Generalized Fibonacci Numbers: Sum Formulas of the Squares of Terms

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=1}^{n} kW_k^2$ and $\sum_{k=1}^{n} kW_{-k}^2$ for the squares of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize second order recurrence relations.

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1 Introduction

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. Specifically, there are so many studies in the literature that concern about special second order recurrence sequences such as Fibonacci and Lucas. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ n \geq 2, \quad F_0 = 0, F_1 = 1.$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \ n \geq 2, \quad L_0 = 2, L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [8] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \ W_0 = a, W_1 = b, \quad (n \geq 2) \tag{1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [7], [9] and [10]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

$Name of sequence$	Notation: $W_n(a, b; r, s)$	OEIS: [17]
$Fibonacci$	$F_n = W_n(0, 1; 1, 1)$	A000045
$Lucas$	$L_n = W_n(2, 1; 1, 1)$	A000032
$Pell$	$P_n = W_n(0, 1; 2, 1)$	A000129
$Pell - Lucas$	$Q_n = W_n(2, 2; 2, 1)$	A002203
$Jacobsthal$	$J_n = W_n(0, 1; 1, 2)$	A001045
$Jacobsthal - Lucas$	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=1}^{n} kP_k^2 = \frac{1}{8}(-P_{n+2}^2 - (9+8n)\, P_{n+1}^2 + 2\,(3+2n)\, P_{n+2}P_{n+1} + 1)$$

and

$$\sum_{k=1}^{n} kF_{-k}^2 = \frac{1}{2}(-F_{-n+1}^2 + (-1+2n)F_{-n}^2 + (1-2n)F_{-n+1}F_{-n} + 1).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

$Name of sequence$	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[1,2,6,11,12,18]	[5,19]	[3,4,13]
Generalized Tribonacci	[15]		
Generalized Tetranacci	[14,16]		

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 1.1 *For $n \geq 1$ we have the following formulas:if $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} W_k^2 = \frac{(1-s)W_{n+2}^2 + (1-s-r^2-r^2s)W_{n+1}^2 + 2rsW_{n+1}W_{n+2} + (s-1)\,W_1^2 + s^2\,(s-1)\,W_0^2 - 2rsW_1W_0}{(s+1)\,(r+s-1)\,(r-s+1)}.$$

(b)

$$\sum_{k=1}^{n} W_{k+1}W_k = \frac{rW_{n+2}^2 + rs^2W_{n+1}^2 + (1-r^2-s^2)W_{n+1}W_{n+2} - rW_1^2 - rs^2W_0^2 + s(-r^2+s^2-1)W_1W_0}{(s+1)\,(r+s-1)\,(r-s+1)}.$$

Proof. This is given in [18].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 1.2 *For $n \geq 1$ we have the following formulas: If $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} W_{-k}^2 = \frac{\begin{array}{c}(s-1)W_{-n+1}^2 + (r^2+r^2s+s-1)W_{-n}^2 - 2rsW_{-n+1}W_{-n} + 2rsW_1W_0 \\ +(1-s)W_1^2 + (1-s-r^2-r^2s)W_0^2\end{array}}{(s+1)\,(r+s-1)\,(r-s+1)}$$

(b)

$$\sum_{k=1}^{n} W_{-k+1}W_{-k} = \frac{-rW_{-n+1}^2 - rs^2W_{-n}^2 + (r^2+s^2-1)W_{-n+1}W_{-n} + (1-r^2-s^2)W_1W_0 + rW_1^2 + rs^2W_0^2}{(s+1)(r+s-1)(r-s+1)}$$

Proof. This is given in [18].

2 Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1 *For $n \geq 1$ we have the following formulas:if $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} kW_k^2 = \frac{\Lambda_1}{(s+1)^2(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned}
\Lambda_1 &= (-(s-1)(s+1)(r+s-1)(r-s+1)n - r^2s^2 - 2r^2s + r^2 - 2s^2 + 4s - 2)W_{n+2}^2 \\
&+(-(s+1)(r-s+1)(r+s-1)(s+r^2s+r^2-1)n - r^4s^2 - 2r^2s^3 \\
&-2r^4s - r^4 - s^4 - 2r^2s^2 + 2s^3 + 2r^2 - 2s^2 + 2s - 1)W_{n+1}^2 \\
&+2rs((s+1)(r-s+1)(r+s-1)n + r^2s + 2r^2 + s^2 + 2s - 3)W_{n+2}W_{n+1} \\
&+(s^4 - 2s^3 + 2r^2s + 2s^2 - 2s + 1)W_1^2 + s^2(r^2s^2 + 2r^2s - r^2 + 2s^2 - 4s + 2)W_0^2 \\
&-2rs(r^2 + s^3 + s - 2)W_1W_0.
\end{aligned}$$

(b)

$$\sum_{k=1}^{n} kW_{k+1}W_k = \frac{\Lambda_2}{(s+1)^2(r-s+1)^2(r+s-1)^2}$$

where

$$\begin{aligned}
\Lambda_2 &= r((s+1)(r-s+1)(r+s-1)n + s^3 + r^2 + s - 2)W_{n+2}^2 \\
&+rs^2((s+1)(r-s+1)(r+s-1)n + r^2s + 2r^2 + s^2 + 2s - 3)W_{n+1}^2 \\
&+(-(s+1)(r-s+1)(r+s-1)(r^2+s^2-1)n - 2r^2s^3 - r^4 - s^4 \\
&-2r^2s^2 - 2r^2s + 2r^2 + 2s^2 - 1)W_{n+2}W_{n+1} \\
&+r(r^2s - 2s^3 + s^2 + 1)W_1^2 - rs^2(s^3 + r^2 + s - 2)W_0^2 \\
&+s(2r^2s^2 - r^4 + s^4 + 2r^2 - 2s^2 + 1)W_1W_0.
\end{aligned}$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$\begin{aligned} sW_n &= W_{n+2} - rW_{n+1}, \\ s^2W_n^2 &= (W_{n+2} - rW_{n+1})^2 = W_{n+2}^2 + r^2W_{n+1}^2 - 2rW_{n+2}W_{n+1} \end{aligned}$$

we obtain

$$\begin{aligned} s^2nW_n^2 &= nW_{n+2}^2 + nr^2W_{n+1}^2 - 2r \times nW_{n+2}W_{n+1} \\ s^2(n-1)W_{n-1}^2 &= (n-1)W_{n+1}^2 + (n-1)r^2W_n^2 - 2r \times (n-1)W_{n+1}W_n \\ s^2(n-2)W_{n-2}^2 &= (n-2)W_n^2 + (n-2)r^2W_{n-1}^2 - 2r \times (n-2)W_nW_{n-1} \\ s^2(n-3)W_{n-3}^2 &= (n-3)W_{n-1}^2 + (n-3)r^2W_{n-2}^2 - 2r \times (n-3)W_{n-1}W_{n-2} \\ &\vdots \\ s^2 3W_3^2 &= 3W_5^2 + 3r^2W_4^2 - 2r \times 3W_5W_4 \\ s^2 2W_2^2 &= 2W_4^2 + 2r^2W_3^2 - 2r \times 2W_4W_3 \\ s^2W_1^2 &= W_3^2 + r^2W_2^2 - 2rW_3W_2. \end{aligned}$$

If we add the above equations by side by, we get

$$s^2\sum_{k=1}^{n} kW_k^2 = \sum_{k=3}^{n+2}(k-2)W_k^2 + r^2\sum_{k=2}^{n+1}(k-1)W_k^2 - 2r\sum_{k=2}^{n+1}(k-1)W_{k+1}W_k. \tag{2}$$

Note that

$$\begin{aligned} \sum_{k=3}^{n+2}(k-2)W_k^2 &= W_1^2 + (n-1)W_{n+1}^2 + nW_{n+2}^2 + \sum_{k=1}^{n} kW_k^2 - 2\sum_{k=1}^{n} W_k^2 \\ \sum_{k=2}^{n+1}(k-1)W_k^2 &= nW_{n+1}^2 + \sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_k^2 \\ \sum_{k=2}^{n+1}(k-1)W_{k+1}W_k &= nW_{n+2}W_{n+1} + \sum_{k=1}^{n} kW_{k+1}W_k - \sum_{k=1}^{n} W_{k+1}W_k. \end{aligned}$$

If we put them into the (2), we get

$$\begin{aligned} s^2\sum_{k=1}^{n} kW_k^2 &= (W_1^2 + (n-1)W_{n+1}^2 + nW_{n+2}^2 + \sum_{k=1}^{n} kW_k^2 - 2\sum_{k=1}^{n} W_k^2) \\ &\quad + r^2(nW_{n+1}^2 + \sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_k^2) \\ &\quad - 2r(nW_{n+2}W_{n+1} + \sum_{k=1}^{n} kW_{k+1}W_k - \sum_{k=1}^{n} W_{k+1}W_k) \\ &\Rightarrow \\ s^2\sum_{k=1}^{n} kW_k^2 - r^2\sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} kW_k^2 &= -r^2\sum_{k=1}^{n} W_k^2 - 2\sum_{k=1}^{n} W_k^2 + 2r\sum_{k=1}^{n} W_kW_{k+1} \\ &\quad - 2r\sum_{k=1}^{n} kW_kW_{k+1} + nW_{n+1}^2 + W_1^2 - W_{n+1}^2 \\ &\quad + nW_{n+2}^2 + nr^2W_{n+1}^2 - 2nrW_{n+1}W_{n+2} \end{aligned}$$

ans so

$$\begin{aligned} (s^2 - r^2 - 1)\sum_{k=1}^{n} kW_k^2 &= (-r^2 - 2)\sum_{k=1}^{n} W_k^2 + 2r\sum_{k=1}^{n} W_kW_{k+1} - 2r\sum_{k=1}^{n} kW_kW_{k+1} + nW_{n+1}^2 \\ &\quad + W_1^2 - W_{n+1}^2 + nW_{n+2}^2 + nr^2W_{n+1}^2 - 2nrW_{n+1}W_{n+2}. \end{aligned} \tag{3}$$

Next we calculate $\sum_{k=1}^{n} kW_{k+1}W_k$. Multiplying the both side of the relation

$$sW_n = W_{n+2} - rW_{n+1}$$

by W_{n+1} we obtain

$$sW_{n+1}W_n = W_{n+2}W_{n+1} - rW_{n+1}^2$$

and so

$$\begin{aligned}
snW_{n+1}W_n &= nW_{n+2}W_{n+1} - r \times nW_{n+1}^2 \\
s(n-1)W_nW_{n-1} &= (n-1)W_{n+1}W_n - r \times (n-1)W_n^2 \\
s(n-2)W_{n-1}W_{n-2} &= (n-2)W_nW_{n-1} - r(n-2)W_{n-1}^2 \\
s(n-3)W_{n-2}W_{n-3} &= (n-2)W_{n-1}W_{n-2} - r(n-2)W_{n-2}^2 \\
s(n-4)W_{n-3}W_{n-4} &= (n-4)W_{n-2}W_{n-3} - r(n-4)W_{n-3}^2 \\
&\vdots \\
s \times 4W_5W_4 &= 4W_6W_5 - r \times 4W_5^2 \\
s \times 3W_4W_3 &= 3W_5W_4 - r \times 3W_4^2 \\
s \times 2W_3W_2 &= 2W_4W_3 - r \times 2W_3^2 \\
sW_2W_1 &= W_3W_2 - rW_2^2
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n} kW_{k+1}W_k = \sum_{k=2}^{n+1}(k-1)W_{k+1}W_k - r\sum_{k=2}^{n+1}(k-1)W_k^2. \tag{4}$$

Note that

$$\begin{aligned}
\sum_{k=2}^{n+1}(k-1)W_{k+1}W_k &= nW_{n+2}W_{n+1} + \sum_{k=1}^{n} kW_{k+1}W_k - \sum_{k=1}^{n} W_{k+1}W_k, \\
\sum_{k=2}^{n+1}(k-1)W_k^2 &= nW_{n+1}^2 + \sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_k^2.
\end{aligned}$$

We put them in (4) we obtain

$$\begin{aligned}
s\sum_{k=1}^{n} kW_{k+1}W_k &= (nW_{n+2}W_{n+1} + \sum_{k=1}^{n} kW_{k+1}W_k - \sum_{k=1}^{n} W_{k+1}W_k) \\
&\quad -r(nW_{n+1}^2 + \sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_k^2) \\
&\Rightarrow \\
s\sum_{k=1}^{n} kW_{k+1}W_k - \sum_{k=1}^{n} kW_{k+1}W_k &= -r\sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_{k+1}W_k \\
&\quad +r\sum_{k=1}^{n} W_k^2 - nrW_{n+1}^2 + nW_{n+2}W_{n+1}
\end{aligned}$$

and so

$$(s-1)\sum_{k=1}^{n} kW_{k+1}W_k = -r\sum_{k=1}^{n} kW_k^2 - \sum_{k=1}^{n} W_{k+1}W_k + r\sum_{k=1}^{n} W_k^2 - nrW_{n+1}^2 + nW_{n+2}W_{n+1}. \tag{5}$$

Then, using

$$W_2 = (rW_1 + sW_0)$$

and Theorem 1.1 and solving the system (3)-(5), the required results of (a) and (b) follow.

Taking $r = s = 1$ in Theorem 1.1 (a) and (b), we obtain the following proposition.

Proposition 2.2 *If $r = s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_k^2 = \frac{1}{2}(-W_{n+2}^2 - (3+2n)W_{n+1}^2 + (3+2n)W_{n+2}W_{n+1} + W_1^2 + W_0^2 - W_1W_0)$.

(b) $\sum_{k=1}^{n} kW_{k+1}W_k = \frac{1}{4}((1+2n)\,W_{n+2}^2 + (3+2n)\,W_{n+1}^2 - (5+2n)\,W_{n+2}W_{n+1} + W_1^2 - W_0^2 + 3W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.3 *For $n \geq 1$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} kF_k^2 = \frac{1}{2}(-F_{n+2}^2 - (3+2n)F_{n+1}^2 + (3+2n)F_{n+2}F_{n+1} + 1)$.

(b) $\sum_{k=1}^{n} kF_{k+1}F_k = \frac{1}{4}((1+2n)\,F_{n+2}^2 + (3+2n)\,F_{n+1}^2 - (5+2n)\,F_{n+2}F_{n+1} + 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.4 *For $n \geq 1$, Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kL_k^2 = \frac{1}{2}(-L_{n+2}^2 - (3+2n)L_{n+1}^2 + (3+2n)L_{n+2}L_{n+1} + 3)$.

(b) $\sum_{k=1}^{n} kL_{k+1}L_k = \frac{1}{4}((1+2n)\,L_{n+2}^2 + (3+2n)\,L_{n+1}^2 - (5+2n)\,L_{n+2}L_{n+1} + 3)$.

Taking $r = 2, s = 1$ in Theorem 1.1 (a) and (b), we obtain the following proposition.

Proposition 2.5 *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_k^2 = \frac{1}{8}(-W_{n+2}^2 - (9+8n)\,W_{n+1}^2 + 2\,(3+2n)\,W_{n+2}W_{n+1} + (W_1 - W_0)^2)$.

(b) $\sum_{k=1}^{n} kW_{k+1}W_k = \frac{1}{8}((1+2n)\,W_{n+2}^2 + (3+2n)\,W_{n+1}^2 - 4\,(1+n)\,W_{n+2}W_{n+1} + W_1^2 - W_0^2)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.6 *For $n \geq 1$, Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kP_k^2 = \frac{1}{8}(-P_{n+2}^2 - (9+8n)\,P_{n+1}^2 + 2\,(3+2n)\,P_{n+2}P_{n+1} + 1)$.

(b) $\sum_{k=1}^{n} kP_{k+1}P_k = \frac{1}{8}((1+2n)\,P_{n+2}^2 + (3+2n)\,P_{n+1}^2 - 4\,(1+n)\,P_{n+2}P_{n+1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.7 *For $n \geq 1$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kQ_k^2 = \frac{1}{8}(-Q_{n+2}^2 - (9+8n)\,Q_{n+1}^2 + 2\,(3+2n)\,Q_{n+2}Q_{n+1})$.

(b) $\sum_{k=1}^{n} kQ_{k+1}Q_k = \frac{1}{8}((1+2n)\,Q_{n+2}^2 + (3+2n)\,Q_{n+1}^2 - 4\,(1+n)\,Q_{n+2}Q_{n+1})$.

If $r = 1, s = 2$ then $(s+1)(r+s-1)(r-s+1) = 0$ so we can't use Theorem 2.1. In other words, the method of the proof Theorem 2.1 can't be used to find $\sum_{k=1}^{n} kW_k^2$ and $\sum_{k=1}^{n} kW_{k+1}W_k$. Therefore we need another method to find them which is given in the following theorem.

Theorem 2.8 *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_k^2 = \frac{1}{162}((2+27n)W_{n+2}^2 + 2(-5+9n)W_{n+1}^2 - 4(2+9n)W_{n+2}W_{n+1}$
$+8\,(2W_1 - W_0)\,(W_1 + W_0) + 9\,(W_1 - 2W_0)^2\,n^2)$.

(b) $\sum_{k=1}^{n} kW_{k+1}W_k = \frac{1}{162}((10+9n)W_{n+2}^2 + 4(10-9n)W_{n+1}^2 + 2(-29+27n)W_{n+2}W_{n+1}$
$+4\,(W_1 + 10W_0)\,(2W_1 - W_0) - 9\,(W_1 - 2W_0)^2\,n^2)$.

Proof.

(a) The proof will be by induction on n. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} + 2W_{n-2};\ W_0 = a, W_1 = b, \quad (n \geq 2) \tag{6}$$

with the initial values W_0, W_1 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (6) holds for all integer n. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal numbers

n	W_n	W_{-n}
0	W_0	...
1	W_1	$-\frac{1}{2}W_0 + \frac{1}{2}W_1$
2	$2W_0 + W_1$	$\frac{3}{4}W_0 - \frac{1}{4}W_1$
3	$2W_0 + 3W_1$	$-\frac{5}{8}W_0 + \frac{3}{8}W_1$
4	$6W_0 + 5W_1$	$\frac{11}{16}W_0 - \frac{5}{16}W_1$
5	$10W_0 + 11W_1$	$-\frac{21}{32}W_0 + \frac{11}{32}W_1$
6	$22W_0 + 21W_1$	$\frac{43}{64}W_0 - \frac{21}{64}W_1$

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$t^2 - t - 2 = 0.$$

The roots of characteristic equation are

$$\alpha = 2, \quad \beta = -1$$

and the roots satisfy the following

$$\alpha+\beta=1, \alpha\beta=-2, \alpha-\beta=3.$$

Using these roots and the recurrence relation, Binet formula can be given as

$$W_n=\frac{A\alpha^n-B\beta^n}{\alpha-\beta}=\frac{A\times 2^n-B(-1)^n}{3} \tag{7}$$

where $A=W_1-W_0\beta=W_1+W_0$ and $B=W_1-W_0\alpha=W_1-2W_0$.

We now prove (a) by induction on n. If $n=1$ wee that the sum formula reduces to the relation

$$W_1^2=\frac{1}{162}(29W_3^2+8W_2^2+25W_1^2+28W_0^2-44W_3W_2-28W_1W_0). \tag{8}$$

Since

$$\begin{aligned} W_2 &= 2W_0+W_1, \\ W_3 &= 2W_0+3W_1, \end{aligned}$$

(8) is true. Assume that the relation in (a) is true for $n=m$, i.e.,

$$\begin{aligned} \sum_{k=1}^{m} kW_k^2 &= \frac{1}{162}((2+27m)W_{m+2}^2+2(-5+9m)W_{m+1}^2-4(2+9m)W_{m+2}W_{m+1} \\ &\quad +8\left(2W_1-W_0\right)\left(W_1+W_0\right)+9\left(W_1-2W_0\right)^2 m^2). \end{aligned}$$

Then we get

$$\begin{aligned} \sum_{k=1}^{m+1} kW_k^2 &= (m+1)W_{m+1}^2+\sum_{k=1}^{m} kW_k^2 \\ &= \frac{1}{162}((2+27m)W_{m+2}^2+4\left(45m+38\right)W_{m+1}^2-4(2+9m)W_{m+2}W_{m+1} \\ &\quad -9\left(W_1-2W_0\right)^2\left(1+2m\right)+8\left(2W_1-W_0\right)\left(W_1+W_0\right)+9\left(W_1-2W_0\right)^2\left(m+1\right)^2) \\ &= \frac{1}{162}((29+27m)W_{m+3}^2+2(4+9m)W_{m+2}^2-4(11+9m)W_{m+3}W_{m+2} \\ &\quad +8\left(2W_1-W_0\right)\left(W_1+W_0\right)+9\left(W_1-2W_0\right)^2\left(m+1\right)^2) \\ &= \frac{1}{162}((2+27(m+1))W_{(m+1)+2}^2+2(-5+9(m+1))W_{(m+1)+1}^2 \\ &\quad -4(2+9(m+1))W_{(m+1)+2}W_{(m+1)+1}+8\left(2W_1-W_0\right)\left(W_1+W_0\right)+9\left(W_1-2W_0\right)^2\left(m+1\right)^2) \end{aligned}$$

where

$$\begin{aligned} &(2+27m)W_{m+2}^2+4\left(45m+38\right)W_{m+1}^2-4(2+9m)W_{m+2}W_{m+1}-9\left(W_1-2W_0\right)^2\left(1+2m\right) \qquad (9) \\ =\ &(29+27m)W_{m+3}^2+2(4+9m)W_{m+2}^2-4(11+9m)W_{m+3}W_{m+2}. \end{aligned}$$

(9) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n=m+1$.

(b) We now prove (b) by induction on n. If $n=1$ wee that the sum formula reduces to the relation

$$W_2W_1=\frac{1}{162}(19W_3^2+4W_2^2-W_1^2-76W_0^2-4W_2W_3+112W_0W_1). \tag{10}$$

Since

$$\begin{aligned} W_2 &= 2W_0+W_1, \\ W_3 &= 2W_0+3W_1, \end{aligned}$$

(10) is true. Assume that the relation in (b) is true for $n=m$, i.e.,

$$\begin{aligned}\sum_{k=1}^{m} kW_{k+1}W_k &= \frac{1}{162}((10+9m)W_{m+2}^2+4(10-9m)W_{m+1}^2+2(-29+27m)W_{m+2}W_{m+1}\\ &\quad +4\left(W_1+10W_0\right)\left(2W_1-W_0\right)-9\left(W_1-2W_0\right)^2 m^2).\end{aligned}$$

Then we get

$$\begin{aligned}\sum_{k=1}^{m+1} kW_{k+1}W_k &= (m+1)W_{m+2}W_{m+1}+\sum_{k=1}^{m} W_{k+1}W_k\\ &= \frac{1}{162}((10+9m)W_{m+2}^2+4(10-9m)W_{m+1}^2+2(-29+27m)W_{m+2}W_{m+1}\\ &\quad +162(m+1)W_{m+2}W_{m+1}+9\left(W_1-2W_0\right)^2(1+2m)\\ &\quad +4\left(W_1+10W_0\right)\left(2W_1-W_0\right)-9\left(W_1-2W_0\right)^2(m+1)^2)\\ &= \frac{1}{162}((19+9m)W_{m+3}^2+4(1-9m)W_{m+2}^2+2(-2+27m)W_{m+3}W_{m+2}\\ &\quad +4\left(W_1+10W_0\right)\left(2W_1-W_0\right)-9\left(W_1-2W_0\right)^2(m+1)^2)\\ &= \frac{1}{162}((10+9(m+1))W_{(m+1)+2}^2+4(10-9(m+1))W_{(m+1)+1}^2\\ &\quad +2(-29+27(m+1))W_{(m+1)+2}W_{(m+1)+1}\\ &\quad +4\left(W_1+10W_0\right)\left(2W_1-W_0\right)-9\left(W_1-2W_0\right)^2(m+1)^2)\end{aligned}$$

where

$$\begin{aligned}&(10+9m)W_{m+2}^2+4(10-9m)W_{m+1}^2+2(-29+27m)W_{m+2}W_{m+1}\\ &+162(m+1)W_{m+2}W_{m+1}+9\left(W_1-2W_0\right)^2(1+2m)\\ =\ &(19+9m)W_{m+3}^2+4(1-9m)W_{m+2}^2+2(-2+27m)W_{m+3}W_{m+2}.\end{aligned} \tag{11}$$

(11) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n=m+1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 2.9 *For $n\geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} kJ_k^2=\frac{1}{162}((2+27n)J_{n+2}^2+2(-5+9n)J_{n+1}^2-4(2+9n)J_{n+2}J_{n+1}+16+9n^2)$.

(b) $\sum_{k=1}^{n} kJ_{k+1}J_k=\frac{1}{162}((10+9n)J_{n+2}^2+4(10-9n)J_{n+1}^2+2(-29+27n)J_{n+2}J_{n+1}+8-9n^2)$

Taking $W_n=j_n$ with $j_0=2, j_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.10 *For $n\geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} kj_k^2=\frac{1}{162}((2+27n)j_{n+2}^2+2(-5+9n)j_{n+1}^2-4(2+9n)j_{n+2}j_{n+1}+81n^2)$.

(b) $\sum_{k=1}^{n} kj_{k+1}j_k=\frac{1}{162}((10+9n)j_{n+2}^2+4(10-9n)j_{n+1}^2+2(-29+27n)j_{n+2}j_{n+1}-81n^2)$.

3 Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1 *For $n \geq 1$ we have the following formulas: If $(s+1)(r+s-1)(r-s+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} kW_{-k}^2 = \frac{\Lambda_3}{(s+1)^2 (r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned}
\Lambda_3 &= (n(s-1)(s+1)(r-s+1)(r+s-1) - s^4 + 2s^3 - 2r^2 s - 2s^2 + 2s - 1)W_{-n+1}^2 \\
&\quad +((s+1)(r-s+1)(s+r^2 s+r^2-1)(r+s-1)n + r^2 s^2 - 2r^2 s^3 - r^2 s^4 - 2s^2 + 4s^3 - 2s^4)W_{-n}^2 \\
&\quad +2rs((s+1)(-r+s-1)(r+s-1)n + s^3 + r^2 + s - 2)W_{-n+1}W_{-n} \\
&\quad +(s^4 - 2s^3 + 2r^2 s + 2s^2 - 2s + 1)W_1^2 + s^2(r^2 s^2 + 2r^2 s - r^2 + 2s^2 - 4s + 2)W_0^2 \\
&\quad -2rs(s^3 + r^2 + s - 2)W_1 W_0.
\end{aligned}$$

(b)

$$\sum_{k=1}^{n} kW_{-k+1}W_{-k} = \frac{\Lambda_4}{(s+1)^2 (r-s+1)^2 (r+s-1)^2}$$

where

$$\begin{aligned}
\Lambda_4 &= r(-(s+1)(r-s+1)(r+s-1)n + 2s^3 - r^2 s - s^2 - 1)W_{-n+1}^2 \\
&\quad +rs^2(-(s+1)(r-s+1)(r+s-1)n + s^3 + r^2 + s - 2)W_{-n}^2 \\
&\quad +((s+1)(r-s+1)(r+s-1)(r^2+s^2-1)n - s^5 + r^4 s - 2r^2 s^3 - 2r^2 s + 2s^3 - s)W_{-n+1}W_{-n} \\
&\quad +rW_1^2(-2s^3 + r^2 s + s^2 + 1) - rs^2(s + r^2 + s^3 - 2)W_0^2 \\
&\quad +s\left(-r^4 + s^4 + 2r^2 s^2 + 2r^2 - 2s^2 + 1\right)W_1 W_0.
\end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n+2} = r \times W_{-n+1} + s \times W_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

and using

$$\begin{aligned}
s^2 W_{-n}^2 &= W_{-n+2}^2 + r^2 W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1}, \\
s^2 W_{-n+1}^2 &= W_{-n+3}^2 + r^2 W_{-n+2}^2 - 2rW_{-n+3}W_{-n+2}, \\
s^2 W_{-n+2}^2 &= W_{-n+4}^2 + r^2 W_{-n+3}^2 - 2rW_{-n+4}W_{-n+3},
\end{aligned}$$

we obtain

$$\begin{aligned} s^2\times(n+2)W_{-n-2}^2 &= (n+2)W_{-n}^2+r^2\times(n+2)W_{-n-1}^2-2r\times(n+2)W_{-n}W_{-n-1} \\ s^2(n+1)W_{-n-1}^2 &= (n+1)W_{-n+1}^2+r^2\times(n+1)W_{-n}^2-2r\times(n+1)W_{-n+1}W_{-n} \end{aligned}$$

$$\begin{aligned} s^2\times nW_{-n}^2 &= nW_{-n+2}^2+r^2\times nW_{-n+1}^2-2r\times nW_{-n+2}W_{-n+1} \\ s^2\times(n-1)W_{-n+1}^2 &= (n-1)W_{-n+3}^2+r^2\times(n-1)W_{-n+2}^2-2r\times(n-1)W_{-n+3}W_{-n+2} \\ s^2\times(n-2)W_{-n+2}^2 &= (n-2)W_{-n+4}^2+r^2\times(n-2)W_{-n+3}^2-2r\times(n-2)W_{-n+4}W_{-n+3} \\ s^2\times(n-3)W_{-n+3}^2 &= (n-3)W_{-n+5}^2+r^2\times(n-3)W_{-n+4}^2-2r\times(n-3)W_{-n+5}W_{-n+4} \\ &\vdots \\ s^2\times 3W_{-3}^2 &= 3W_{-1}^2+r^2\times 3W_{-2}^2-2r\times 3W_{-1}W_{-2} \\ s^2\times 2W_{-2}^2 &= 2W_0^2+r^2\times 2W_{-1}^2-2r\times 2W_0W_{-1} \\ s^2W_{-1}^2 &= W_1^2+r^2W_0^2-2rW_1W_0. \end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned} s^2\sum_{k=1}^{n}kW_{-k}^2 &= (W_1^2+2W_0^2-(n+1)W_{-n+1}^2-(n+2)W_{-n}^2+\sum_{k=1}^{n}(k+2)W_{-k}^2) \\ &\quad +r^2(W_0^2-(n+1)W_{-n}^2+\sum_{k=1}^{n}(k+1)W_{-k}^2) \\ &\quad -2r(W_1W_0-(n+1)W_{-n+1}W_{-n}+\sum_{k=1}^{n}(k+1)W_{-k+1}W_{-k}) \end{aligned}$$

and so

$$\begin{aligned} s^2\sum_{k=1}^{n}kW_{-k}^2 &= (W_1^2+2W_0^2-(n+1)W_{-n+1}^2-(n+2)W_{-n}^2+\sum_{k=1}^{n}kW_{-k}^2+2\sum_{k=1}^{n}W_{-k}^2) \\ &\quad +r^2(W_0^2-(n+1)W_{-n}^2+\sum_{k=1}^{n}kW_{-k}^2+\sum_{k=1}^{n}W_{-k}^2)-2r(W_1W_0-(n+1)W_{-n+1}W_{-n} \\ &\quad +\sum_{k=1}^{n}kW_{-k+1}W_{-k}+\sum_{k=1}^{n}W_{-k+1}W_{-k}) \end{aligned} \tag{12}$$

Next we calculate $\sum_{k=1}^{n}kW_{-k+1}W_{-k}$. Using the recurrence relation

$$W_{-n+2}=r\times W_{-n+1}+s\times W_{-n}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

and multiplying the both side of the last relations by W_{-n+1} we obtain

$$sW_{-n+1}W_{-n}=W_{-n+2}W_{-n+1}-rW_{-n+1}^2$$

and so

$$\begin{aligned} s\times nW_{-n+1}W_{-n} &= nW_{-n+2}W_{-n+1}-r\times nW_{-n+1}^2 \\ s\times(n-1)W_{-n+2}W_{-n+1} &= (n-1)W_{-n+3}W_{-n+2}-r\times(n-1)W_{-n+2}^2 \end{aligned}$$

$$
\begin{aligned}
s\times(n-2)W_{-n+3}W_{-n+2} &= (n-2)W_{-n+4}W_{-n+3}-r\times(n-2)W_{-n+3}^{2}\\
s\times(n-3)W_{-n+4}W_{-n+3} &= (n-3)W_{-n+5}W_{-n+4}-r\times(n-3)W_{-n+4}^{2}\\
s\times(n-4)W_{-n+5}W_{-n+4} &= (n-4)W_{-n+6}W_{-n+5}-r\times(n-4)W_{-n+5}^{2}\\
&\vdots\\
s\times 4W_{-3}W_{-4} &= 4W_{-2}W_{-3}-r\times 4W_{-3}^{2}\\
s\times 3W_{-2}W_{-3} &= 3W_{-1}W_{-2}-r\times 3W_{-2}^{2}\\
s\times 2W_{-1}W_{-2} &= 2W_{0}W_{-1}-r\times 2W_{-1}^{2}\\
sW_{0}W_{-1} &= W_{1}W_{0}-r\times W_{0}^{2}
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{aligned}
s\sum_{k=1}^{n}kW_{-k+1}W_{-k} &= (W_{1}W_{0}-(n+1)W_{-n+1}W_{-n}+\sum_{k=1}^{n}kW_{-k+1}W_{-k}+\sum_{k=1}^{n}W_{-k+1}W_{-k}) \\
&\quad -r(W_{0}^{2}-(n+1)W_{-n}^{2}+\sum_{k=1}^{n}kW_{-k}^{2}+\sum_{k=1}^{n}W_{-k}^{2})
\end{aligned}
\tag{13}
$$

Then, using Theorem 1.2 and solving the system (12)-(13), the required results of (a) and (b) follow.

Taking $r=s=1$ in Theorem 1.2 (a) and (b), we obtain the following proposition.

Proposition 3.2 *If $r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}kW_{-k}^{2}=\frac{1}{2}(-W_{-n+1}^{2}+(-1+2n)W_{-n}^{2}+(1-2n)W_{-n+1}W_{-n}+W_{1}^{2}+W_{0}^{2}-W_{1}W_{0})$.

(b) $\sum_{k=1}^{n}kW_{-k+1}W_{-k}=\frac{1}{4}((-1-2n)W_{-n+1}^{2}+(1-2n)W_{-n}^{2}+(-3+2n)W_{-n+1}W_{-n}+W_{1}^{2}-W_{0}^{2}+3W_{1}W_{0})$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_{n}=F_{n}$ with $F_{0}=0,F_{1}=1$).

Corollary 3.3 *For $n\geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n}kF_{-k}^{2}=\frac{1}{2}(-F_{-n+1}^{2}+(-1+2n)F_{-n}^{2}+(1-2n)F_{-n+1}F_{-n}+1)$.

(b) $\sum_{k=1}^{n}kF_{-k+1}F_{-k}=\frac{1}{4}((-1-2n)F_{-n+1}^{2}+(1-2n)F_{-n}^{2}+(-3+2n)F_{-n+1}F_{-n}+1)$.

Taking $W_{n}=L_{n}$ with $L_{0}=2,L_{1}=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.4 *For $n\geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}kL_{-k}^{2}=\frac{1}{2}(-L_{-n+1}^{2}+(-1+2n)L_{-n}^{2}+(1-2n)L_{-n+1}L_{-n}+3)$.

(b) $\sum_{k=1}^{n}kL_{-k+1}L_{-k}=\frac{1}{4}((-1-2n)L_{-n+1}^{2}+(1-2n)L_{-n}^{2}+(-3+2n)L_{-n+1}L_{-n}+3)$.

Taking $r=2,s=1$ in Theorem 1.2 (a) and (b), we obtain the following proposition.

Proposition 3.5 *If $r=2,s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k}^{2} = \frac{1}{8}(-W_{-n+1}^{2} + (-1+8n)\, W_{-n}^{2} + 2\,(1-2n)\, W_{-n+1}W_{-n} + (W_1 - W_0)^2)$.

(b) $\sum_{k=1}^{n} kW_{-k+1}W_{-k} = \frac{1}{8}((-1-2n)\, W_{-n+1}^{2} + (1-2n)\, W_{-n}^{2} + 4nW_{-n+1}W_{-n} + (W_1^2 - W_0^2))$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.6 *For $n \geq 1$, Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} kP_{-k}^{2} = \frac{1}{8}(-P_{-n+1}^{2} + (-1+8n)\, P_{-n}^{2} + 2\,(1-2n)\, P_{-n+1}P_{-n} + 1)$.

(b) $\sum_{k=1}^{n} kP_{-k+1}P_{-k} = \frac{1}{8}((-1-2n)\, P_{-n+1}^{2} + (1-2n)\, P_{-n}^{2} + 4nP_{-n+1}P_{-n} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.7 *For $n \geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} kQ_{-k}^{2} = \frac{1}{8}(-Q_{-n+1}^{2} + (-1+8n)\, Q_{-n}^{2} + 2\,(1-2n)\, Q_{-n+1}Q_{-n})$.

(b) $\sum_{k=1}^{n} kQ_{-k+1}Q_{-k} = \frac{1}{8}((-1-2n)\, Q_{-n+1}^{2} + (1-2n)\, Q_{-n}^{2} + 4nQ_{-n+1}Q_{-n})$.

If $r = 1, s = 2$ then $(s+1)\,(r+s-1)\,(r-s+1) = 0$ so we can't use Theorem 3.1. In other words, the method of the proof Theorem 3.1 can't be used to find $\sum_{k=1}^{n} kW_{-k}^{2}$ and $\sum_{k=1}^{n} kW_{-k+1}W_{-k}$. Therefore we need another method to find them which is given in the following theorem.

Theorem 3.8 *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k}^{2} = \frac{1}{162}(-(16+9n)W_{-n+1}^{2} + 2(4+27n)W_{-n}^{2} - 4(2+9n)W_{-n+1}W_{-n}$
$+8\,(2W_1 - W_0)\,(W_1 + W_0) + 9\,(W_1 - 2W_0)^2\, n^2)$.

(b) $\sum_{k=1}^{n} kW_{-k+1}W_{-k} = \frac{1}{162}(-(8+27n)W_{-n+1}^{2} + 4(10-9n)W_{-n}^{2} + 2(-38+9n)W_{-n+1}W_{-n} + 4(2W_1 - W_0)(W_1 + 10W_0) - 9(W_1 - 2W_0)^2 n^2)$.

Proof. (a) and (b) can be proved by mathematical induction.

(a) (a) We prove (a). The proof will be by induction on n. We now prove (a) by induction on n. If $n = 1$ wee that the sum formula reduces to the relation

$$W_{-1}^{2} = \frac{1}{162}(25W_1^2 + 28W_0^2 + 62W_{-1}^2 - 28W_0W_1 - 25W_0^2 - 44W_0W_{-1}). \tag{14}$$

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1)$$

(14) is true. Assume that the relation in (a) is true for $n = m$, i.e.,

$$\begin{aligned}\sum_{k=1}^{m} kW_{-k}^{2} &= \frac{1}{162}(-(16+9m)W_{-m+1}^{2} + 2(4+27m)W_{-m}^{2} - 4(2+9m)W_{-m+1}W_{-m} \\ &\quad +8\,(2W_1 - W_0)\,(W_1 + W_0) + 9\,(W_1 - 2W_0)^2\, m^2).\end{aligned}$$

Then we get

$$\begin{aligned}\sum_{k=1}^{m+1} kW_{-k}^2 &= (m+1)W_{-(m+1)}^2 + \sum_{k=1}^{m} W_{-k}^2 \\ &= (m+1)W_{-m-1}^2 + \frac{1}{162}(-(16+9m)W_{-m+1}^2 + 2(4+27m)W_{-m}^2 \\ &\quad -4(2+9m)W_{-m+1}W_{-m} + 8(2W_1-W_0)(W_1+W_0) + 9(W_1-2W_0)^2 m^2) \\ &= \frac{1}{162}(-(16+9m)W_{-m+1}^2 + 2(4+27m)W_{-m}^2 + 162(m+1)W_{-m-1}^2 - 4(2+9m)W_{-m+1}W_{-m} \\ &\quad -9(W_1-2W_0)^2(1+2m) + 8(2W_1-W_0)(W_1+W_0) + 9(W_1-2W_0)^2(m+1)^2) \\ &= \frac{1}{162}(-(25+9m)W_{-m}^2 + 2(31+27m)W_{-m-1}^2 - 4(11+9m)W_{-m}W_{-m-1} \\ &\quad +8(2W_1-W_0)(W_1+W_0) + 9(W_1-2W_0)^2(m+1)^2)\end{aligned}$$

where

$$\begin{aligned}& -(16+9m)W_{-m+1}^2 + 2(4+27m)W_{-m}^2 + 162(m+1)W_{-m-1}^2 \\ & -4(2+9m)W_{-m+1}W_{-m} - 9(W_1-2W_0)^2(1+2m) \\ = {} & -(25+9m)W_{-m}^2 + 2(31+27m)W_{-m-1}^2 - 4(11+9m)W_{-m}W_{-m-1}.\end{aligned} \tag{15}$$

(15) can be proved by using Binet formula of W_n. Hence, the relation in (a) holds also for $n=m+1$.

(b) We now prove (b) by induction on n. If $n=1$ wee see that the sum formula reduces to the relation

$$W_0W_{-1} = \frac{1}{162}(-W_1^2 - 111W_0^2 + 4W_{-1}^2 + 112W_0W_1 - 58W_0W_{-1}) \tag{16}$$

Since

$$W_{-1} = (-\frac{1}{2}W_0 + \frac{1}{2}W_1),$$

(16) is true. Assume that the relation in (b) is true for $n=m$ i,e.,

$$\begin{aligned}\sum_{k=1}^{m} kW_{-k+1}W_{-k} &= \frac{1}{162}(-(8+27m)W_{-m+1}^2 + 4(10-9m)W_{-m}^2 \\ &\quad +2(-38+9m)W_{-m+1}W_{-m} + 4(2W_1-W_0)(W_1+10W_0) - 9(W_1-2W_0)^2m^2).\end{aligned}$$

Then we get

$$\begin{aligned}\sum_{k=1}^{m+1} W_{-k+1}W_{-k} &= (m+1)W_{-(m+1)+1}W_{-(m+1)} + \sum_{k=1}^{m} W_{-k+1}W_{-k} \\ &= (m+1)W_{-m}W_{-m-1} + \frac{1}{162}(-(8+27m)W_{-m+1}^2 + 4(10-9m)W_{-m}^2 \\ &\quad +2(-38+9m)W_{-m+1}W_{-m} + 4(2W_1-W_0)(W_1+10W_0) - 9(W_1-2W_0)^2m^2) \\ &= \frac{1}{162}(-(8+27m)W_{-m+1}^2 + 4(10-9m)W_{-m}^2 + 2(-38+9m)W_{-m+1}W_{-m} \\ &\quad +162(m+1)W_{-m}W_{-m-1} \\ &\quad +9(W_1-2W_0)^2(1+2m) + 4(2W_1-W_0)(W_1+10W_0) - 9(W_1-2W_0)^2(m+1)^2) \\ &= \frac{1}{162}(-(35+27m)W_{-m}^2 + 4(1-9m)W_{-m-1}^2 + 2(-29+9m)W_{-m}W_{-m-1} \\ &\quad +4(2W_1-W_0)(W_1+10W_0) - 9(W_1-2W_0)^2(m+1)^2)\end{aligned}$$

where

$$\begin{aligned} & -(8+27m)W_{-m+1}^2+4(10-9m)W_{-m}^2+2(-38+9m)W_{-m+1}W_{-m} \\ & +162(m+1)W_{-m}W_{-m-1}+9\left(W_1-2W_0\right)^2(1+2m) \\ = & -(35+27m)W_{-m}^2+4(1-9m)W_{-m-1}^2+2(-29+9m)W_{-m}W_{-m-1}. \end{aligned} \tag{17}$$

(17) can be proved by using Binet formula of W_n. Hence, the relation in (b) holds also for $n=m+1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 3.9 *For $n\geq 1$, Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n} kJ_{-k}^2=\frac{1}{162}(-(16+9n)J_{-n+1}^2+2(4+27n)J_{-n}^2-4(2+9n)J_{-n+1}J_{-n}+16+9n^2).$

(b) $\sum_{k=1}^{n} kJ_{-k+1}J_{-k}=\frac{1}{162}(-(8+27n)J_{-n+1}^2+4(10-9n)J_{-n}^2+2(-38+9n)J_{-n+1}J_{-n}+8-9n^2)$

Taking $W_n=j_n$ with $j_0=2, j_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.10 *For $n\geq 1$, Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n} kj_{-k}^2=\frac{1}{162}(-(16+9n)j_{-n+1}^2+2(4+27n)j_{-n}^2-4(2+9n)j_{-n+1}j_{-n}+81n^2).$

(b) $\sum_{k=1}^{n} kj_{-k+1}j_{-k}=\frac{1}{162}(-(8+27n)j_{-n+1}^2+4(10-9n)j_{-n}^2+2(-38+9n)j_{-n+1}j_{-n}-81n^2).$

4 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

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