Analysis of nonlinear neutral pantograph differential equations with ψ-Hilfer fractional derivative

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Abstract
In this paper, we discuss the existence, uniqueness and stability of nonlinear neutral pantograph equation with ψ-Hilfer fractional derivative. The arguments are based upon Schauder fixed point theorem and Banach contraction principle. Moreover we discuss the Ulam-Hyers type stability.

Indexing terms/Keywords: ψ-Hilfer fractional derivative, Existence, Stability.

Subject Classification: 26A33, 34A12, 49K40.

Language: English

Date of Submission: 2018-04-25

Date of Acceptance: 2018-010

Date of Publication: 2018-08-30

ISSN:

Volume: 1 Issue: 2

Journal: MathLAB Journal

Website: https://purkh.com

1 Introduction

Fractional differential equations (FDEs) have gained considerable importance due to their applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al [12], Hilfer [9] and Podlubny [16]. The significant attention on the existence of solutions of FDEs with classical fractional derivatives can be seen in [2, 6, 4, 8]. A special kind of delay differential equations is so called pantograph equations is initiated by Balachandran [3] and led a foundation to neutral type pantograph equation in the remark. The analysis of pantograph equations and interesting remarks can be found in [3, 7, 11, 17, 18]. Recently, stability of FDEs has attracted increasing interest. A complete work on the Hyers Ulam Rassias stability for nonlinear FDEs has been developed by many researchers [1, 10, 14, 21]. In the recent days, ψ-Hilfer fractional derivative has introduced in [19], wherein the kernel is in terms of function. This incorporate many fractional derivatives concerning to the rate assigned to the function. The purpose of the paper is to investigate the existence, uniqueness and stability of nonlinear neutral pantograph equation involving ψ-Hilfer fractional derivative of the form

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t) = \mathfrak{g}(t,\mathfrak{u}(t),\mathfrak{u}(\kappa t),\mathfrak{D}^{\alpha,\beta}\mathfrak{u}(\kappa t)), \quad t \in J := [a,b], \tag{1.1}$$

$$\mathfrak{I}^{1-\gamma;\psi}x(a) = x_a, \tag{1.2}$$

where $\mathfrak{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha(0<\alpha<1)$ and type $\beta(0\le\beta\le1)$ and $\mathfrak{I}^{1-\gamma;\psi}$ is the ψ-fractional integral of order $1-\gamma(\gamma=\alpha+\beta-\alpha\beta)$. Let R be a Banach space, $\mathfrak{g}: J\times R\times R\times R\to R$ is a given continuous function and $0<\kappa<1$.

The outline of the paper is as follows. In Section 2, we give some basic definitions and results concerning the ψ-Hilfer fractional derivative. In Section 3, we present our existence and uniqueness results by using Schauder's fixed point theorem and Banach contraction principle. In section 4, we discuss four kinds of Ulam stability.

2 Preliminaries

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [12, 19].

$$C_{\gamma,\psi}[a,b] = \{\mathfrak{g}:(a,b]\to R : (\psi(t)-\psi(a))^{\gamma}\,\mathfrak{g}(t)\in C[a,b]\}, 0\le\gamma<1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\gamma,\psi}} = \left\|(\psi(t)-\psi(a))^{\gamma}\,\mathfrak{g}(t)\right\|_{C[a,b]} = \max_{t\in J}\left|(\psi(t)-\psi(a))^{\gamma}\,\mathfrak{g}(t)\right|.$$

The weighted space $C^n_{\gamma,\psi}[a,b]$ of functions $\mathfrak{g}$ on $(a,b]$ is defined by

$$C^n_{\gamma,\psi}[a,b] = \left\{\mathfrak{g}: J\to R : \mathfrak{g}(t)\in C^{n-1}[a,b]; \mathfrak{g}(t)\in C_{\gamma,\psi}[a,b]\right\}, 0\le\gamma<1,$$

with the norm

$$\|\mathfrak{g}\|_{C^n_{\gamma,\psi}[a,b]} = \sum_{k=0}^{n-1}\left\|\mathfrak{g}^k\right\|_{C[a,b]} + \|\mathfrak{g}^n\|_{C_{\gamma,\psi}[a,b]}.$$

For $n=0$, we have, $C^0_{\gamma,\psi}[a,b] = C_{\gamma,\psi}[a,b]$.

Definition 2.1 *The left-sided fractional integral of a function $\mathfrak{g}$ with respect to another function ψ on [a, b] is defined by*

$$\left(\mathfrak{I}^{\alpha;\psi}\right)\mathfrak{g}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,\mathfrak{g}(s)ds, \; t>a. \tag{2.1}$$

Definition 2.2 *Let $\psi^{'}(x) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0$, $n \in N$. The Riemann-Liouville fractional derivative of a function $\mathfrak{g}$ with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by*

$$\left(\mathfrak{D}^{\alpha;\psi}\mathfrak{g}\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^{n}\int_{a}^{t}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1}\mathfrak{g}(s)ds, \tag{2.2}$$

where $n = [\alpha] + 1$.

Definition 2.3 *Let $\alpha > 0$, $n \in N$, $I = [a,b]$ is the interval $(-\infty \leq a < t < b < \infty)$, $\mathfrak{g}, \psi \in C^{n}([a,b], R)$ two functions such that ψ is increasing and $\psi^{'}(x) \neq 0$, for all $x \in I$. The left ψ-Caputo derivative of $\mathfrak{g}$ of order α is given by*

$$\left(\mathfrak{D}^{\alpha;\psi}\mathfrak{g}\right)(t) = \mathfrak{I}^{n-\alpha;\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^{n}\mathfrak{g}(t) \tag{2.3}$$

where $n = [\alpha] + 1$ for $\alpha \notin N$ and $\alpha = n$ for $\alpha \in N$.

Definition 2.4 *The ψ-Hilfer fractional derivative of function $\mathfrak{g}$ oforder α is given by,*

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g}(t). \tag{2.4}$$

The ψ-Hilfer fractional derivative as above defined, can be written in the following

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\gamma-\alpha;\psi}\mathfrak{D}^{\gamma;\psi}\mathfrak{g}(t).$$

Lemma 2.5 *Let $\alpha, \beta > 0$, Then we have the following semigroup property*

$$(\mathfrak{I}^{\alpha;\psi}\mathfrak{I}^{\beta;\psi}\mathfrak{g})(t) = (\mathfrak{I}^{\alpha+\beta;\psi}\mathfrak{g})(t),$$

and

$$(\mathfrak{D}^{\alpha;\psi}\mathfrak{I}^{\alpha;\psi}\mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.6 *Let $\alpha, \beta > 0$, and*

1. *If $\mathfrak{g}(t) = (\psi(t) - \psi(a))^{\beta-1}$, then*

$$\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\psi(t)-\psi(a)\right)^{\alpha+\beta-1}.$$

2. *If $g(t) = (\psi(t) - \psi(a))^{\alpha-1}$, then*

$$\mathfrak{D}^{\alpha;\psi}g(t) = 0.$$

Lemma 2.7 *Let $0 < \alpha < 1$. If $\mathfrak{g} \in C^{n}[a,b]$, then*

$$\left(\mathfrak{I}^{\alpha;\psi}D^{\alpha;\psi}\mathfrak{g}\right)(t) = \mathfrak{g}(t) - \frac{\left(\mathfrak{I}_{a+}^{1-\alpha;\psi}\mathfrak{g}\right)(a)}{\Gamma(\alpha)}\left(\psi(t)-\psi(a)\right)^{\alpha-1},$$

for all $t \in (a,b]$.

Lemma 2.8 *Let $n-1 \leq \gamma < n$ and $f \in C_{\gamma}[a,b]$. Then*

$$\left(\mathfrak{I}^{\alpha;\psi}\mathfrak{g}\right)(a) = \lim_{t\to a^{+}}\left(\mathfrak{I}^{\alpha;\psi}\right)\mathfrak{g}(t) = 0.$$

Lemma 2.9 *[22] Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $a \le t < b$ (some $b \le \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \le t < b$, such that $g(t) \le K$ for some constant K. Further let $\mathfrak{u}(t)$ be a nonnegative locally integrable on $a \le t < b$ function with*

$$|\mathfrak{u}(t)| \le a(t) + g(t) \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1}\, \mathfrak{u}(s)ds, \quad t \in J$$

with some $\alpha > 0$. Then

$$|\mathfrak{u}(t)| \le a(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s)\,(\psi(t) - \psi(s))^{n\alpha-1} \right] \mathfrak{u}(s)ds, \quad a \le t < b.$$

Lemma 2.10 *A function $\mathfrak{u}$ is the solution of fractional initial value problem*

$$\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t) = f(t), \\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(a) = \mathfrak{u}_a, \end{cases}$$

if and only if $\mathfrak{u}$ satisfies the following Volterra integral equation

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\gamma)}\,(\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1} f(s)ds.$$

Theorem 2.11 *(Schauder fixed point theorem [5]) Let B be closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has atleast one fixed point in B.*

3 Existence results

Now we give our existence result for problem (1.1)-(1.2). Before starting and proving this result, we list the following hypotheses:

(H1) There exist constants $K > 0$ and $L > 0$ such that

$$|\mathfrak{g}(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w}) - \mathfrak{g}(t, \overline{\mathfrak{u}}, \overline{\mathfrak{v}}, \overline{\mathfrak{w}})| \le K\,(|\mathfrak{u} - \overline{\mathfrak{u}}| + |\mathfrak{v} - \overline{\mathfrak{v}}|) + L\,|\mathfrak{w} - \overline{\mathfrak{w}}|,$$

for any $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}}, \overline{\mathfrak{w}} \in R$, and $t \in J$.

(H2) There exist $l, m, n, p \in C(J, R)$ with $l^* = \sup_{t \in J} l(t) < 1$ such that

$$|\mathfrak{g}(t, \mathfrak{u}, \mathfrak{v}, \mathfrak{w})| \le l(t) + m(t)\,|\mathfrak{u}| + n(t)\,|\mathfrak{v}| + p(t)\,|\mathfrak{w}|,$$

for $t \in J$ and $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in R$.

(H3) There exists an increasing function $\varphi \in C(J, R)$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$\mathfrak{I}^{\alpha;\psi}\varphi(t) \le \lambda_\varphi \varphi(t).$$

Theorem 3.1 *Assume that (H1)-(H2) hold. Then the problem* (1.1)-(1.2) *has at least one solution.*

Consider the operator $\mathfrak{K} : C_{1-\gamma,\psi}(J, R) \to C_{1-\gamma,\psi}(J, R)$ defined by

$$(\mathfrak{K}\mathfrak{u})(t) = \frac{\mathfrak{u}_a}{\Gamma(\gamma)}\,(\psi(t) - \psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1}\, \mathfrak{g}(s, \mathfrak{u}(s), \mathfrak{u}(\kappa s), \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(\kappa s))ds. \tag{3.1}$$

It can be written as

$$(\mathfrak{K}\mathfrak{u})(t) = \frac{\mathfrak{u}_a}{\Gamma(\gamma)}\,(\psi(t) - \psi(a))^{\gamma-1} + \left[\mathfrak{I}^{\alpha;\psi} g_{\mathfrak{u}}(s)\right](t), \tag{3.2}$$

where $g_{\mathfrak{u}}(t) := \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t) = \mathfrak{g}(t,\mathfrak{u}(t),\mathfrak{u}(\kappa t),g_{\mathfrak{u}}(t))$.

Clearly, the fixed points of the operator $\mathfrak{K}$ are solutions of the problem (1.1).
For any $\mathfrak{u} \in C_{1-\gamma,\psi}(J,R)$ and each $t \in J$, we have

$$\left|(\psi(t)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t)\right| \leq \frac{|\mathfrak{u}_a|}{\Gamma(\gamma)} + \frac{(\psi(t)-\psi(a))^{1-\gamma}}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|g_{\mathfrak{u}}(s)|ds. \tag{3.3}$$

By (H2), for each $t \in J$, we have

$$\begin{aligned}|g_{\mathfrak{u}}(t)| &= |\mathfrak{g}(t,\mathfrak{u}(t),\mathfrak{u}(\kappa t),g_{\mathfrak{u}}(t))| \\ &\leq l(t)+m(t)|\mathfrak{u}(t)|+n(t)|\mathfrak{u}(\kappa t)|+p(t)|g_{\mathfrak{u}}(\kappa t)| \\ &\leq l^*+m^*|\mathfrak{u}(t)|+n^*|\mathfrak{u}(\kappa t)|+p^*|g_{\mathfrak{u}}(\kappa t)| \\ &\leq \frac{l^*+m^*|\mathfrak{u}(t)|+n^*|\mathfrak{u}(\kappa t)|}{1-p^*}. \end{aligned} \tag{3.4}$$

By replacing (3.4) in the inequality (3.3), we get

$$\begin{aligned}&\left|(\psi(t)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t)\right| \\ &\leq \frac{|\mathfrak{u}_a|}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left[\frac{l^*+m^*|\mathfrak{u}(s)|+n^*|\mathfrak{u}(\kappa s)|}{1-p^*}\right]ds \\ &\leq \frac{|\mathfrak{u}_a|}{\Gamma(\gamma)} + \frac{1}{(1-p^*)\Gamma(\alpha)}(\psi(t)-\psi(a))^{1-\gamma}\left(l^*\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}ds\right. \\ &\quad \left.+m^*\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{u}(s)|ds + n^*\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))|\mathfrak{u}(\kappa s)|ds\right) \\ &\leq \frac{|\mathfrak{u}_a|}{\Gamma(\gamma)} + \frac{l^*}{(1-p^*)\Gamma(\alpha+1)}(\psi(t)-\psi(a))^{\alpha-\gamma+1} + \frac{B(\gamma,\alpha)}{(1-p^*)\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha}(m^*+n^*)\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathfrak{K}\mathfrak{u}\|_{C_{1-\gamma}} &\leq \frac{|\mathfrak{u}_a|}{\Gamma(\gamma)} + \frac{l^*}{(1-p^*)\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-\gamma+1} \\ &\quad + \frac{B(\gamma,\alpha)}{(1-p^*)\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha}(m^*+n^*)\|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \\ &:= r^{'}.\end{aligned}$$

This proves that $\mathfrak{K}$ transforms the ball $\mathfrak{B}_{r'} := \left\{\mathfrak{u} \in C_{1-\gamma,\psi}(J,R) : \|\mathfrak{u}\| \leq r^{'}\right\}$ into itself. We shall show that the operator $\mathfrak{K} : \mathfrak{B}_{r'} \to \mathfrak{B}_{r'}$ satisfies all the conditions of Schauder fixed point theorem. The proof will be given in several steps.

Step 1: $\mathfrak{K} : \mathfrak{B}_{r'} \to \mathfrak{B}_{r'}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J,R)$. Then for each $t \in J$,

$$\begin{aligned}&\left|(\psi(t)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u}_n)(t)-(\psi(t)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t)\right| \\ &\leq \frac{1}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{1-\gamma}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|g_{\mathfrak{u}_n}(\cdot)-g_{\mathfrak{u}}(\cdot)\right|ds \\ &\leq \frac{1}{\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha}B(\gamma,\alpha)\left\|g_{\mathfrak{u}_n}(\cdot)-g_{\mathfrak{u}}(\cdot)\right\|_{C_{1-\gamma,\psi}}.\end{aligned}$$

Since $\mathfrak{u}_n \to \mathfrak{u}$ as $n \to \infty$ and $g_{\mathfrak{u}}$ is continuous (i.e., f is continuous), then by the Lebesgue dominated convergence theorem, we have

$$\|\mathfrak{K}\mathfrak{u}_n - \mathfrak{K}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: $\mathfrak{K}(\mathfrak{B}_{r'})$ is uniformly bounded.
This is clear since $\mathfrak{K}(\mathfrak{B}_{r'}) \subset \mathfrak{B}_{r'}$ and $\mathfrak{B}_{r'}$ is bounded.
Step 3: $\mathfrak{K}(\mathfrak{B}_{r'})$ is equicontinuous.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and let $\mathfrak{u} \in \mathfrak{B}_{r'}$. Thus, we have

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t_2)-(\psi(t_1)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t_1)\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}(\psi(t_2)-\psi(a))^{1-\gamma}\int_a^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}g_{\mathfrak{u}}(s)ds\right. \\
&\quad \left.-\frac{1}{\Gamma(\alpha)}(\psi(t_1)-\psi(a))^{1-\gamma}\int_a^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}g_{\mathfrak{u}}(s)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}(\psi(t_2)-\psi(a))^{1-\gamma}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\left[\frac{l^*+m^*|\mathfrak{u}(s)|+n^*|\mathfrak{u}(\kappa s)|}{1-p^*}\right]ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big[(\psi(t_2)-\psi(a))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1} \\
&\quad -(\psi(t_1)-\psi(a))^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\Big]\psi'(s)\left[\frac{l^*+m^*|\mathfrak{u}(s)|+n^*|\mathfrak{u}(\kappa s)|}{1-p^*}\right]ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t_2)-(\psi(t_1)-\psi(a))^{1-\gamma}(\mathfrak{K}\mathfrak{u})(t_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}(\psi(t_2)-\psi(a))^{1-\gamma}\left[\frac{l^*(\psi(t_2)-\psi(t_1))^{\alpha}}{\alpha(1-p^*)}+\frac{(m^*+n^*)B(\gamma,\alpha)}{1-p^*}(\psi(t_2)-\psi(t_1))^{\alpha+\gamma-1}\|\mathfrak{u}\|C_{1-\gamma,\psi}\right] \\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big[(\psi(t_2)-\psi(a))^{1-\gamma}(\psi(t_2)-\psi(s))^{\alpha-1} \\
&\quad -(\psi(t_1)-\psi(a))^{1-\gamma}(\psi(t_1)-\psi(s))^{\alpha-1}\Big]\psi'(s)\left[\frac{l^*+m^*|\mathfrak{u}(s)|+n^*|\mathfrak{u}(\kappa s)|}{1-p^*}\right]ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Step 1-3 together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{K}$ is continuous and compact. From an application of Schauder's theorem, we deduce that $\mathfrak{K}$ has a fixed point $\mathfrak{u}$ which is a solution of the problem (1.1)-(1.2).

Lemma 3.2 *Assume that (H1),(H2) hold. If*

$$\left(\frac{2K(\psi(b)-\psi(a))^{\alpha}}{(1-L)\Gamma(\alpha)}B(\gamma,\alpha)\right) < 1, \tag{3.5}$$

then the problem (1.1)-(1.2) *has a unique solution.*

4 Stability analysis

In this section, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.1)-(1.2).

Definition 4.1 *The equation* (1.1) *is Ulam-Hyers stable if there exists a real number* $C_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $\mathfrak{v} \in C_{1-\gamma,\psi}(J,R)$ *of the inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t)-\mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \leq \epsilon, \quad t \in J,$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J,R)$ *of equation* (1.1) *with*

$$|\mathfrak{v}(t)-\mathfrak{u}(t)| \leq C_f\epsilon, \quad t \in J.$$

Definition 4.2 *The equation* (1.1) *is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0,\infty),[0,\infty))$, $\psi_f(0) = 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\psi}(J,R)$ of the inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \epsilon, \quad t \in J,$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J,R)$ of equation (1.1) *with*

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le \psi_f \epsilon, \quad t \in J.$$

Definition 4.3 *The equation* (1.1) *is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma}(J,R)$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{v} \in C_{1-\gamma,\psi}(J,R)$ of the inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \epsilon\varphi(t), \quad t \in J, \tag{4.1}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J,R)$ of equation (1.1) *with*

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 4.4 *The equation* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma}(J,R)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\psi}(J,R)$ of the inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \varphi(t), \quad t \in J, \tag{4.2}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J,R)$ of equation (1.1) *with*

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \le C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5 *A function $\mathfrak{v} \in C_{1-\gamma,\psi}(J,R)$ is a solution of the inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma,\psi}(J,R)$ (which depends on solution $\mathfrak{u}$) such that

1. $|g(t)| \le \epsilon, \quad \forall\ t \in J.$
2. $\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) = \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t)) + g(t), \quad t \in J.$

One can have similar remarks for the inequalities (4.1) and (4.2).

Remark 4.6 *It is clear that:*

1. *Definition 4.1$\Rightarrow$ Definition 4.2.*
2. *Definition 4.3$\Rightarrow$ Definition 4.4.*

Remark 4.7 *A solution of the ψ-Hilfer type nonlinear neutral pantograph inequality*

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \epsilon, \quad t \in J,$$

is called an fractional ϵ-solution of the problem (1.1).

Theorem 4.8 *Assume that (H1),(H3) and* (3.5) *hold. Then the problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Let $\epsilon > 0$ and let $\mathfrak{v}$ be a function which satisfies the inequality:

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t))\right| \le \epsilon, \quad \text{for any } t \in J, \tag{4.3}$$

and $\mathfrak{u}$ is the unique solution of the following nonlinear neutral pantograph differential equation

$$\begin{aligned}
&\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) = \mathfrak{g}(t,\mathfrak{v}(t),\mathfrak{v}(\kappa t),\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(\kappa t)),\ t \in J,\\
&\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(a) = \mathfrak{I}^{1-\gamma;\psi}\mathfrak{v}(a) = \mathfrak{u}_a,
\end{aligned}$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.
Using Lemma 2.10, we obtain

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} + \left[\mathfrak{I}^{\alpha;\psi} g_{\mathfrak{u}}\right](t).$$

By integration of the inequality (4.3) and using Remark 4.5, we obtain

$$\left| \mathfrak{v}(t) - \frac{\mathfrak{u}_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} g_{\mathfrak{v}}(s) ds \right| \leq \frac{\epsilon}{\Gamma(\alpha+1)} (\psi(b) - \psi(a))^{\alpha}.$$

We have

$$\begin{aligned}
&|\mathfrak{v}(t) - \mathfrak{u}(t)| \\
&\leq \left| \mathfrak{v}(t) - \frac{\mathfrak{u}_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} g_{\mathfrak{v}}(s) ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} (g_{\mathfrak{v}}(s) - g_{\mathfrak{u}}(s)) ds \right| \\
&\leq \frac{\epsilon (\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} |g_{\mathfrak{v}}(s) - g_{\mathfrak{u}}(s)| ds \\
&\leq \frac{\epsilon (\psi(t) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{2K}{(1-L)\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{v}(s) - \mathfrak{u}(s)| ds,
\end{aligned}$$

and to apply Lemma 2.9, we obtain

$$\begin{aligned}
|\mathfrak{v}(t) - \mathfrak{u}(t)| &\leq \frac{1}{\Gamma(\alpha+1)} (\psi(b) - \psi(a))^{\alpha} \left(1 + \frac{\nu 2K}{(1-L)\Gamma(\alpha+1)} (\psi(b) - \psi(a))^{\alpha} \right) \epsilon \\
&:= C_f \epsilon.
\end{aligned}$$

Where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = C_f \epsilon$; $\psi(0) = 0$, then the problem (1.1) is generalized Ulam-Hyers stable.

Theorem 4.9 *Assume that (H1), (H3) and* (3.5) *hold. Then, the problem* (1.1)-(1.2) *is Ulam-Hyers-Rassias stable.*

Let $\mathfrak{v} \in C_{1-\gamma}(J, R)$ be solution of the inequality

$$\left| \mathfrak{D}^{\alpha,\beta;\psi} \mathfrak{v}(t) - \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), \mathfrak{D}^{\alpha,\beta;\psi} \mathfrak{v}(\kappa t)) \right| \leq \epsilon \varphi(t), \quad t \in J,\ \epsilon > 0, \tag{4.4}$$

and let $\mathfrak{u} \in C_{1-\gamma}(J, R)$ the unique solution of the following nonlinear neutral pantograph differential equation

$$\begin{aligned}
&\mathfrak{D}^{\alpha,\beta;\psi} \mathfrak{u}(t) = \mathfrak{g}(t, \mathfrak{v}(t), \mathfrak{v}(\kappa t), \mathfrak{D}^{\alpha,\beta;\psi} \mathfrak{v}(\kappa t)), \quad t \in J, \\
&\mathfrak{I}^{1-\gamma;\psi} \mathfrak{u}(a) = \mathfrak{I}^{1-\gamma;\psi} \mathfrak{v}(a) = \mathfrak{u}_a,
\end{aligned}$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.
Using Lemma 2.10, we get

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\gamma)} (\psi(b) - \psi(a))^{\gamma-1} + \left[\mathfrak{I}^{\alpha;\psi} g_{\mathfrak{u}}\right](t).$$

By integration of the inequality (4.4), we get

$$\left| \mathfrak{v}(t) - \frac{\mathfrak{u}_a}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) (\psi(t) - \psi(s))^{\alpha-1} g_{\mathfrak{v}}(s) ds \right| \leq \epsilon \lambda_{\varphi} \varphi(t). \tag{4.5}$$

On the other hand, we have

$$
\begin{aligned}
&|\mathfrak{v}(t)-\mathfrak{u}(t)| \\
&\leq \left|\mathfrak{v}(t)-\frac{\mathfrak{u}_a}{\Gamma(\gamma)}(\psi(b)-\psi(a))^{\gamma-1}-\frac{1}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}g_{\mathfrak{v}}(s)ds\right| \\
&+\frac{2K}{(1-L)\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{v}(s)-\mathfrak{u}(s)|ds \\
&\leq \epsilon\lambda_\varphi\varphi(t)+\frac{2K}{(1-L)\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{v}(s)-\mathfrak{u}(s)|ds.
\end{aligned}
$$

By applying Lemma 2.9, we get

$$|\mathfrak{v}(t)-\mathfrak{u}(t)| \leq \epsilon\lambda_\varphi\varphi(t)+\frac{2K\nu_1}{1-L}\epsilon\lambda_\varphi^2\varphi(t).$$

Then for any $t\in J$, and by (H3), we have

$$
\begin{aligned}
|\mathfrak{v}(t)-\mathfrak{u}(t)| &\leq \left[\left(1+\frac{2K\nu_1\lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\epsilon\varphi(t) \\
&= C_f\epsilon\varphi(t),
\end{aligned}
$$

where $\nu_1=\nu_1(\alpha)$ is a constant. which completes the proof of Theorem 4.9.

5 An example

Example 5.1 *Consider the following ψ-Hilfer type nonlinear neutral pantograph problem*

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t)=\frac{e^{-t}}{9+e^t}\left(\mathfrak{u}(t)+\mathfrak{u}\left(\frac{t}{2}\right)+\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}\left(\frac{t}{2}\right)\right),\quad t\in J:=[0,1], \tag{5.1}$$

$$\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(0)=0,\quad \gamma=\alpha+\beta-\alpha\beta. \tag{5.2}$$

Where $0<\kappa<1$, $a=0$, $b=1$, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $\kappa=\frac{1}{2}$, and choose $\gamma=\frac{2}{3}$.
Set

$$f(t,\mathfrak{u},\mathfrak{v},\mathfrak{w})=\frac{e^{-t}}{9+e^t}(\mathfrak{u}+\mathfrak{v}+\mathfrak{w}),\quad \text{for any } \mathfrak{u},\mathfrak{v},\mathfrak{w}\in R.\ t\in J.$$

Clearly, the function f satisfies the hypotheses of Theorem 3.1.
For any $\mathfrak{u},\mathfrak{v},\mathfrak{w},\overline{\mathfrak{u}},\overline{\mathfrak{v}},\overline{\mathfrak{w}}\in R$ and $t\in J$.

$$|f(t,\mathfrak{u},\mathfrak{v},\mathfrak{w})-f(t,\overline{\mathfrak{u}},\overline{\mathfrak{v}},\overline{\mathfrak{w}})|\leq\frac{1}{10}(|\mathfrak{u}-\overline{\mathfrak{u}}|+|\mathfrak{v}-\overline{\mathfrak{v}}|)+\frac{1}{10}|\mathfrak{w}-\overline{\mathfrak{w}}|.$$

Hence the hypothesis (H1) is satisfied with $K=L=\frac{1}{10}$.
Thus condition from (3.5)

$$\left(\frac{2K}{(1-L)\Gamma(\alpha)}(\psi(b)-\psi(a))^\alpha B(\gamma,\alpha)\right)=0.1622<1,$$

It follows from Lemma 3.2 that the problem (5.1)-(5.2) has a unique solution.

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