the Pythagorean factor $\sqrt{2}$ is a constructible magnitude in plane geometry.

Figure (10) provides justification of the statement that there is a constructible magnitude of order $\sqrt{2}$ in any geometrical plane. The use of the computer-aided form of analysis provides two forms of information from figure (10); the power of using the compass equivalence theorem in comparison of similar magnitudes, and the actual values which measure the particular magnitudes. Starting with the application of compass equivalence theorem, the green curvature through vertices A and C drown using radius $\overline{HI}$ and center D shows that $\overline{HI} \cong \overline{AB}$, with $\overline{AB}$ as a side of square $ABCD$. The diagonals $\overline{FG}$ and $\overline{BH}$ intersect at I (diagonals of a square make right angles), implying $\overline{HI} \cong \overline{IG} \cong \overline{IF} \cong \overline{IG} \cong \overline{AB}$. Applying Euclid's notion of testing for commensurability between the diagonal of a square and its side and using the diagonal $\overline{FG}$ and side $\overline{FB}$ on figure (10) we get:

$$\frac{\overline{FG}}{\overline{FG}} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad (9)$$

The factor $\sqrt{2}$ is a symbolic representation of the measure of a diagonal (diagonal of the smaller square), while the diagonal of the larger square is obtained from the construction as $1.999999999999997 \cong 2$, to 15 decimals with respect to the result of figure (10). The error margin between 1.999999999999997 and 2 is probably due to the design of the software, and it simply proves that measurements would not be exactly accurate in plane geometry. The result of $\sqrt{2}$ from equation (9) shows that $\sqrt{2}$ is a geometrical constant which propagates along any construction plane.

3.2. Geometrical Duplication of a Cube (Justification on Rationality of $\sqrt[3]{2}$)

This section provides simple proof on the possibility of constructing a line segment of magnitude 1.2599 … ≅ 1.26 where 1.26 is equivalent to the rational approximate fraction $^{63}/_{50}$. Therefore the solution 1.2599 … presents the very "near perfect" solution available today, for doubling the volume of a cube [8]. This nearly perfect solution is aimed at showing that the solution to the Delian constant $\sqrt[3]{2}$ is attainable for a finite number of construction steps, given that one can measure the ever shortest distance between two preceding constructible points that define a line segment. However, it is geometrically difficult to attain such a measurement using compass-straightedge constructions, and since the use of measurements does not form proof, this limitation implies the problem of using numbers (analysis techniques) as means of proving geometric concepts. Thus this approximate solution provides the most reasonable argument to this day, that there is some magnitude between a number of points making a straight line segment, which is geometrically constructible.

Claim 4: It is geometrically possible to construct $\sqrt[3]{2}$ for compass-straightedge construction

Consider the follows construction steps (modified from [8])

1. Given a cube of unit length, construct square $ABCD$.
2. Using the radius $\overline{AB}$ of the constructed square, place the compass at point A and make an arc through vertices D and B.
3. Without adjusting the compass, position the compass spike at point B and make another arc from point A through C. Label E the point of intersection of the two arcs inside the square.
4. Join point E to point C using a straight line, and construct its bisection at point F, to cut curve $\widehat{\mathrm{EC}}$ at a point G.
5. Using $\overline{\mathrm{EF}}$, place the compass at A and mark a point H on the extended edge $\overline{\mathrm{BA}}$.
6. Again using chord $\overline{\mathrm{EG}}$, place the compass at A and mark another arc at a point I on the extended line.
7. Construct the bisection of $\overline{\mathrm{HI}}$ at J as shown in figure (11).

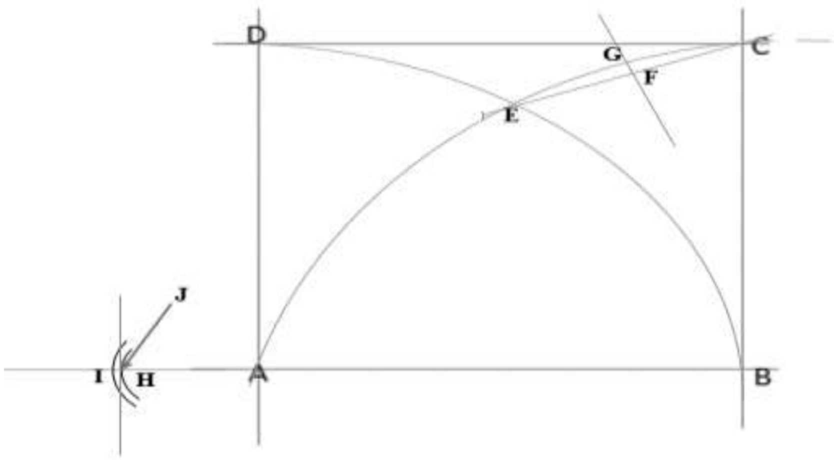

Figure 11: How to construct a line segment of length $1.2599 \cong \sqrt[3]{2}$

It has been proven using both algebra and geometry methods that the magnitude $JB \cong 1.2599 \ldots \sqrt[3]{2}$ [8]. This is the required result to illustrate that if it was possible to construct the shortest straight line segment, it would be possible to illustrate the measurements of all lengths, geometrically. The inability to construct the shortest distance (shortest limit of a straight line) is not only a geometric problem but equally a philosophical problem, as addressed in appendix 4. As we cannot conceive geometric construction of the shortest line segments with compass-straightedge constructions (with respect to the definition of a point), so it is that no known arithmetic which can provide the coordinates of the finitely closest preceding points on a straight line. The symbolic representation of magnitudes has no significance in the ordinary geometric constructions (in both practice and reasoning). However, symbolic representation of magnitudes is applied in this work to establish the relationship between geometry and algebra, and carefully to establish the differences where necessary. It was earlier established that in some way, the irrationality argument provides the same conclusion for the factors $\sqrt{2}$ and $\sqrt[3]{2}$, a statement which has no geometric cogency to justify that magnitudes of the form $\sqrt[3]{n}$ are not geometrically solvable, where n is not perfect cube.

4. Discussion on Geometrical Constructability of the Factors $\sqrt{2}$ and $\sqrt[3]{2}$

Throughout this paper, an attempt has been made to prove that all irrational statements in regard to the factors $\sqrt{2}$ and $\sqrt[3]{2}$ imposed into geometry from algebra are invalid. It has also been proven that the existing algebraic proofs do not meet Euclid's threshold of defining commensurability (rational) or incommensurability (irrational) in geometry, of employing the use of both straight line and their squares. The proofs are majorly based on a condition confined only to the comparison of straight lines without considering their squares. The paper begins by illustrating that the factor $\sqrt{2}$ is a consequence of the Pythagorean Theorem as depicted in figure (1). By considering figure (1), one can deduce that the factor $\sqrt{2}$ is simply a definition of some magnitude (length of a diagonal in a square) for a unity square. This proves $\sqrt{2}$ to be a plane geometric factor whose one significant effect is the duplication of a square, as shown in figure (7). A straight line segment was used to prove that there exists an irrational constructible magnitude between any two geometric points, in regard to the factor $\sqrt{2}$. The reason for using magnitudes unity and $\sqrt{2}$ was because it is not geometrically feasible to conceive a measure of the finitely smallest magnitude using compass-straightedge constructions. Equally, it has also been proven that there is a relationship between the factor $\sqrt{2}$ and $\sqrt[3]{2}$ in that, both require to be expressed in the form x/y when treated as numbers, where x and y represent integers with no other common factor except 1. Figure (5) provides a simple geometric configuration based on a straight line, to demonstrate that both the rational and the irrational magnitudes exist in some ordered formation in a plane. The goal of figure (5) was to translate the algebraic claim; that there exists an irrational number between any to rational numbers, or that there exists a rational number between any two irrational numbers into geometric form, based on magnitudes. It has been established that the shortest distance between two preceding points is a straight line. Taking one of these points as rational and the other as irrational, this means that the difference (the ever shortest straight line segment) between the two points is irrational.

This line is a valid geometric argument, for it must exist in a plane (and magnitudes commensurate to zero do not exist in compass-straightedge constructions). Considerably, the case would be equally true if we assume the existence of a rational number between any two irrational numbers. To this point, it is geometrically clear that based on the notion of using the positive integers or real numbers (the number zero excluded) the ever shortest straight line that can be constructed using compass and straightedge is irrational, and this refutes the algebraic argument that the factor $\sqrt[3]{2}$ has no geometric solution. This paper has treated all real positive numbers as geometric magnitudes, since ratios could as well be treated as fractions of magnitudes, and this mode of analysis is not restricted for Euclidean geometry. Throughout the paper, the primary goal has been to demonstrate the constructability of all geometric magnitudes, and correct a misconception that Euclid proved the $\sqrt{2}$ as an irrational magnitude (diagonal of a square). A careful analysis of the Euclid's Elements show that the irrationality of $\sqrt{2}$ (as imposed from algebra to geometry) is not comprehensively geometrically proven. It has been shown that the two factors $\sqrt{2}$ and $\sqrt[3]{2}$ are algebraically related and thus the statements of $\sqrt[3]{2}$ as a non-constructible magnitude is equally invalid, as it is based on one condition, the algebraic definition of rational numbers.

5. Conclusion

The notion of considering the factors $\sqrt{2}$ and $\sqrt[3]{2}$ as irrational numbers in algebra and equally as irrational geometric magnitudes suffers a serious misconception. In this paper, it has been established that the irrationality proofs for $\sqrt{2}$ and $\sqrt[3]{2}$ got no geometrical cogency to state algebraic irrationalities as geometric irrationalities. Figure (5) was used as proof that in the simplest form possible (the case of a straight line), all points are constructible in a two-dimensional plane, implying all numbers are constructible (they exist in this configuration). These numbers have been applied ordinarily to represent geometric magnitudes throughout this work. The result in figure (10) shows the possibility to construct a line of magnitude 2 from $\sqrt{2}$, while as proven in [8] which provides $x = 1.25993$ ..., it is geometrically possible to construct the magnitude 2 from $\sqrt[3]{2}$ such that $2 \cong 2.00004$. The result 2.00004 shows that in a finite multi-step construction in which the geometric construct of any two successive points is possible, then a line of length $\sqrt[3]{2}$ is possible. These results have been obtained by solving two problems, the geometric duplication of a square, and the geometric duplication of a cube. Thus, only geometrically translated algebraic factors of the form $\sqrt{n}$ and $\sqrt[3]{x}$ [15, 16] have been dealt with, in which case, n and x are not perfect squares and perfect cubes respectively. Throughout the work it has been demonstrated that the algebraic proof of irrationality is not geometrically consistent, and that equally, the existing ordinary geometry (Euclidean geometry) does not prove that in the simplest configuration of a straight line, all magnitudes exist beginning with the ever shortest straight line (that can exist), being irrational. This claim is established in this work, objectively to establish the relationship between algebraic analysis and geometric analysis methods. Appendix 3 provides a simple account showing that the development of geometry from time of Euclid to the modern times unfortunately, has not taken into consideration that all geometric magnitudes are finitely possible to construct. The paper is concluded by Appendix-**4**, which puts forward the philosophical perspective on constructability of magnitudes involving straight lines. The discussion in this section widely shows the misconception and problems associated with use of algebraic means or concepts as methods of proving geometric concepts.

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Appendices

The contents in this paper are based on the following common notions and postulates of Euclidean geometry.

Appendix-1: Common Notions [1, 2, 16]

1. Things which are equal to the same thing are also equal to one another.

2. If equals are added to equals, the wholes are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. Things which coincide with one another are equal to one another.

5. The whole is greater than the part

Appendix-2: Common Postulates [1, 2, 16]

1. It is possible to draw a straight line from any point to any other point.
2. It is possible to extend a line segment continuously in both directions.
3. It is possible to describe a circle with any center and any radius.

Appendix-3: Teaching Geometry According to Euclid (A Simple Historic Milestone on Development of Mathematics) [17]

This paper acknowledges the work of Profesor Robin Hartshorne called Teaching Geometry According to Euclid [17], which provides simple and yet an elegant analysis of how geometry is taught today. Hartshorne does acknowledge that geometry is an independent subject, although he expressed contentments with Hilbert's axioms which somewhat seem to establish some kind of solid relationship between geometry and algebra. This paper cast misgivings in Hilbert's axioms, as they do not provide proper geometric accounts on cases of commensurability and incommensurability, besides what is established in Euclid's Elements. Thus Hilbert's axioms (from which the modern relationship between geometry and algebra is based) do not fully establish the relationship between geometric concepts, and algebraic concepts and this equally justify the assertion that Hilbert's axioms are not sufficiently strong enough to define geometric concepts.

Appendix-4: Philosophical Viewpoint on Constructability of Magnitudes of Forms $\sqrt{n}$ and $\sqrt[3]{x}$

(Where n not a perfect is square and x is not a perfect cube)

Euclidean geometry is a philosophically founded subject in which its nature has been diluted in the modern days, and the cognitive understanding of the ordinary nature of geometry appears to be fading. Perhaps, the problem stems from the misunderstanding of the philosophical perception the early mathematicians and scientists had in approaching the subject. These problems are mainly based on believes that some concepts (such as algebra) were only intuitive to the early mathematicians and philosophers, and such is meant to imply that the natural meaning of geometry is faulty. This is false perception which can be rectified, and separate the algebraic means of proof from the algebraic forms of proof, thus establishing ordinary geometry (Euclidean geometry) as an independent subject. To achieve this, one can perhaps respond to two main philosophical questions: What is the length of the ever shortest straight line segment? How should one prove a geometric concept?

This section responds to these two questions separately. It assumes that the understanding of a concept begins with a definition of the language and metaphor used in constructing a particular concept. The aim of this section therefore, is to establish the desire for scholars to consider the use of the proper approaches in developing geometrical proofs. This is established by considering two important philosophical approaches, based on a metaphorical approach, and the language ideology approach.

Metaphorical approach: metaphorical approach concerns dealing with a situation or problem, according to oxford advanced learner's dictionary [18]. This approach utilizes characteristics of a similar kind for overcoming a situation or problem. On the other hand, Language ideology: show how linguistic choices and language changes are affected by how people formulate and reformulate, and use language. It also reconnoiters the circulation of and struggles over dominant conceptualizations of language and its functions (Discourse analysis, p 66) [19]. Language ideology provides a special association between words and their applications. By relating the two approaches, it is found that the metaphorical approach provides special means of using words and thus, the metaphorical approach is one side of reading while language ideology provides another side of reading. By adopting these two approaches, then one can consider the follows points:

1. One clear problem of using algebraic language as geometrical language is largely based on use of the terms "commensurability" and "incommensurability" in respect to Euclid's notion. As it is required in the philosophical rigor, Euclid did not define these very important geometric terms. The use of unknown language or metaphor would always lead to problems, and this is the case of applying algebra as means of defining geometric concepts. As established earlier, the algebraic notion of "incommensurability" is incomplete, and this is true because such methods of proofs are not comprehensively based on the Euclid's way of using "commensurable" and "incommensurable.

2. Based on the discussion provided through this work, it can be deduced that the algebraic means of proof being applied to geometry as a means of geometric proves, lack the genetic geometric and philosophical test. This is because, the algebraic proofs simply translate the problems from their genetic geometric formulations, into algebraic definitions. With this form of approach, the geometric metaphor and language are lost. For instance, the use of unity in geometry does not imply geometric proofs should be based on unity. A unity magnitude precisely shows the practicability of geometric constructions based on the use of compass and straightedge tools. It does not imply, compass-straightedge constructions cannot finitely conceive smaller magnitudes than unity. Algebraic analysis (used and means of proof) does not consider the smallest line that can be, or that cannot be drawn using straightedge constructions. Any attempts to show this using algebra (or mechanical means) just violates all compass-straightedge construction conditions.

3. It is established in many accounts that all constructible numbers have algebraic characteristics, and so is all constructible points and geometric magnitudes. However, this paper has proven that such notion is geometrically invalid, since all magnitudes involving straight lines are geometrically constructible. To resolve this absurdity, we redefine the statement of constructible magnitudes to assert: "**All measurable magnitudes are geometrically constructible, and not all geometrically constructible magnitudes are measurable**". This statement significantly corrects the notion that compass-straightedge constructions are limited to some forms of measurements (starting with unity) in respect to numbers. However, as demonstrated in the following sections, a straight line is infinitely divisible (at least, one can infinitely bisect any straight line using only compass and straightedge) and some straight line magnitudes would never be measurable using numbers, as some magnitudes would ever be irrational. This observation proves limitations of the concept of measure which assumes some sort of bounds in the ends of a straight line based on use of numbers. With the exception of the diagonals of some quadrilaterals, any other magnitude which cannot be expressed as a rational number is defined as an impossible construction. This paper considers the relationship between constructability and measure as a misconception, and thus corrects the case by redefining the relationship between the two concepts based on the fact that compass-straightedge constructions are genetically not founded on measurability.

Appendix-4.1: Philosophical Proof that the Irrationality Statements for Magnitudes of Forms $\sqrt{n}$ and $\sqrt[3]{x}$ is Faulty

The primary goal of this section is to contrast the perception that the factors $\sqrt{n}$ and $\sqrt[3]{x}$ are geometric irrational magnitudes, in which cases, n and x are not perfect squares and cubes respectively. The proof employs both the metaphorical and language ideologies. An elementary way to expose the flaws in the "irrational assertion" can be established by considering the concepts of 'constructability' and 'measure' as defined in section (2.1). Generally, the concept of construction in Euclidean geometry typically involves use of two tool; classical compass and straightedge, and it is the genetic design of these two tools, which form the foundation of plane geometry. Considering compass-straightedge constructions, this part of the proof begins by answering the question, "what is the measure of the ever shortest straight line that can be constructed?" In principle, a straightedge has no marking and so, the language of measurements in a construction is ruled out. Second, if it was possible to set up the magnitude of the ever shortest straight line segment in some ways (say use of coordinates), then the modern concept of measure, which induce bounds to a magnitude from an assumed point $(0,0)$ becomes invalid, because the number zero does not represent a geometric magnitude. In geometry, any point can be set as reference, and it is not necessarily that such a point have the coordinate $(0,0)$ and, it is not necessarily a geometric requirement that a reference point be set at the point zero. Finally, one can figure out if there exists the shortest straight line. The simplest answer to this statement would be "No". If there was the ever shortest constructible straight line then considerably, the concept of bisecting straight lines would be restricted, but we know that any straight line (however short it might be) can be bisected using compass-straightedge methods. Or if one use arithmetic, it is equally true that it is possible to divide any number by two, except the number zero (at least, this agree with the idea of omitting use of "zero" in this work, as zero is not a geometric magnitude). So in essence, this discovery proves that the language of geometry is very consistent and not flawed. This then implies (as shown in the case of cube duplication), one can construct to the most pleasing limit, any required magnitude using the compass-straightedge approach. To this effect, we can conclude that the concept of defining geometric concept as irrational using algebraic means suffer misconceptions on what the genetic nature of geometry is, and so such proves are geometrically invalid. In essence, the concept of constructability in geometry is not only limited to rational numbers. At least, there is no inherently proper geometric proof for this. In a different argument, it is pellucid that any straight line magnitude can be bisected. So the notion of treating the compass-straightedge constructions as the only constructions that conceive real numbers ratios in the case of comparing magnitudes is an ill-defined notion. We are not restricted from performing bisection of straight lines, and this is the nature of compass straightedge geometry. Based on this understanding, one can then affirm that the statement of the factors $\sqrt{n}$ and $\sqrt[3]{x}$ (with n and x not perfect square and not perfect cube respectively) as "irrational magnitudes" is geometrically invalid.

Appendix-4.2: Philosophical Viewpoint on How to Prove Geometric Concepts

The question of how to prove geometric concepts require thorough discussion and understanding and this paper hopes to set up the flashpoint in a simplified fashion, on how to prove geometric concepts. One of an evincing technique used by Euclid and other early mathematicians as means of proving geometric problems is the use of ratios, a technique which involves comparing magnitudes of a similar kind. In the case of straight lines, complete test for commensurability and as a method of proof, squares of the lines were used. Any other form prove has to be inherently geometrical. Thus, the discovery that the ratio between the diagonal of a square and side length of the square as incommensurate did not completely prove incommensurable magnitudes are equally incommensurable in their squares or the possibility of an impossible construction. Indeed, a handy example to justify this statement and to prove that geometric means of proof are rich enough to prove geometric concepts is the problem of squaring a circle. This problem asks "given a circle of some content (area), construct a square whose content (area) is commensurate to the content (area) of the given circle". Before the problem was justified and labeled as an impossible construct using algebra, it was already known using geometric means that a curve and a straight line are incommensurate. This was formulated by Descartes in his Descartes Géométrie [21, 22]. Descartes established the axiom of incommensurability of the straight line and the curved lines, which can be used to show that segments and circular arcs are incommensurable magnitudes or, at least, they are magnitudes that stay to each other in an exactly unknowable proportion. Descartes also proved that this type of magnitudes does not rely on any proof or argument and so geometric methods of proof are strong enough in verifying geometric concepts. Archimedes's Dimensio circuli [22] also demonstrated that the circle-squaring problem is reducible to the rectification of the circumference. Considerably, there is perfect coherence with Descartes' view and some other geometric accounts about the incomparability between straight lines and curves, which then shows that Descartes umpired the incommensurability in geometry. Equally, in terms of their squares, the contents of a circle and the contents of a square would ever be incommensurate. This is how ordinary geometric concepts should be proven in cases involving lines, and when the use of numbers in any other forms besides using them as ratios is introduced, then such an approach would not be proper geometric proof, and it can best serve as means of analysis. This declaration at least goes back to the time of Descartes. Thus, the problem on the constructability of magnitudes of forms $\sqrt{n}$ and $\sqrt[3]{x}$ has been geometrically solved in this paper.

Appendix-5: Conflict of Interest

The authors declare no conflict of interest.

Appendix-6: Author Biography

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Kimuya M Alex is a physicist and a scholar, passionate in academic research innovations in the areas of: Physics (computation and electronics), Computer Vision and Machine Learning, Geometry and Physics, and Signal Processing.

2. Munyambu .C. June

Munyambu .C. June is a graduand, with degree in education (Physics and Mathematics). Her research interest is in the interface between mathematics and scientific research.