A general study on Langevin equations of arbitrary order

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Abstract

In this paper, the broad study depends on Langevin differential equations (LDE) of arbitrary order. The fractional order is in terms of ψ-Hilfer fractional operator. This work reveals the dynamical behaviour such as existence, uniqueness and stability solutions for LDE involving ψ-Hilfer fractional erivative (HFD). Thus the fractional LDE with boundary condition, impulsive effect and nonlocal conditions are taken in account to prove the results.

Keywords: Langevin differential equations, Fractional calculus, Existence, Stability.

1. Introduction

The physical phenomena in fluctuating environments are described using Langevin equation. The Langevin equation is a powerful tool for the study of dynamical properties of many interesting systems in physics, chemistry and engineering. The generalized Langevin equation was introduced by Kubo in 1966. Since then the generalized LDE has become a searing research topic. The literature on LDE is huge; dynamical analysis and important results can be seen in [1, 2, 3, 7, 9].

The fractional derivatives make the fractional-order models more realistic than the classical integer-order model. Since the development of fractional calculus in various fields such as thermodynamics, biophysics, aerodynamics, viscoelasticity, capacitor theory, etc., there was been an intensive development in fractional derivative with singular and non-singular kernals. The Riemann-Liouville, Caputo, Hadamard, Hilfer, etc., are just a few fractional derivatives. Later HFD is fused with ψ-fractional derivative with kernel of function and pulled out a new fractional derivative known as ψ-HFD. The ψ-HFD integrate numerous fractional derivative with their properties are discussed in [10]. The dynamical behaviour and development of differential equation with different fractional derivatives, we refer to [5, 6, 8, 11, 12, 13].

Motivated by the works mentioned, here LDE with ψ-HFD involving boundary, impulsive and nonlocal conditions are studied. Stability criteria is an important aspect of differential equations of arbitrary order. The stable solution of fractional LDE is provided by utilizing the idea provided by Ulam. Thus, we discuss the generalized Ulam-Hyers-Rassias (g-UHR) stable for fractional LDE. For

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detailed study on fractional Ulam stability and its generalization one can refer to [14, 15, 16, 17].
First, we study the dynamical behaviour of LDE with ψ-HFD involving the boundary conditions is as follows

$$\begin{cases} \mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi}+\lambda\right)\mathfrak{u}(t)=\mathfrak{g}(t,\mathfrak{u}(t)), \quad t\in J:=[0,T], \\ a\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=0}+b\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=T}=c, \end{cases} \tag{1}$$

where $\mathscr{D}^{\alpha_1,\beta;\psi}$ and $\mathscr{D}^{\alpha_2,\beta;\psi}$ are two ψ-HFD of orders α_1 and α_2 and type β, $\mathscr{I}^{1-\gamma;\psi}$ is ψ-fractional integral of order $1-\gamma(\gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta)$ is any real number. Let $\mathfrak{g}:J\times R\rightarrow R$ is given continuous function.
Next, we study the impulsive LDE with ψ-HFD involving is given by

$$\begin{cases} \mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi}+\lambda\right)\mathfrak{u}(t)=\mathfrak{g}(t,\mathfrak{u}(t)), \quad t\in J^{'}:=J\setminus\{t_1,...,t_m\}, \\ \Delta\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=t_k}=I_k\mathfrak{u}(t_k), \\ \mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=0}=\mathfrak{u}_0, \end{cases} \tag{2}$$

where $I_k:R\rightarrow R,$ and $0=t_0<t_1<...<t_m<t_{m+1}=T$, $\Delta\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=t_k}=\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t_k^+)-\mathscr{I}^{1-\gamma;\psi}\mathfrak{v}(t_k^-)$, $\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t_k^+)=\lim_{h\rightarrow0+}\mathfrak{u}(t_k+h)$ and $\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t_k^-)=\lim_{h\rightarrow0-}\mathfrak{u}(t_k+h)$ represent the right and left limits of $\mathfrak{u}(t)$ at $t=t_k$.
Finally, we study the existence, uniqueness and stability of nonlocal LDE with ψ-HFD of the form

$$\begin{cases} \mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi}+\lambda\right)\mathfrak{u}(t)=\mathfrak{g}(t,\mathfrak{u}(t)), \\ \mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)=\sum_{i=1}^{m}c_i\mathfrak{u}(\tau_i), \quad \tau_i\in J \end{cases} \tag{3}$$

where $\tau_i, i=0,1,...,m$ are prefixed points satisfying $0<\tau_1\leq...\leq\tau_m<T$ and c_i is real numbers. Here, nonlocal condition $\mathfrak{u}(0)=\sum_{i=1}^{m}c_i\mathfrak{u}(\tau_i)$ can be applied in physical problems yields better effect than the initial conditions $\mathfrak{u}(0)=\mathfrak{u}_0$.
The effort in this paper is systematic as follows: In Section 2, some preliminari definitions and lemmas that used throughout the paper is provided. In Section 3, we establish existence, uniqueness and stability for fractional LDE involving ψ-HFD. In Section 4, impulsive fractional LDE is analyzed. In Section 5, finally nonlocal fractional LDE is discussed.

2. Preliminaries

In this section, we consider the following spaces, definitions, results and theorems that are used in the paper.

Definition 2.1. *Consider the space C of continuous functions with norm*

$$\|\mathfrak{u}\|=\sup\{|\mathfrak{u}(t)|:t\in J\}.$$

The weighted space $C_{\gamma,\psi}$ of functions $\mathfrak{g}$ on J is defined by

$$C_{\gamma,\psi}=\{\mathfrak{g}:J\rightarrow R:(\psi(t)-\psi(0))^{\gamma}\mathfrak{g}(t)\in C[a,b]\},0\leq\gamma<1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\gamma,\psi}}=\|(\psi(t)-\psi(0))^{\gamma}\mathfrak{g}(t)\|_{C}=\max_{t\in J}|(\psi(t)-\psi(0))^{\gamma}\mathfrak{g}(t)|.$$

Definition 2.2. *Consider the piecewise continuous spaces on J is given by*

$$PC = \left\{ \mathfrak{u} : J \to R : \mathfrak{u}(t) \in C(t_k, t_{k+1}], k = 0, ..., m; \ there \ \ exists \ \mathfrak{u}(t_k^+) and \mathfrak{u}(t_k^-) \right\}.$$

Now, we consider the weighted space PC_γ.

$$PC_{\gamma,\psi} = \left\{ \mathfrak{u} : (\psi(t) - \psi(t_k))^\gamma \, \mathfrak{u}|_{[t_k, t_{k+1}]} \in C[t_k, t_{k+1}], k = 0, ..., m \ where \ 0 \leq \gamma < 1 \right\}.$$

Obviously, which is a Banach space with norm

$$\|\mathfrak{u}\|_{PC_{\gamma,\psi}} = \sup_{(t_k, t_{k+1}]} \left| (\psi(t) - \psi(t_k))^\gamma \, \mathfrak{u}(t) \right|.$$

Definition 2.3. *The left-sided fractional integral of a function $\mathfrak{g}$ with respect to another function ψ on [a, b] is defined by*

$$\left(\mathscr{I}^{\alpha;\psi} \mathfrak{g} \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t) - \psi(s) \right)^{\alpha - 1} \mathfrak{g}(s) ds, \ t > a. \tag{4}$$

Definition 2.4. *Let $\psi^{'}(\mathfrak{u}) \neq 0$ $(-\infty \leq t < b < \infty)$ and $\alpha > 0$, $n \in N$. The Riemann-Liouville fractional derivative of a function $\mathfrak{g}$ with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by*

$$\left(\mathscr{D}^{\alpha;\psi} \mathfrak{g} \right)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi^{'}(s) \left(\psi(t) - \psi(s) \right)^{n - \alpha - 1} \mathfrak{g}(s) ds, \tag{5}$$

where $n = [\alpha] + 1$.

Definition 2.5. *Let $\alpha > 0$, $n \in N$, J is the interval $(-\infty \leq t < T < \infty)$, $f, \psi \in C^n(J, R)$ two functions such that ψ is increasing and $\psi^{'}(\mathfrak{u}) \neq 0$, for all $\mathfrak{u} \in J$. The left ψ-Caputo derivative of $\mathfrak{g}$ of order α is given by*

$$\left(\mathscr{D}^{\alpha;\psi} \mathfrak{g} \right)(t) = \mathscr{I}^{n - \alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathfrak{g}(t) \tag{6}$$

Definition 2.6. *The ψ-Hilfer fractional derivative of function $\mathfrak{g}$ oforder α is given by,*

$$\mathscr{D}^{\alpha,\beta;\psi} \mathfrak{g}(t) = \mathscr{I}^{\beta(1-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) \mathscr{I}^{(1-\beta)(1-\alpha);\psi} \mathfrak{g}(t). \tag{7}$$

The ψ-Hilfer fractional derivative as above defined, can be written in the following

$$\mathscr{D}^{\alpha,\beta;\psi} \mathfrak{g}(t) = \mathscr{I}^{\gamma - \alpha;\psi} \mathscr{D}^{\gamma;\psi} \mathfrak{g}(t).$$

Lemma 2.7. *Let $\alpha, \beta > 0$, Then we have the following semigroup property*

$$(\mathscr{I}^{\alpha;\psi} \mathscr{I}^{\beta;\psi} \mathfrak{g})(t) = (\mathscr{I}^{\alpha+\beta;\psi} \mathfrak{g})(t),$$

and

$$(\mathscr{D}^{\alpha;\psi} \mathscr{I}^{\alpha;\psi} \mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.8. *Let $\alpha, \beta > 0$, and*

1. *If* $\mathfrak{g}(t) = (\psi(t) - \psi(a))^{\beta - 1}$, *then*

$$\mathscr{I}^{\alpha;\psi}\mathfrak{g}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1}.$$

2. *If* $\overline{\mathfrak{g}}(t) = (\psi(t) - \psi(a))^{\alpha - 1}$, *then*

$$\mathscr{D}^{\alpha;\psi}\overline{\mathfrak{g}}(t) = 0.$$

Lemma 2.9. *Let* $0 < \alpha < 1$. *If* $\mathfrak{g} \in C_n$, *then*

$$\left(\mathscr{I}^{\alpha;\psi}\mathscr{D}^{\alpha;\psi}\mathfrak{g}\right)(t) = \mathfrak{g}(t) - \frac{\left(\mathscr{I}_{a+}^{1-\alpha;\psi}\mathfrak{g}\right)(a)}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\alpha-1}.$$

Lemma 2.10. *Suppose* $\alpha > 0$, $a(t)$ *is a nonnegative function locally integrable on* $0 \leq t < T$ *(some* $T \leq \infty$*), and let* $g(t)$ *be a nonnegative, nondecreasing continuous function defined on* $0 \leq t < T$, *such that* $g(t) \leq K$ *for some constant* K. *Further let* $\mathfrak{u}(t)$ *be a nonnegative locally integrable on* $0 \leq t < T$ *function with*

$$|\mathfrak{u}(t)| \leq a(t) + g(t)\int_a^t \psi^{'}(s)\,(\psi(t) - \psi(s))^{\alpha-1}\,\mathfrak{u}(s)ds, \quad t \in J$$

with some $\alpha > 0$. *Then*

$$|\mathfrak{u}(t)| \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\,(\psi(t) - \psi(s))^{n\alpha-1}\right]\mathfrak{u}(s)ds.$$

Theorem 2.11. *[4](Schauder fixed point theorem) Let* B *be closed, convex and nonempty subset of a Banach space* E. *Let* $N : B \to B$ *be a continuous mapping such that* $N(B)$ *is a relatively compact subset of* E. *Then* N *has atleast one fixed point in* B.

Theorem 2.12. *[4](Schaefer's Fixed Point Theorem) Let* R *be a Banach space and let* $\mathfrak{P} : R \to R$ *be completely continuous operator. If the set* $\{\mathfrak{h} \in R : \mathfrak{h} = \delta\mathfrak{P}\mathfrak{h} \text{ for some } \delta \in (0,1)\}$ *is bounded, then* $\mathfrak{P}$ *has a fixed point.*

Theorem 2.13. *[4](Krasnoselskii's fixed point theorem) Let* X *be a Banach space, let* Ω *be a bounded closed convex subset of* X *and let* T_1, T_2 *be mapping from* Ω *into* X *such that* $T_1x + T_2y, \in \Omega$ *for every pair* $x, y \in \Omega$. *If* T_1 *is contraction and* T_2 *is completely continuous, then the equation* $T_1x + T_2x = x$ *has a solution on* Ω.

Theorem 2.14. *[4](Banach Fixed Point Theorem) Suppose* Q *be a non-empty closed subset of a Banach space* E. *Then any contraction mapping* $\mathfrak{P}$ *from* Q *into itself has a unique fixed point.*

Theorem 2.15. *(Arzela-Ascoli theorem) Let* X *be a Banach space, let* Ω *be a bounded closed convex subset of* X *and let* T_1, T_2 *be mapping from* Ω *into* X *such that* $T_1x + T_2y, \in \Omega$ *for every pair* $x, y \in \Omega$. *If* T_1 *is contraction and* T_2 *is completely continuous, then the equation* $T_1x + T_2x = x$ *has a solution on* Ω.

Before stating and proving the main results, we introduce the following hypotheses:

(H1) Let $\mathfrak{g}: J \times R \to R$ be a continuous function such that, there exists a positive constant $\ell_{\mathfrak{g}} > 0$ such that

$$|\mathfrak{g}(\cdot,\mathfrak{u}(\cdot)) - \mathfrak{g}(\cdot,\overline{\mathfrak{u}}(\cdot))| \leq \ell_{\mathfrak{g}} |\mathfrak{u} - \overline{\mathfrak{u}}|, \quad \text{for all } \mathfrak{u}, \overline{\mathfrak{u}} \in R.$$

(H2) The exist $M > 0$ and $N \geq 0$ such that we have

$$|\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))| \leq M |\mathfrak{u}| + N.$$

(H3) Suppose that there exists $\lambda_\varphi > 0$ such that

$$\mathscr{I}^{\alpha_1+\alpha_2;\psi}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

(H4) Let $I : J \to R$ be a continuous function such that, there exists a positive constant ℓ_I, we have

$$|I_k \mathfrak{u}(t_k) - I_k \mathfrak{v}(t_k)| \leq \ell_I |\mathfrak{u} - \mathfrak{v}|.$$

(H5) Set $\mathfrak{g}(t,0) = L_1$, and $I_k(0) = L_2$.

(H6)

$$\begin{aligned}\rho := \Bigg[&\lambda \frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)} \left(|T| \sum_{i=1}^{m} c_i \left(\psi(\tau_i) - \psi(0)\right)^{\alpha_2+\gamma-1} + \left(\psi(t) - \psi(0)\right)^{\alpha_2} \right) \\ &+ \ell_{\mathfrak{g}} \frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(|T| \sum_{i=1}^{m} c_i \left(\psi(\tau_i) - \psi(0)\right)^{\alpha_1+\alpha_2+\gamma-1} + \left(\psi(t) - \psi(0)\right)^{\alpha_1+\alpha_2} \right) \Bigg] < 1.\end{aligned}$$

3. Langevin equation with boundary condition

Lemma 3.1. *A function* $\mathfrak{u}$ *is the solution of* (1) *if and only if* $\mathfrak{u}$ *satisfies the following integral equation:*

$$\begin{aligned}\mathfrak{u}(t) =& \frac{\left(\psi(t) - \psi(0)\right)^{\gamma-1}}{(a+b)\Gamma(\alpha)} \left(c + b\lambda \mathscr{I}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(T) - b\mathscr{I}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T)) \right) \\ &- \lambda \mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t) + \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)).\end{aligned}$$

Theorem 3.2. *(Existence) Assume that [H1] and [H2] are satisfied. Then, Eq.*(1) *has at least one solution.*

Proof. Consider the operator $\mathscr{N} : C_{1-\gamma,\psi} \to C_{1-\gamma,\psi}$, it is well defined and given by

$$(\mathscr{N}\mathfrak{u})(t) = \begin{cases} \frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} \left(c + b\lambda \mathscr{I}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(T) - b\mathscr{I}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T)) \right) \\ -\lambda \mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t) + \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)). \end{cases} \tag{8}$$

Befine $B_r = \left\{ \mathfrak{u} \in C_{1-\gamma,\psi} : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq r \right\}$.
To apply Theorem 2.11 we derive the following steps.
Step 1. We show that $\mathscr{N} B_r \subset B_r$.

$$\left| \left(\mathscr{N}\mathfrak{u}\right)(t) \left(\psi(t) - \psi(0)\right)^{1-\gamma} \right|$$

$$
\begin{aligned}
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b\lambda \mathscr{I}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\left|u(T)\right|}{(a+b)\Gamma(\gamma)} + \frac{b\mathscr{I}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\left|\mathfrak{g}(T,u(T))\right|}{(a+b)\Gamma(\gamma)} \\
&\quad + \lambda\left(\psi(t)-\psi(0)\right)^{1-\gamma}\mathscr{I}^{\alpha_2;\psi}\left|u(t)\right| + \left(\psi(t)-\psi(0)\right)^{1-\gamma}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\left|\mathfrak{g}(t,u(t))\right| \\
&\leq \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b\lambda}{(a+b)\Gamma(\gamma)}\frac{B(\gamma,(1-\alpha_1)(1-\beta)+\alpha_2\beta)}{\Gamma((1-\alpha_1)(1-\beta)+\alpha_2\beta)}\left(\psi(T)-\psi(0)\right)^{\alpha_2}\|u\|_{C_{1-\gamma,\psi}} \\
&\quad + \frac{bN}{(a+b)\Gamma(\gamma)}\frac{\left(\psi(T)-\psi(0)\right)^{1+(\alpha_1+\alpha_2-1)\beta}}{\Gamma(2+(\alpha_1+\alpha_2-1)\beta)} \\
&\quad + \frac{bM}{(a+b)\Gamma(\gamma)}\frac{B(\gamma,1+(\alpha_1+\alpha_2-1)\beta)}{\Gamma(1+(\alpha_1+\alpha_2-1)\beta)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}\|u\|_{C_{1-\gamma,\psi}} \\
&\quad + \lambda\frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_2}\|u\|_{C_{1-\gamma,\psi}} + \frac{N}{\Gamma(\alpha_1+\alpha_2+1)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2+1-\gamma} \\
&\quad + \frac{MB(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}\|u\|_{C_{1-\gamma,\psi}} \\
&\leq r.
\end{aligned}
$$

Step 2. The operator $\mathscr{N}$ is continuous.

$$
\begin{aligned}
&\left|\left(\mathscr{N}\mathfrak{u}_n(t)-\mathscr{N}\mathfrak{u}(t)\right)\left(\psi(t)-\psi(0)\right)^{1-\gamma}\right| \\
&\leq \frac{b}{(a+b)\Gamma(\gamma)}\left(\lambda\mathscr{I}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\left|u_n(T)-u(T)\right| + \mathscr{I}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\left|\mathfrak{g}(T,u_n(T))-\mathfrak{g}(T,u(T))\right|\right) \\
&\quad + \left(\psi(t)-\psi(0)\right)^{1-\gamma}\lambda\mathscr{I}^{\alpha_2;\psi}\left|u_n(t)-u(t)\right| + \left(\psi(t)-\psi(0)\right)^{1-\gamma}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\left|\mathfrak{g}(t,u_n(t))-\mathfrak{g}(t,u(t))\right|.
\end{aligned}
$$

It implies that operator $\mathscr{N}$ is continuous.

Step 3. Operator $\mathscr{N}B_r$ is relatively compact.

$$
\begin{aligned}
&\left|\left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\mathscr{N}\mathfrak{u}(t_1) - \left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\mathscr{N}\mathfrak{u}(t_2)\right| \\
&\leq \Big|-\lambda\left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\lambda\mathscr{I}^{\alpha_2;\psi}u(t_1) + \left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t_1,u(t_1)) \\
&\quad +\lambda\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\lambda\mathscr{I}^{\alpha_2;\psi}u(t_2) - \left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t_2,u(t_2))\Big|.
\end{aligned}
$$

tending to zero as $t_1 \to t_2$. That is $\mathscr{N}$ is equicontinuous. Therefore by Theorem 2.15 and Theorem 2.11 we conclude the proof. □

Theorem 3.3. *If hypothesis (H1) and*

$$
\begin{aligned}
&\Bigg(\frac{b}{(a+b)\Gamma(\gamma)}\Bigg(\lambda\frac{B(\gamma,(1-\alpha_1)(1-\beta)+\alpha_2\beta)}{\Gamma((1-\alpha_1)(1-\beta)+\alpha_2\beta)}\left(\psi(T)-\psi(0)\right)^{\alpha_2} \\
&\quad +\ell\frac{B(\gamma,1+(\alpha_1+\alpha_2-1)\beta)}{\Gamma(1+(\alpha_1+\alpha_2-1)\beta)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}\Bigg) \\
&\quad +\lambda\frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_2} + \frac{\ell B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}\Bigg) < 1.
\end{aligned}
$$

hold. Then, Eq. (1) *has a unique solution.*

Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$
\left|\mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi}+\lambda\right)\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varphi(t). \tag{9}
$$

Definition 3.4. *The Eq.* (1) *is g-UHR stable with respect to* $\varphi \in C_{1-\gamma,\psi}$ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each solution* $\mathfrak{v} \in C_{1-\gamma,\psi}$ *of the inequality* (9) *there exists a solution* $\mathfrak{u} \in C_{1-\gamma,\psi}$ *of Eq.* (1) *with*

$$|\mathfrak{v}(t) - \mathfrak{u}(t)| \leq C_{f,\varphi}\varphi(t).$$

Theorem 3.5. *The hypotheses [H1] and [H3] are satisfied. Then Eq.*(1) *is g-UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (9) and by Theorem 3.3, $\mathfrak{u}$ is a unique solution of the problem

$$\mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi} + \lambda\right)\mathfrak{u}(t) = \mathfrak{g}(t,\mathfrak{u}(t)),$$
$$a\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=0} + b\mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=T} = c = a\mathscr{I}^{1-\gamma;\psi}\mathfrak{v}(t)|_{t=0} + b\mathscr{I}^{1-\gamma;\psi}\mathfrak{v}(t)|_{t=T},$$

Then, we have

$$\begin{aligned}\mathfrak{u}(t) =& \frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)}\left(c + b\lambda\mathscr{I}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(T) - b\mathscr{I}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T))\right)\\ &- \lambda\mathscr{I}^{\alpha_2;\psi}u(t) + \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,u(t))\\ =& A_{\mathfrak{u}} - \lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t) + \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)).\end{aligned}$$

Thus, we have $A_{\mathfrak{u}} = A_{\mathfrak{v}}$. By differentiating inequality (9), we have

$$\left|\mathfrak{v}(t) - A_{\mathfrak{u}} + \lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{v}(t) - \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \left|\mathscr{I}^{\alpha_1+\alpha_2;\psi}\varphi(t)\right| \leq \lambda_\varphi\varphi(t).$$

Hence it follows that,

$$\begin{aligned}|\mathfrak{v}(t) - \mathfrak{u}(t)| \leq& \left|\mathfrak{v}(t) - A_{\mathfrak{u}} + \lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t) - \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\right|\\ \leq& \left|\mathfrak{v}(t) - A_{\mathfrak{u}} + \lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{v}(t) - \mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{v}(t))\right|\\ &+ \lambda\mathscr{I}^{\alpha_2;\psi}\left|\mathfrak{v}(t) - \mathfrak{u}(t)\right| + \mathscr{I}^{\alpha_1+\alpha_2;\psi}\left|\mathfrak{g}(t,\mathfrak{v}(t)) - \mathfrak{g}(t,\mathfrak{u}(t))\right|\\ \leq& \lambda_\varphi\varphi(t) + \lambda\mathscr{I}^{\alpha_2;\psi}\left|\mathfrak{v}(t) - \mathfrak{u}(t)\right| + \ell\mathscr{I}^{\alpha_1+\alpha_2;\psi}\left|\mathfrak{v}(t) - \mathfrak{u}(t)\right|\end{aligned}$$

By Lemma 2.9, there exists a constant $M^* > 0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|\mathfrak{v}(t) - \mathfrak{v}(t)| \leq M^*\varphi(t).$$

Thus, Eq.(1) is g-UHR. □

4. Langevin equation with impulsive effect

Lemma 4.1. *A function* $\mathfrak{u}$ *is the solution of* (2) *if and only if* $\mathfrak{u}$ *satisfies the following integral equation:*

$$\begin{aligned}\mathfrak{u}(t) =& \frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\Bigg[\mathfrak{u}_a + \sum_{0<t<t_k} I_k\mathfrak{u}(t_k) - \lambda\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(t_k)\\ &+ \sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{u}(t_k))\Bigg] - \lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\mathfrak{u}(t) + \mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\end{aligned}$$

Consider the operator $\mathscr{P} : PC_{1-\gamma,\psi} \to PC_{1-\gamma,\psi}$, it is well defined and given by

$$\mathfrak{u}(t) = \mathscr{P}\mathfrak{u}(t),$$

where

$$(\mathscr{P}\mathfrak{u})(t) = \begin{cases} \frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)} \Big[\mathfrak{u}_a + \sum_{0<t<t_k} I_k\mathfrak{u}(t_k) - \lambda \sum_{0<t<t_k} \mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(t_k) \\ + \sum_{0<t<t_k} \mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{u}(t_k))\Big] - \lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\mathfrak{u}(t) + \mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)) \end{cases} \tag{10}$$

Lemma 4.2. *The operator $\mathscr{P}$ is continuous.*

Proof. Let $\mathscr{P}_n$ be a sequence such that $\mathscr{P}_n \to \mathscr{P}$ in $PC_{1-\gamma,\psi}$. Then for each $t \in J$,

$$\begin{aligned} &\left|(\mathscr{P}\mathfrak{u}_n(t) - (\mathscr{P}\mathfrak{u})(t))\,(\psi(t)-\psi(t_k))^{1-\gamma}\right| \\ &\le \frac{1}{\Gamma(\gamma)}\Bigg[\sum_{0<t<t_k} |I_k\mathfrak{u}_n(t_k) - I_k\mathfrak{u}(t_k)| + \lambda \sum_{0<t<t_k} \mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}|\mathfrak{u}_n(t_k)-\mathfrak{u}(t_k)| \\ &\quad + \sum_{0<t<t_k} \mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}|\mathfrak{g}(t_k,\mathfrak{u}_n(t_k)) - \mathfrak{g}(t_k,\mathfrak{u}(t_k))|\Bigg] \\ &\quad + \lambda\,(\psi(t)-\psi(t_k))^{1-\gamma}\,\mathscr{I}_{t_k}^{\alpha_2;\psi}|\mathfrak{u}_n(t)-\mathfrak{u}(t)| \\ &\quad + (\psi(t)-\psi(t_k))^{1-\gamma}\,\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}|\mathfrak{g}(t,\mathfrak{u}_n(t)) - \mathfrak{g}(t,\mathfrak{u}(t))| \end{aligned}$$

since $\mathfrak{g}$ is continuous, then we have

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the continuity of $\mathscr{P}$. □

Lemma 4.3. *The operator $\mathscr{P}$ maps bounded sets into bounded sets in $PC_{1-\gamma,\psi}$.*

Proof. Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that $B_r = \left\{\mathfrak{u} \in PC_{1-\gamma,\psi} : \|(\mathscr{P}\mathfrak{u})(t)\|_{PC_{1-\gamma,\psi}} \le r\right\}$, we have $\|(\mathscr{P}\mathfrak{u})\|_{PC_{1-\gamma,\psi}} \le l$.

$$\begin{aligned} &\left|(\mathscr{P}\mathfrak{u})(t)\,(\psi(t)-\psi(t_k))^{1-\gamma}\right| \\ &\le \frac{1}{\Gamma(\gamma)}\Big[\mathfrak{u}_a + (\psi(T)-\psi(0))^{\gamma-1} m\ell_I \|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} + mL_2 \\ &\quad +\lambda m \frac{B(\gamma,(1-\alpha_1)(1-\beta)+\alpha_2\beta)}{\Gamma((1-\alpha_1)(1-\beta)+\alpha_2\beta)}(\psi(T)-\psi(0))^{\alpha_2}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\ &\quad +m\ell_{\mathfrak{g}}\frac{B(\gamma,1+(\alpha_1+\alpha_2-1)\beta)}{\Gamma(1+(\alpha_1+\alpha_2-1)\beta)}(\psi(T)-\psi(0))^{\alpha_1+\alpha_2}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\ &\quad +\frac{L_1 m\,(\psi(T)-\psi(0))^{1+(\alpha_1+\alpha_2-1)\beta}}{\Gamma(2+(\alpha_1+\alpha_2-1)\beta)}\Big] \\ &\quad +\frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}(\psi(T)-\psi(0))^{\alpha_2}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\ &\quad +\frac{\ell_{\mathfrak{g}} B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}(\psi(T)-\psi(0))^{\alpha_1+\alpha_2}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\ &\quad +\frac{L_1\,(\psi(T)-\psi(0))^{\alpha_1+\alpha_2-\gamma+1}}{\Gamma(\alpha_1+\alpha_2+1)} \end{aligned}$$

$$\leq l.$$

That is $\mathscr{P}$ is bounded. $\square$

Lemma 4.4. *The operator $\mathscr{P}$ maps bounded sets into equicontinuous set of $PC_{1-\gamma,\psi}$.*

Proof. Let $t_1, t_2 \in J, t_1 < t_2, B_r$ be a bounded set of $PC_{1-\gamma,\psi}$ as in 4.3, and $\mathfrak{u} \in B_r$. Then,

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(t_1)\left(\psi(t_1)-\psi(t_k)\right)^{1-\gamma}-(\mathscr{P}\mathfrak{u})(t_2)\left(\psi(t_2)-\psi(t_k)\right)^{1-\gamma}\right| \\
&\leq \frac{1}{\Gamma(\gamma)}\left[\sum_{0<t_1-t_2<t_k} I_k\mathfrak{u}(t_k)+\lambda\sum_{0<t_1-t_2<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(t_k)\right. \\
&\qquad \left.+\sum_{0<t_1-t_2<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{u}(t_k))\right] \\
&\qquad +\left|\left(\psi(t_1)-\psi(t_k)\right)^{1-\gamma}\lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t_1)-\left(\psi(t_2)-\psi(t_k)\right)^{1-\gamma}\lambda\mathscr{I}^{\alpha_2;\psi}\left|\mathfrak{u}(t_2)\right|\right| \\
&\qquad +\left|\left(\psi(t_1)-\psi(t_k)\right)^{1-\gamma}\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t_1,\mathfrak{u}(t_1))-\left(\psi(t_2)-\psi(t_k)\right)^{1-\gamma}\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t_2,\mathfrak{u}(t_2))\right|
\end{aligned}$$

As $t_1 \to t_2$, the above inequality tends to zero. By Lemma 4.2 - 4.4 with Theorem 2.15, we conclude that $\mathscr{P}$ is continuous and completely continuous. $\square$

Theorem 4.5. *(Existence) Assume that [H1], [H4] and [H5] are satisfied. Then, Eq.* (2) *has at least one solution.*

Proof. It is continuous and bounded from Lemma 4.2 - Lemma 4.4. Now, it remains to show that the set

$$\omega = \{\mathfrak{u} \in PC_{1-\gamma,\psi} : \mathfrak{u} = \delta\mathscr{P}(\mathfrak{u}), 0 < \delta < 1\}$$

is bounded set.

Let $\mathfrak{u} \in \omega,\ \mathfrak{u} = \delta\mathscr{P}(\mathfrak{u})$ for some $0 < \delta < 1$. Thus for each $t \in J$. We have

$$\begin{aligned}
\mathfrak{u}(t) = \delta&\left[\frac{\left(\psi(t)-\psi(t_k)\right)^{\gamma-1}}{\Gamma(\gamma)}\left[\mathfrak{u}_a+\sum_{0<t<t_k} I_k\mathfrak{u}(t_k)-\lambda\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(t_k)\right.\right. \\
&\left.\left.+\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{u}(t_k))\right]-\lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\mathfrak{u}(t)+\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\right].
\end{aligned}$$

This shows that the set ω is bounded. By Theorem 2.12, that $\mathscr{P}$ has a fixed point which is a solution of problem (2). $\square$

Theorem 4.6. *If hypotheses [H1] and [H4]*

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)}\left(m\ell_I\left(\psi(T)-\psi(0)\right)^{\gamma-1}+m\lambda\left(\psi(T)-\psi(0)\right)^{\alpha_2}\frac{B(\gamma,(1-\alpha_1)(1-\beta)+\alpha_1\beta)}{\Gamma((1-\alpha_1)(1-\beta)+\alpha_1\beta)}\right. \\
&\left.+m\ell\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}\frac{B(\gamma,1+(\alpha_1+\alpha_2-1)\beta)}{\Gamma(1+(\alpha_1+\alpha_2-1)\beta)}\right)+\frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_2} \\
&+\frac{\lambda B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\psi(T)-\psi(0)\right)^{\alpha_1+\alpha_2}<1.
\end{aligned}$$

are satisfied. Then, Eq. (2) *has a unique solution.*

Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$\begin{cases} \left|\mathscr{D}^{\alpha_1,\beta;\psi}\left(\mathscr{D}^{\alpha_2,\beta;\psi}+\lambda\right)\mathfrak{v}(t)-\mathfrak{g}(t,\mathfrak{v}(t))\right| \le \varphi(t) \\ \Delta \mathscr{I}^{1-\gamma;\psi}\mathfrak{u}(t)|_{t=t_k} - I_k\mathfrak{u}(t_k) \le \varphi(t). \end{cases} \tag{11}$$

Definition 4.7. *The Eq.* (2) *is g-UHR stable with respect to* $\varphi \in PC_{1-\gamma,\psi}$ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each solution* $\mathfrak{v} \in PC_{1-\gamma,\psi}$ *of the inequality* (11) *there exists a solution* $\mathfrak{u} \in PC_{1-\gamma,\psi}$ *of Eq.* (2) *with*

$$|\mathfrak{v}(t)-\mathfrak{u}(t)| \le C_{f,\varphi}\varphi(t).$$

Theorem 4.8. *The hypotheses [H1] and [H4] holds. Then Eq.* (2) *is g-UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (11) and by Theorem 4.6, $\mathfrak{u}$ is a unique solution of problem (2). By integrating (11), we obtain

$$\begin{aligned}
&\Bigg|\mathfrak{v}(t)-\frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\Bigg[\mathfrak{u}_a+\sum_{0<t<t_k} I_k\mathfrak{v}(t_k)-\lambda\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{v}(t_k) \\
&\qquad+\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{v}(t_k))\Bigg]+\lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\mathfrak{v}(t)-\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{v}(t))\Bigg| \\
&\le \frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\sum_{0<t<t_k} g_k+\frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}g(t_k)+\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}g(t) \\
&\le m\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\varphi(t)+m\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\lambda_\varphi\varphi(t)+\lambda_\varphi\varphi(t).
\end{aligned}$$

Hence it follows that,

$$\begin{aligned}
&|\mathfrak{v}(t)-\mathfrak{u}(t)| \\
&\le \Bigg|\mathfrak{v}(t)-\frac{(\psi(t)-\psi(t_k))^{\gamma-1}}{\Gamma(\gamma)}\Bigg[\mathfrak{u}_a+\sum_{0<t<t_k} I_k\mathfrak{u}(t_k)-\lambda\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\mathfrak{u}(t_k) \\
&\qquad+\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\mathfrak{g}(t_k,\mathfrak{u}(t_k))\Bigg]+\lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\mathfrak{u}(t)-\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\Bigg| \\
&\le \left(\mathfrak{v}(t)m\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\varphi(t)+m\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\lambda_\varphi\varphi(t)+\lambda_\varphi\varphi(t)\right) \\
&\qquad+\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\Bigg(m\ell_I\,|\mathfrak{v}(t)-\mathfrak{u}(t)|+\lambda\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{(1-\alpha_1)(1-\beta)+\alpha_2\beta;\psi}\,|\mathfrak{v}(t_k)-\mathfrak{u}(t_k)| \\
&\qquad+\ell_{\mathfrak{g}}\sum_{0<t<t_k}\mathscr{I}_{t_{k-1}}^{1+(\alpha_1+\alpha_2-1)\beta;\psi}\,|\mathfrak{v}(t_k)-\mathfrak{u}(t_k)|\Bigg)+\lambda\mathscr{I}_{t_k}^{\alpha_2;\psi}\,|\mathfrak{v}(t)-\mathfrak{u}(t)|+\ell_{\mathfrak{g}}\mathscr{I}_{t_k}^{\alpha_1+\alpha_2;\psi}\,|\mathfrak{v}(t)-\mathfrak{u}(t)|
\end{aligned}$$

By Lemma 2.9, there exists a constant $M^* > 0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|\mathfrak{v}(t)-\mathfrak{v}(t)| \le M^*\varphi(t).$$

Thus, Eq.(2) is g-UHR. □

5. Langevin equation with nonlocal initial condition

Lemma 5.1. *A function* $\mathfrak{u}$ *is the solution of* (3) *iff* $\mathfrak{u}$ *satisfies the following integral equation:*

$$\begin{aligned}\mathfrak{u}(t) =& T\left(\psi(t)-\psi(0)\right)^{\gamma-1}\left(-\lambda\sum_{i=1}^{m}c_i\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(\tau_i)+\sum_{i=1}^{m}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(\tau_i,\mathfrak{u}(\tau_i))\right)\\ &-\lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t)+\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)).\end{aligned}$$

where

$$T=\frac{1}{\Gamma(\gamma)-\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\gamma-1}}.$$

Consider the operator $\mathscr{R}: C_{1-\gamma,\psi}\to C_{1-\gamma,\psi}$, it is well defined and given by

$$(\mathscr{R}\mathfrak{u})(t)=\begin{cases}T\left(\psi(t)-\psi(0)\right)^{\gamma-1}\left(-\lambda\sum_{i=1}^{m}c_i\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(\tau_i)+\sum_{i=1}^{m}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(\tau_i,\mathfrak{u}(\tau_i))\right)\\ -\lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t)+\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\end{cases}\tag{12}$$

Consider the ball $B_r=\left\{\mathfrak{u}\in C_{1-\gamma,\psi}:\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}\le r\right\}$. Set $\tilde{\mathfrak{g}}(t)=\mathfrak{g}(t,0)$ and

$$\omega=\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(|T|\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha_1+\alpha_2+\gamma-1}+\left(\psi(t)-\psi(0)\right)^{\alpha_1+\alpha_2}\right)\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}.$$

Now we subdivide the operator $\mathscr{R}$ into two operator $\mathscr{R}_1$ and $\mathscr{R}_2$ on B_r as follows

$$(\mathscr{R}_1\mathfrak{u})(t)=T\left(\psi(t)-\psi(0)\right)^{\gamma-1}\left(-\lambda\sum_{i=1}^{m}c_i\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(\tau_i)+\sum_{i=1}^{m}\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(\tau_i,\mathfrak{u}(\tau_i))\right),$$

and

$$(\mathscr{R}_2\mathfrak{u})(t)=-\lambda\mathscr{I}^{\alpha_2;\psi}\mathfrak{u}(t)+\mathscr{I}^{\alpha_1+\alpha_2;\psi}\mathfrak{g}(t,\mathfrak{u}(t)).$$

Lemma 5.2. *The operator* $\mathscr{R}_1\mathfrak{u}+\mathscr{R}_2\mathfrak{v}\in B_r$.

Proof.

$$\begin{aligned}&\left|\left(\mathscr{R}_1\mathfrak{u}(t)+\mathscr{R}_2\mathfrak{v}(t)\right)\left(\psi(t)-\psi(0)\right)^{1-\gamma}\right|\\ &\le\Bigg[\lambda\frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(|T|\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha_2+\gamma-1}+\left(\psi(t)-\psi(0)\right)^{\alpha_2}\right)\\ &\quad+\ell_{\mathfrak{g}}\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(|T|\sum_{i=1}^{m}\left(\psi(\tau_i)-\psi(0)\right)^{\alpha_1+\alpha_2+\gamma-1}+\left(\psi(t)-\psi(0)\right)^{\alpha_1+\alpha_2}\right)\Bigg]r\\ &\quad+\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(|T|\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha_1+\alpha_2+\gamma-1}+\left(\psi(t)-\psi(0)\right)^{\alpha_1+\alpha_2}\right)\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\\ &\le\rho r+\omega.\end{aligned}$$

□

Lemma 5.3. *The operator* $\mathscr{R}_1$ *is a contration mapping.*

Proof.

$$\left|(\mathscr{R}_1\mathfrak{u}(t)+\mathscr{R}_1\mathfrak{v}(t))\,(\psi(t)-\psi(0))^{1-\gamma}\right|$$
$$\leq |T|\sum_{i=1}^{m} c_i\left(\lambda\frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\,(\psi(\tau_i)-\psi(0))^{\alpha_2+\gamma-1}+\ell_{\mathfrak{g}}\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\,(\psi(\tau_i)-\psi(0))^{\alpha_1+\alpha_2+\gamma-1}\right)\|\mathfrak{u}-\mathfrak{v}\|_{C_{1-\gamma,\psi}}.$$

The operator $\mathscr{R}_1$ is contraction mapping due to hypothesis [H6]. □

Lemma 5.4. *The operator $\mathscr{R}_2$ is compact and continuous.*

Proof. First we show that the operator $\mathscr{R}_2$ is uniformly bounded.

$$\begin{aligned}\|\mathscr{R}_2\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq&\lambda\frac{B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\,(\psi(T)-\psi(0))^{\alpha_2+\gamma-1}\,\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}\\ &+\frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\,(\psi(T)-\psi(0))^{\alpha_1+\alpha_2}\left(\ell_{\mathfrak{g}}\,\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right)\end{aligned}$$

Next, we verify the compactness of operator $\mathscr{R}_2$.

$$|(\mathscr{R}_2\mathfrak{u}(t_1)+\mathscr{R}_2\mathfrak{u}(t_2))|\leq\lambda\mathscr{I}^{\alpha_2;\psi}\,|\mathfrak{u}(t_1)-\mathfrak{u}(t_2)|+\mathscr{I}^{\alpha_1+\alpha_2;\psi}\,|\mathfrak{g}(t_1,\mathfrak{u}(t_1))-\mathfrak{g}(t_2,\mathfrak{u}(t_2))|$$

tending to zero as $t_1\to t_2$. Thus $\mathscr{R}_2$ is equicontinuous. Hence, the operator $\mathscr{R}_2$ is compact on B_r by Theorem 2.15. □

Theorem 5.5. *Assume that [H1] and [H6] are satisfied. Then, Eq.* (3) *has at most one solution.*

Proof. In Lemma 5.2 - Lemma 5.4 all the conditions of Theorem 2.13 are verified and thus by the Theorem 2.13 nonlocal problem (3) is the unique solution. □

Definition 5.6. *The Eq.* (3) *is g-UHR stable with respect to $\varphi\in C_{1-\gamma,\psi}$ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $\mathfrak{v}\in C_{1-\gamma,\psi}$ of the inequality* (9) *there exists a solution $\mathfrak{u}\in C_{1-\gamma,\psi}$ of Eq.* (3) *with*

$$|\mathfrak{v}(t)-\mathfrak{u}(t)|\leq C_{f,\varphi}\varphi(t).$$

Theorem 5.7. *The hypotheses [H1] hold. Then Eq.* (3) *is g-UHR stable.*

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