Robust RLS Wiener State Estimators in Linear Discrete-Time Stochastic Systems with Uncertain Parameters

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Abstract

This paper proposes the robust estimation technique for the signal and the state variables with respect to the state-space model having the general types of system matrices in linear discrete-time stochastic systems with the uncertain parameters. It is assumed that the signal and degraded signal processes are fitted to the finite order autoregressive (AR) models. By fitting the signal process to the AR model, the system matrix for the signal is transformed to the controllable canonical form. By using the system matrix, the existing robust RLS Wiener filter and fixed-point smoother are adopted to estimate the signal. Concerning the state estimation, the existing robust RLS Wiener filter calculates the filtering estimate of the signal. By replacing the observed value with the robust filtering estimate of the signal in the existing RLS Wiener filtering and fixed-point smoothing algorithms, the robust filtering and fixed-point smoothing estimates of the signal and the state variables are calculated.

The simulation result shows the superior estimation characteristics of the proposed robust estimation technique for the signal and the state variables in comparison with the existing H_∞ RLS Wiener filter, the robust Kalman filter, and the existing robust RLS Wiener filter and fixed-point smoother.

Keywords: Robust RLS Wiener estimators, state estimation, Wiener-Hopf equation, uncertain parameters, degraded signal

2010 AMS Subject Classification: 60G35, 93E10, 93E11, 94A12

1. Introduction

The Kalman filter is designed for the estimation of the state variables by using the observed values, generated by the accurate state-space model. Given the observed values generated by the state-space model with the uncertain parameters, the robust estimation techniques have been studied extensively [1]-[22] in linear discrete-time stochastic systems. In [3], the robust state filtering technique is devised based on the minimization of the bound on the state error variance. In [6], [7], by solving the convex optimization problem with the linear matrix inequality (LMI) method, the adaptive robust Kalman filtering algorithm is proposed in linear time-varying systems with stochastic parametric uncertainties. In [8], the robust finite-horizon Kalman filter is devised for the uncertain systems with both additive and multiplicative noises. In [9], the robust Kalman filter is proposed for the time-varying stochastic systems with state delay and the possibility of missing measurements. In [10], [11], the regularized robust filters are proposed in linear uncertain stochastic systems. In [11], the LMI method is adopted. In [12], the H_∞ filter is proposed in linear time-varying systems subject to norm-bounded parameter uncertainties. In [13], by solving the constrained minimization problem, the robust filter is presented for the stochastic systems with parameter uncertainties. In [14], the robust Kalman filtering algorithms are presented in linear stochastic systems with the uncertain parameters in the system and observation matrices. Together with the two Riccati-type equations, the filtering estimate is calculated recursively. In [15], the robust filter is proposed under incremental model perturbations characterized by the τ -divergence family by minimizing the mean square error according to the least favorable model.

The recursive least squares (RLS) Wiener filter and fixed-point smoother are proposed for the observations with random unit delays [23] and also with delays and the packet dropout [24] in linear discrete-time stochastic systems. In [20], the H_∞ RLS Wiener filter and fixed-point smoother are proposed in linear continuous-time stochastic systems. The observation equation in the Krein spaces is obtained. In [21], the H_∞ RLS Wiener filter and fixed-point smoother are presented in linear discrete-time stochastic systems. Recently, in [22], the robust RLS Wiener filter and fixed-point smoother are proposed for estimating the signal in linear discrete-time stochastic systems. The estimation accuracy of the robust RLS Wiener filter in [22] is superior to the RLS Wiener filter in [25] and the robust Kalman filter [14]. The robust RLS Wiener estimators [22] assume that the system matrix for the signal is expressed in the controllable canonical form. It is assumed that the signal and degraded signal processes are fitted to the finite order autoregressive (AR) models. The purpose of this paper is to develop the robust estimation technique of the signal and the state variables for the state-space models with the general kind of system matrices. By fitting the signal process to the AR model, the system matrix for the signal is transformed into the controllable canonical form. Then the robust RLS Wiener filter and fixed-point smoother [22] can be applied to the estimation of the signal even if the system matrix, before transformation, has not the controllable canonical form. In section 5, the example on the transformation is explained. Concerning the robust estimations of the state variables, in the first place, the robust filtering estimate of the signal is calculated by the robust RLS Wiener filter in [22]. In the second place, by replacing the observed value with the robust filtering estimate of the signal in the RLS Wiener filtering and fixed-point smoothing algorithms [25], the robust filtering and fixed-point smoothing estimates of the signal and the state variables are obtained. The robust RLS Wiener filtering and fixed-point smoothing algorithms in Theorem 1 [22] require the following information. (1) The system and observation matrices for the signal and the degraded signal. (2) The variance of the state vector for the degraded signal process. (3) The cross-variance of the state vector for the signal with the state vector for the degraded signal. (4) The variance of the observation noise. The RLS Wiener filtering and fixed-point smoothing algorithms, proposed in Theorem 2, require the robust filtering estimate of the signal, the system and observation matrices, and the variances of the state vector and the observation noise.

A simulation example shows the estimation characteristics of the proposed robust estimation method for the signal and the state vector. Here, the signal process, generated by the second-order state equation, and the degraded signal process are fitted to the 5th order AR models. The robust RLS Wiener filter and fixed-point smoother of Theorem 1, for estimating the signal $z(k)$, and the RLS Wiener filter and fixed-lag smoother of Theorem 2, for estimating the signal $z(k)$ with the state variables $x_1(k)$ and $x_2(k)$, are compared in estimation accuracy with the H_∞ RLS Wiener filter in [21], the robust Kalman filter [14] and the robust RLS Wiener filter and fixed-point smoother [22].

2. Robust least-squares fixed-point smoothing problem

Let the state-space model in linear discrete-time stochastic systems be described by

$$\begin{aligned} &y(k) = z(k) + v(k), z(k) = Hx(k), \\ &x(k+1) = \Phi x(k) + \Gamma w(k), \\ &E[v(k)v^T(s)] = R\delta_K(k-s), \\ &E[w(k)w^T(s)] = Q\delta_K(k-s). \end{aligned} \tag{1}$$

Here, $z(k)$ represents the signal to be estimated and $x(k)$ the state vector. H denotes the m by n observation matrix, Γ the n by l input matrix, $v(k)$ the white observation noise with the mean zero and $w(k)$ the input noise with the mean zero. The auto-covariance functions of the observation noise and the input noise are shown in (1). It is assumed that the signal process is uncorrelated with the observation noise process. Let the signal process be expressed by the AR model of the finite order M.

$$\begin{aligned} &z(k) = -\underline{a}_1 z(k-1) - \underline{a}_2 z(k-2) \cdots - \underline{a}_M z(k-M) + \underline{e}(k), \\ &E[\underline{e}(k)\underline{e}^T(s)] = \underline{Q}\delta_K(k-s) \end{aligned} \tag{2}$$

Let $z(k)$ be expressed in terms of the state vector $\underline{x}(k)$ as follows.

$$z(k) = \underline{H}\underline{x}(k), \underline{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0],$$
$$\underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_{M-1}(k) \\ \underline{x}_M(k) \end{bmatrix} = \begin{bmatrix} z(k) \\ z(k+1) \\ \vdots \\ z(k+M-2) \\ z(k+M-1) \end{bmatrix} \tag{3}$$

Then the state equation, corresponding to the AR model (2), is described by

$$\underline{x}(k+1) = \underline{\Phi}\underline{x}(k) + \underline{\Gamma}\underline{w}(k), E[\underline{w}(k)\underline{w}^T(s)] = \underline{Q}\delta_K(k-s),$$
$$\underline{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\underline{a}_M & -\underline{a}_{M-1} & -\underline{a}_{M-2} & \cdots & -\underline{a}_1 \end{bmatrix}, \underline{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix}, \tag{4}$$
$$\underline{w}(k) = \underline{e}(k+N).$$

It should be noted that the system matrix Φ is not necessarily limited to the expression of the controllable canonical form. By introducing the auto-covariance function of the signal $z(k)$, $K_z(k,s) = E[z(k)z^T(s)] = K_z(i)$, $i = k - s$, $0 \le i \le M$, the Yule-Walker equation for the AR parameters $\underline{a}_i$, $1 \le i \le M$, is formalized as

$$K(k,k)\begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_{M-1}^T \\ \underline{a}_M^T \end{bmatrix} = -\begin{bmatrix} K_z^T(1) \\ K_z^T(2) \\ \vdots \\ K_z^T(M-1) \\ K_z^T(M) \end{bmatrix},$$
$$K(k,k) = \begin{bmatrix} K_z(0) & K_z(1) & \cdots & K_z(M-2) & K_z(M-1) \\ K_z^T(1) & K_z(0) & \cdots & K_z(M-3) & K_z(M-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_z^T(M-2) & K_z^T(M-3) & \cdots & K_z(0) & K_z(1) \\ K_z^T(M-1) & K_z^T(M-2) & \cdots & K_z^T(1) & K_z(0) \end{bmatrix}. \tag{5}$$

The purpose of this paper is to develop the robust estimation technique for the state vector $x(k)$ from the degraded measurement data $\breve{y}(k)$, which is generated by the state-space model.

$$\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \bar{H}(k)\breve{x}(k), \bar{H}(k) = H + \Delta H(k),$$
$$\breve{x}(k+1) = \bar{\Phi}(k)\breve{x}(k) + \Gamma w(k), \bar{\Phi}(k) = \Phi + \Delta\Phi(k) \tag{6}$$

In (6) the observation matrix $\bar{H}(k)$ and the system matrix $\bar{\Phi}(k)$ contain the uncertain matrices $\Delta H(k)$ and $\Delta\Phi(k)$ additionally to the observation matrix H and the system matrix Φ, in comparison with the state-space model (1), respectively. Due to the uncertain quantity $\Delta\Phi(k)$, the trajectory of the state vector $\breve{x}(k)$ strays out of the nominal trajectory of $x(k)$. $\breve{z}(k)$ is the degraded signal. The observation matrix $\bar{H}(k)$ contains the uncertain matrix $\Delta H(k)$.

In [22], the robust RLS Wiener filtering and fixed-point smoothing algorithms are proposed for estimating the signal $z(k)$ from the degraded measurement data $\breve{y}(k)$. In [22], the norm-bounded condition [14] posed on the uncertain parameters is not used at all. It is a characteristic that the robust RLS Wiener estimators [22] do not use any information on the uncertain quantities $\Delta\Phi(k)$ and $\Delta H(k)$.

The sequence of the degraded signal $\breve{z}(k)$ is fitted to the AR model of the N-th order.

$$\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2)\cdots - \breve{a}_N\breve{z}(k-N) + \breve{e}(k),$$
$$E[\breve{e}(k)\breve{e}^T(s)] = \breve{Q}\delta_K(k-s) \tag{7}$$

$\breve{z}(k)$ is expressed in terms of the state vector $\breve{x}(k)$ as

$$\breve{z}(k) = \breve{H}\breve{x}(k), \breve{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0],$$
$$\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix}. \tag{8}$$

Hence, the state equation for the state vector $\breve{x}(k)$ is described by

$$\breve{x}(k+1) = \breve{\Phi}\breve{x}(k) + \breve{\Gamma}\zeta(k), E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s),$$
$$\breve{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix}, \breve{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix}, \tag{9}$$
$$\zeta(k) = \breve{e}(k+N).$$

The auto-covariance function $\breve{K}(k,s)$ of the state vector $\breve{x}(k)$ is assumed to have the semi-degenerate kernel form of

$$\breve{K}(k,s) = \begin{cases} A(k)B^T(s), 0 \le s \le k, \\ B(k)A^T(s), 0 \le k \le s, \end{cases}$$
$$A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s). \tag{10}$$

In terms of the auto-covariance function $K_{\breve{z}}(k,s) = E[\breve{z}(k)\breve{z}^T(s)]$ of the degraded signal $\breve{z}(k)$ in wide sense stationary stochastic systems, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is expressed as follows.

$$\breve{K}(k,k) = E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \times [\breve{z}^T(k) \quad \breve{z}^T(k+1) \quad \cdots \quad \breve{z}^T(k+N-2) \quad \breve{z}^T(k+N-1)]\right]$$
$$= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix} \tag{11}$$

By using $K_{\breve{z}}(i)$, $0 \le i \le N$, the Yule-Walker equation for the AR parameters $\breve{a}_i$, $1 \le i \le N$, is formalized as

$$\hat{K}(k,k)\begin{bmatrix} \breve{a}_1^T \\ \breve{a}_2^T \\ \vdots \\ \breve{a}_{N-1}^T \\ \breve{a}_N^T \end{bmatrix} = -\begin{bmatrix} K_{\breve{z}}^T(1) \\ K_{\breve{z}}^T(2) \\ \vdots \\ K_{\breve{z}}^T(N-1) \\ K_{\breve{z}}^T(N) \end{bmatrix},$$
$$\hat{K}(k,k) = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}. \tag{12}$$

Let $K_{\underline{x}\breve{x}}(k,s) = E[\underline{x}(k)\breve{x}^T(s)]$ represent the cross-covariance function of the state vector $\underline{x}(k)$ with $\breve{x}(s)$. $K_{\underline{x}\breve{x}}(k,s)$ is assumed to have the functional form of

$$K_{\underline{x}\breve{x}}(k,s) = \alpha(k)\beta^T(s), 0 \le s \le k,$$
$$\alpha(k) = \underline{\Phi}^k, \beta^T(s) = \underline{\Phi}^{-s}K_{\underline{x}\breve{x}}(s,s) \tag{13}$$

with the system matrix $\underline{\Phi}$ for the state vector $\underline{x}(k)$.

Let the fixed-point smoothing estimate $\hat{\underline{x}}(k,L)$ of the state vector $\underline{x}(k)$ at the fixed point, k be given by

$$\hat{\underline{x}}(k,L)=\sum_{i=1}^{L}h\,(k,i,L)\breve{y}(i) \tag{14}$$

as a linear combination of the impulse response function $h(k,i,L)$ and the observed values $\{\breve{y}(i),1\le i\le L\}$. We consider the least-squares estimation problem, which minimizes the mean-square value (MSV)

$$J=E[||\underline{x}(k)-\hat{\underline{x}}(k,L)||^2] \tag{15}$$

of the fixed-point smoothing error $\underline{x}(k)-\hat{\underline{x}}(k,L)$. From an orthogonal projection lemma [26]

$$\underline{x}(k)-\sum_{i=1}^{L}h\,(k,i,L)\breve{y}(i)\perp\breve{y}(s),1\le s\le L, \tag{16}$$

the Wiener-Hopf equation

$$E[\underline{x}(k)\breve{y}^T(s)]=\sum_{i=1}^{L}h\,(k,i,L)E[\breve{y}(i)\breve{y}^T(s)], \tag{17}$$

which the optimal impulse response function satisfies, is obtained. In (16), '$\perp$' represents the notation of the orthogonality. From (6), (8) and (17), and taking into account of the relationship $E[\underline{x}(k)\breve{y}^T(s)]=K_{\underline{x}\breve{x}}(k,s)\breve{H}^T=K_{\underline{x}\breve{z}}(k,s)$, we obtain

$$h(k,s,L)R=K_{\underline{x}\breve{x}}(k,s)\breve{H}^T-\sum_{i=1}^{L}h\,(k,i,L)\breve{H}\breve{K}(i,s)\breve{H}^T. \tag{18}$$

Here, $K_{\underline{x}\breve{z}}(k,s)$ denote the cross-covariance function of the state vector $\underline{x}(k)$ with the degraded signal $\breve{z}(s)$ as $E[\underline{x}(k)\breve{z}^T(s)]$.

3. Robust RLS Wiener filtering and fixed-point smoothing algorithms

Under the problem formulation on the linear least-squares estimation problem in section 2, Theorem 1 [22] presents the robust RLS Wiener filtering and fixed-point smoothing algorithms for estimating the signal $z(k)$.

Theorem 1 [22] Let the state-space model containing the uncertain quantities $\Delta\Phi$ and ΔH be given by (6) in linear discrete-time stochastic systems. Let the state-space model for the signal $z(k)$ be given by (1). Let the signal $z(k)$ be fitted to the AR model of the order M. Let the degraded signal $\breve{z}(k)$ be fitted to the AR model of the order N. Let the variance $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and the cross-variance $K_{\underline{x}\breve{x}}(k,k)$ of the state vector $x(k)$ for the signal $z(k)$ with the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ be given. Let the variance of the white observation noise $v(k)$ be R. Then, (19)-(29) constitute the robust RLS Wiener estimation algorithms for the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,L)$ of the signal $z(k)$ at the fixed point k.

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L)=\underline{H}\hat{\underline{x}}(k,L) \tag{19}$$

Fixed-point smoothing estimate of the state vector $\underline{x}(k)$: $\hat{\underline{x}}(k,L)$

$$\begin{aligned}&\hat{\underline{x}}(k,L)=\hat{\underline{x}}(k,L-1)+h(k,L,L)(\breve{y}(L)-\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)),\\&\hat{\underline{x}}(k,L)|_{L=k}=\hat{\underline{x}}(k,k)\end{aligned} \tag{20}$$

Smoother gain for $\hat{\underline{x}}(k,L)$ in (20): $h(k,L,L)$

$$\begin{aligned}&h(k,L,L)=[K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T-q(k,L-1)\breve{\Phi}^T\breve{H}^T]\\&\times\{R+\breve{H}[\breve{K}(L,L)-\breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}\end{aligned} \tag{21}$$

$$q(k,L) = q(k,L-1)\breve{\Phi}^T + h(k,L,L)\breve{H}[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T],\quad q(k,k) = S_0(k) \tag{22}$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = \underline{H}\underline{\hat{x}}(k,k) \tag{23}$$

Filtering estimate of $\underline{x}(k)$: $\underline{\hat{x}}(k,k)$

$$\underline{\hat{x}}(k,k) = \underline{\Phi\hat{x}}(k-1,k-1) + G(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\quad \underline{\hat{x}}(0,0) = 0 \tag{24}$$

Filter gain for $\underline{\hat{x}}(k,k)$ in (24): $G(k)$

$$G(k) = [K_{\underline{x}\breve{z}}(k,k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T] \times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1},\quad K_{\underline{x}\breve{z}}(k,k) = K_{\underline{x}\breve{x}}(k,k)\breve{H}^T \tag{25}$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\quad \hat{\breve{x}}(0,0) = 0 \tag{26}$$

Filter gain for $\hat{\breve{x}}(k,k)$ in (26): $g(k)$

$$g(k) = [\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T] \times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1} \tag{27}$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],\quad S_0(0) = 0 \tag{28}$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S(k) = \Phi S(k-1)\breve{\Phi}^T + G(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],\quad S(0) = 0 \tag{29}$$

In the robust RLS Wiener filter and fixed-point smoother [22], the estimation problem of the signal $z(k)$ is the main issue and the estimation of the state variables in the state-space model with the system matrix Φ, in the controllable canonical form, is possible. To be able to deal with the system matrices, not in the controllable canonical form, the signal process needs to be expressed by the AR model of the finite order M as shown in (2). The corresponding state vector and the state-space model are given in (3) and (4). The AR parameters in (2) are calculated by solving the Yule-Walker equation (5). The robust filtering and fixed-point smoothing estimates of the signal $z(k)$ are calculated by the robust RLS Wiener estimation algorithms of Theorem 1 [22]. By replacing the observed value with the robust filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ in the RLS Wiener filtering and fixed-point smoothing algorithms [25], Theorem 2 presents the robust RLS Wiener estimation algorithms for the filtering and fixed-point smoothing estimates of the signal $z(k)$ and the state vector $x(k)$.

Theorem 2 Let the state-space model, without including the uncertain quantities $\Delta\Phi$ and ΔH, be given by (1). Then the robust RLS Wiener filtering and fixed-point smoothing algorithms for the signal $z(k)$ and state vector $x(k)$ consist of (30)-(37). In Theorem 2, the observed value $y(k)$, in the RLS Wiener estimation algorithms [25], is replaced with the robust filtering estimate $\hat{z}(k,k)$, which is calculated by the robust RLS Wiener filter in Theorem 1 [22]. In the filtering and the fixed-point smoothing algorithms of Theorem 2, the filtering estimate $\hat{z}(k,k)$, the system matrix Φ, the observation matrix H, the auto-variance function of $x(k)$, $K_x(k,k) = E[x(k)x^T(k)]$ are used.

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}_{TH2}(k,L)$

$$\hat{z}_{TH2}(k,L) = H\hat{x}(k,L) \tag{30}$$

Fixed-point smoothing estimate of the state vector $x(k)$: $\hat{x}(k,L)$

$$\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(\hat{z}(L,L) - H\Phi\hat{x}(L-1,L-1)),$$
$$\hat{x}(k,L)|_{L=k} = \hat{x}(k,k) \tag{31}$$

Smoother gain for $\hat{x}(k,L)$ in (31): $h(k,L,L)$

$$h(k,L,L) = [K_x(k,k)(\Phi^T)^{L-k}H^T - q(k,L-1)\Phi^T H^T]$$
$$\times \{R + H[K_x(L,L) - \Phi S_x(L-1)\Phi^T]H^T\}^{-1} \tag{32}$$

$$q(k,L) = q(k,L-1)\Phi^T + h(k,L,L)H[K_x(L,L) - \Phi S_x(L-1)\Phi^T],$$
$$q(k,k) = S_x(k) \tag{33}$$

Filtering estimate of the signal $z(k)$: $\hat{z}_{TH2}(k,k)$

$$\hat{z}_{TH2}(k,k) = H\hat{x}(k,k) \tag{34}$$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + G_x(k)(\hat{z}(k,k) - H\Phi\hat{x}(k-1,k-1)),$$
$$\hat{x}(0,0) = 0 \tag{35}$$

Filter gain for $\hat{x}(k,k)$ in (35): $G_x(k)$

$$G_x(k) = [(K_x(k,k) - \Phi S_x(k-1)\Phi^T)H^T]$$
$$\times \{R + H[K_x(k,k) - \Phi S_x(L-1)\Phi^T]H^T\}^{-1} \tag{36}$$

Variance of filtering estimate $\hat{x}(k,k)$: $S_x(k)$

$$S_x(k) = \Phi S_x(k-1)\Phi^T + G_x(k)H[K_x(k,k) - \Phi S_x(k-1)\Phi^T],$$
$$S_x(0) = 0 \tag{37}$$

In section 4, the filtering error variance function for the state vector $x(k)$ is introduced and the existence of the filtering estimate $\hat{x}(k,k)$ Of $x(k)$ is validated.

4. Filtering error variance function of state vector

In this section the filtering error variance function $\tilde{P}_x(k)$ for the state vector $x(k)$ is shown. The filtering error variance function is given by

$$\begin{aligned}\tilde{P}_x(k) &= E[(x(k) - \hat{x}(k,k))(x(k) - \hat{x}(k,k))^T] \\ &= K_x(k,k) - E[\hat{x}(k,k)\hat{x}^T(k,k)] \\ &= K_x(k,k) - S_x(k).\end{aligned} \tag{38}$$

$S_x(k)$ Is calculated by (36) and (37) recursively. Since $\tilde{P}_x(k)$ is the semi-definite function, the filtering variance function $E[\hat{x}(k,k)\hat{x}^T(k,k)]$ is upper bounded by $K_x(k,k)$ and lower bounded by the zero matrices as

$$0 \le E[\hat{x}(k,k)\hat{x}^T(k,k)] \le K_x(k,k). \tag{39}$$

This validates the existence of the filtering estimate $\hat{x}(k,k)$ of the state vector $x(k)$.

5. A numerical simulation example

Let a scalar observation equation and the state equation for $x(k)$ be given by

$$\begin{aligned}&y(k) = z(k) + v(k), z(k) = Hx(k), H = [0.95 \quad -0.4], x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\ &x(k+1) = \Phi x(k) + \Gamma w(k), \Phi = \begin{bmatrix} 0.05 & 0.95 \\ -0.98 & 0.2 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.952 \\ 0.2 \end{bmatrix}, \\ &E[v(k)v(s)] = R\delta_K(k-s), E[w(k)w(s)] = Q\delta_K(k-s), Q = 0.5^2.\end{aligned} \tag{40}$$

As shown in (2), the signal $z(k)$ is fitted to the M-th order AR model. The state-space model of (6) contains the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ as shown in

$$\begin{aligned}&\breve{y}(k)=\breve{z}(k)+v(k),\breve{z}(k)=\bar{H}(k)\bar{x}(k),\bar{x}(k)=\begin{bmatrix}\bar{x}_1(k)\\ \bar{x}_2(k)\end{bmatrix},\\ &\bar{H}(k)=H+\Delta H(k)=[1+\Delta_3(k)\quad 0],\Delta H(k)=[\Delta_3(k)\quad 0],\Delta_3(k)=0.3\zeta_3(k),\\ &\bar{x}(k+1)=\bar{\Phi}(k)\bar{x}(k)+\Gamma w(k),\bar{\Phi}(k)=\Phi+\Delta\Phi(k),\Delta\Phi(k)=\begin{bmatrix}\Delta_1(k) & 0\\ 0 & \Delta_2(k)\end{bmatrix},\\ &\Delta_1(k)=0.1\zeta_1(k),\Delta_2(k)=0.2\zeta_2(k).\end{aligned} \tag{41}$$

It should be noted that the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ are not given as prior information and are not used in the proposed robust estimation algorithms in Theorem 1 [22] and Theorem 2. Their norm-bounded condition [14] is not used in the theorems as well. The task is to estimate the signal $z(k)$ recursively with the scalar observed value $\breve{y}(k)$, which is given as a sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$. $\zeta_1(k)$, $\zeta_2(k)$ and $\zeta_3(k)$ in (41) represent the uniformly distributed random variables taking values between 0 and 1 and are generated by using the MATLAB command "$rand$." $\Delta_1(k)$, $\Delta_2(k)$ and $\Delta_3(k)$ consist of the deterministic mean values and the zero-mean stochastic variables, which are mutually independent. The degraded signal $\breve{z}(k)$ is fitted to the N-th order AR model of (7). The state-space model for the degraded signal $\breve{z}(k)$ is given by (9). The state equation for $\breve{x}(k)$ is given by (9) for $m=1$. $\breve{K}(k,s)=\breve{K}(k-s)$ represents the auto-covariance function of the state vector $\breve{x}(k)$ in wide-sense stationary stochastic systems. $\breve{K}(k,s)$ is expressed in the form of the semi-degenerate function (10). The system matrix $\breve{\Phi}$ for the state vector $\breve{x}(k)$ is given in (9). In terms of the auto-covariance function $K_{\breve{z}}(k-s)=K_{\breve{z}}(s-k)=E[\breve{z}(k)\breve{z}(s)]$ of degraded signal $\breve{z}(k)$, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is expressed as follows.

$$\begin{aligned}\breve{K}(k,k)&=E\left[\begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}[\breve{z}(k)\quad \breve{z}(k+1)\quad\cdots\quad \breve{z}(k+N-2)\quad \breve{z}(k+N-1)]\right]\\ &=\begin{bmatrix}K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0)\end{bmatrix}\end{aligned} \tag{42}$$

Let $K_{z\breve{z}}(k,s)=E[z(k)\breve{z}(s)]$ represent the cross-covariance function between the signal $z(k)$ and the degraded signal $\breve{z}(s)$. From (3) and (13), the cross-covariance function $K_{\underline{x}\breve{x}}(k,s)$ is expressed as

$$K_{\underline{x}\check{x}}(k,s) = \underline{\Phi}^{k-s} K_{\underline{x}\check{x}}(s,s), 0 \le s \le k,$$

$$K_{\underline{x}\check{z}}(k,k) = E\left[\begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_{M-1}(k) \\ \underline{x}_M(k) \end{bmatrix} [\check{z}(k) \quad \check{z}(k+1) \quad \cdots \quad \check{z}(k+N-2) \quad \check{z}(k+N-1)]\right]$$

$$= E\left[\begin{bmatrix} z(k) \\ z(k+1) \\ \vdots \\ z(k+M-2) \\ z(k+M-1) \end{bmatrix} [\check{z}(k) \quad \check{z}(k+1) \quad \cdots \quad \check{z}(k+N-2) \quad \check{z}(k+N-1)]\right] \tag{43}$$

$$= \begin{bmatrix} K_{z\check{z}}(k,k) & K_{z\check{z}}(k,k+1) & \cdots & K_{z\check{z}}(k,k+N-2) & K_{z\check{z}}(k,k+N-1) \\ K_{z\check{z}}(k+1,k) & K_{z\check{z}}(k+1,k+1) & \cdots & K_{z\check{z}}(k+1,k+N-2) & K_{z\check{z}}(k+1,k+N-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{z\check{z}}(k+M-2,k) & K_{z\check{z}}(k+M-2,k+1) & \cdots & K_{z\check{z}}(k+M-2,k+N-2) & K_{z\check{z}}(k+M-2,k+N-1) \\ K_{z\check{z}}(k+M-1,k) & K_{z\check{z}}(k+M-1,k+1) & \cdots & K_{z\check{z}}(k+M-1,k+N-2) & K_{z\check{z}}(k+M-1,k+N-1) \end{bmatrix}.$$

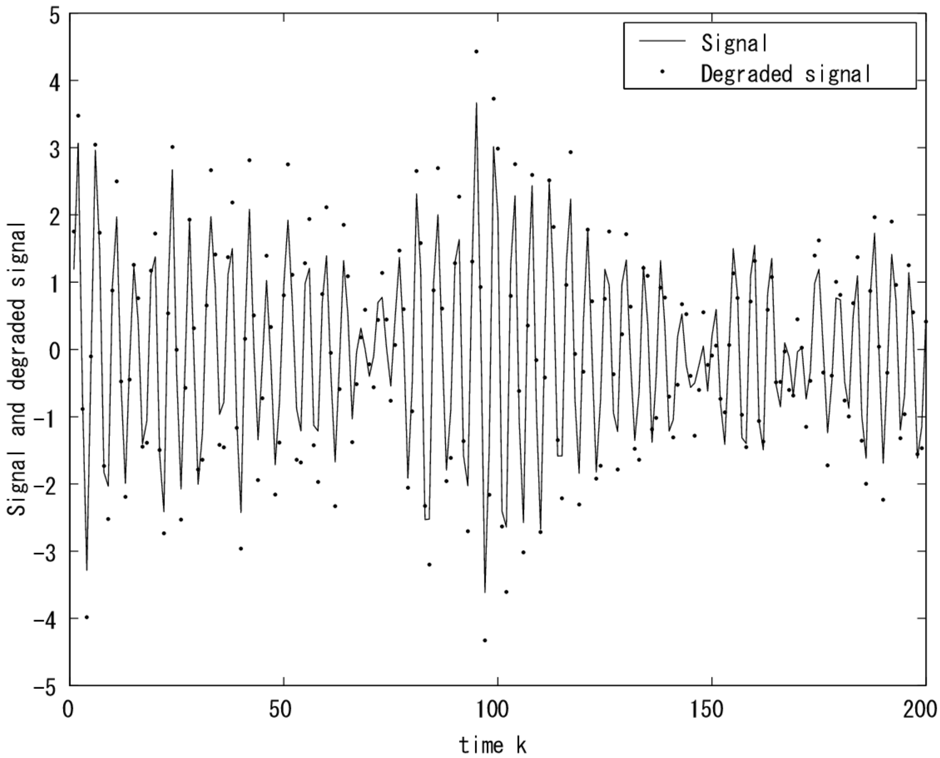

Fig. 1 Signal $\boldsymbol{z(k)}$ and the degraded signal $\boldsymbol{\check{z}(k)}$ vs. $\boldsymbol{k}$, $\boldsymbol{1 \le k \le 200}$, for the white Gaussian observation noise $\boldsymbol{N(0, 0.3^2)}$.

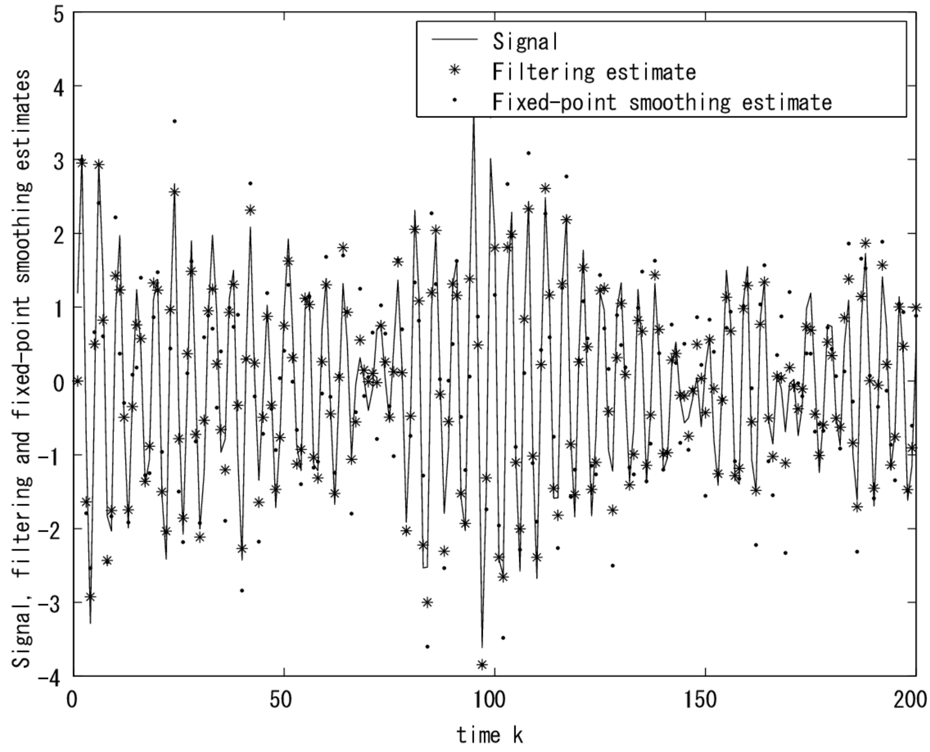

Fig. 2 Signal $\boldsymbol{z(k)}$, the fixed-point smoothing estimate $\hat{\boldsymbol{z}}(\boldsymbol{k},\boldsymbol{k}+\mathbf{3})$ and the filtering estimate $\hat{\boldsymbol{z}}(\boldsymbol{k},\boldsymbol{k})$ of the signal $\boldsymbol{z(k)}$ vs. $\boldsymbol{k}$, $\mathbf{1}\le\boldsymbol{k}\le\mathbf{200}$, for the white Gaussian observation noise $\boldsymbol{N}(\mathbf{0},\mathbf{0.3^2})$ in the case of the AR model orders $\boldsymbol{M}=\mathbf{5}$ and $\boldsymbol{N}=\mathbf{5}$.

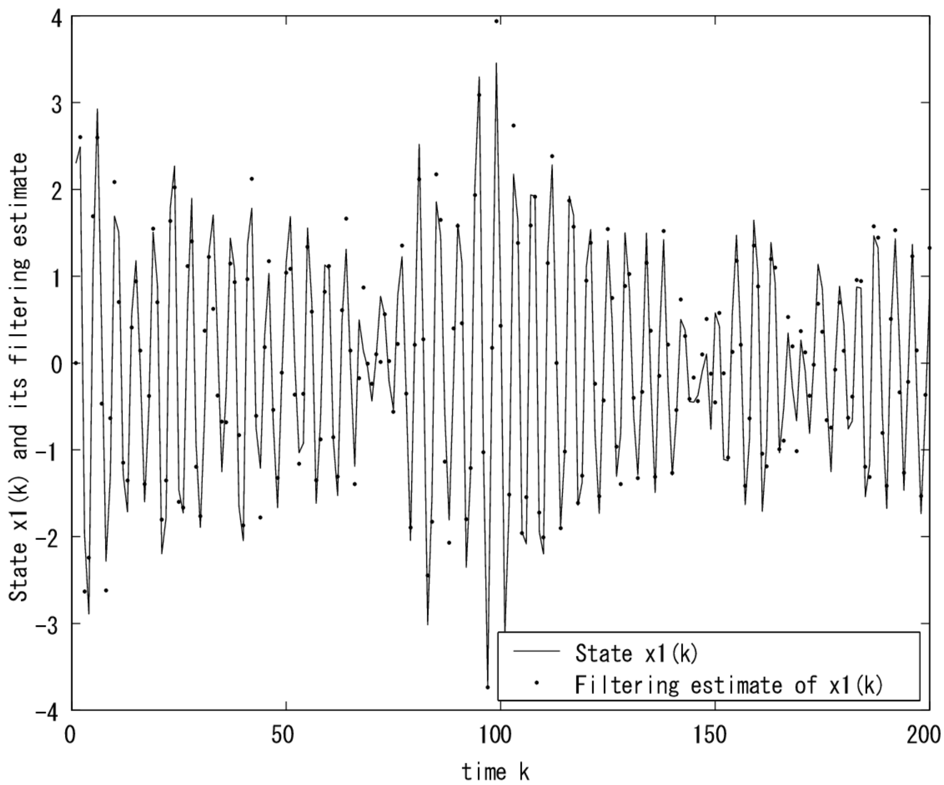

Fig. 3 State variable $\boldsymbol{x_1(k)}$ and its filtering estimate $\hat{\boldsymbol{x}}_{\mathbf{1}}(\boldsymbol{k},\boldsymbol{k})$ vs. $\boldsymbol{k}$, $\mathbf{1}\le\boldsymbol{k}\le\mathbf{200}$, for the white Gaussian observation noise $\boldsymbol{N}(\mathbf{0},\mathbf{0.3^2})$ in the case of the AR model orders $\boldsymbol{M}=\mathbf{5}$ and $\boldsymbol{N}=\mathbf{5}$.

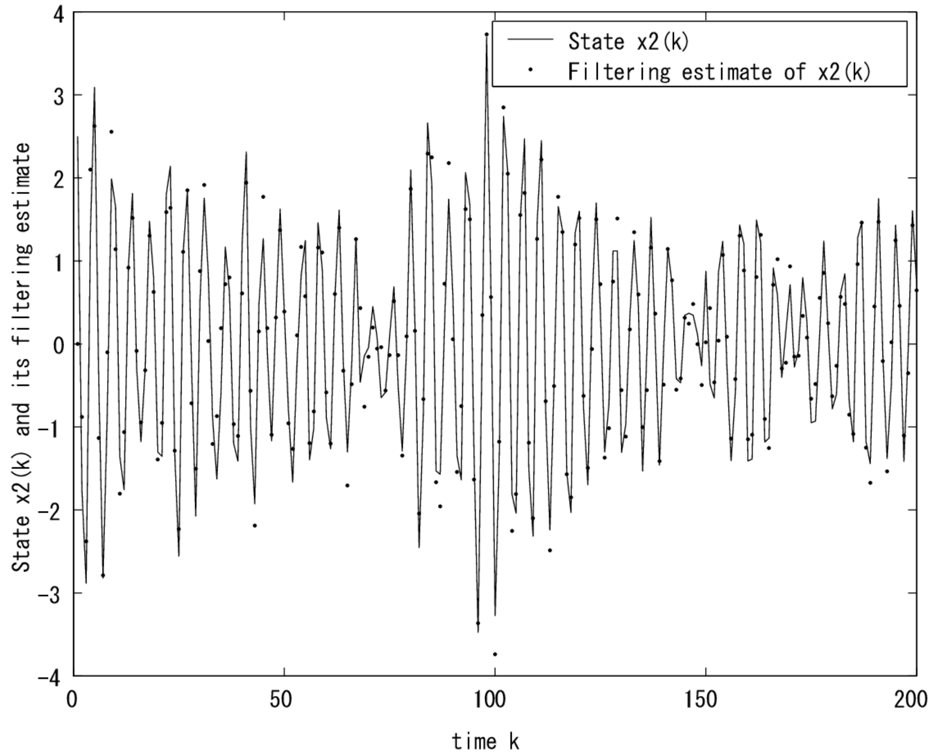

Fig. 4 State variable $x_2(k)$ and its filtering estimate $\hat{x}_2(k,k)$ vs. k, $1 \le k \le 200$, for the white Gaussian observation noise $N(0, 0.3^2)$ in the case of the AR model orders $M = 5$ and $N = 5$.

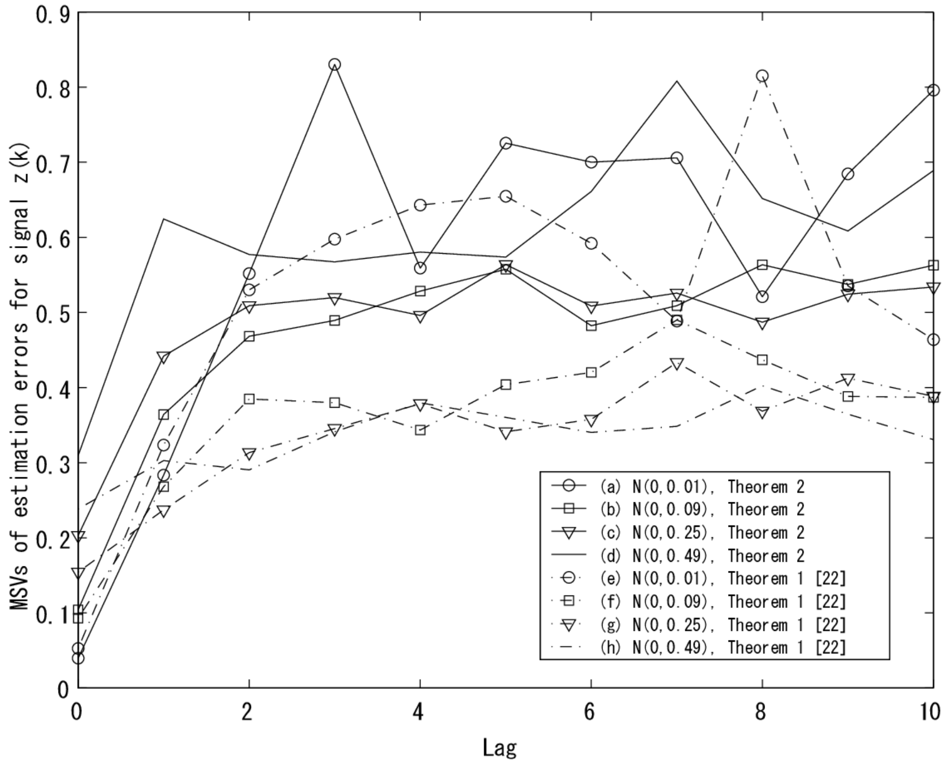

Fig. 5 MSVs of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ by the RLS Wiener filter and fixed-point smoother of Theorem 1 [22] and the MSVs of the filtering errors $z(k) - \hat{z}_{TH2}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}_{TH2}(k, k + Lag)$ by the RLS Wiener filter and fixed-point smoother of Theorem 2 vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ in the case of the AR model orders $M = 5$ and $N = 5$.

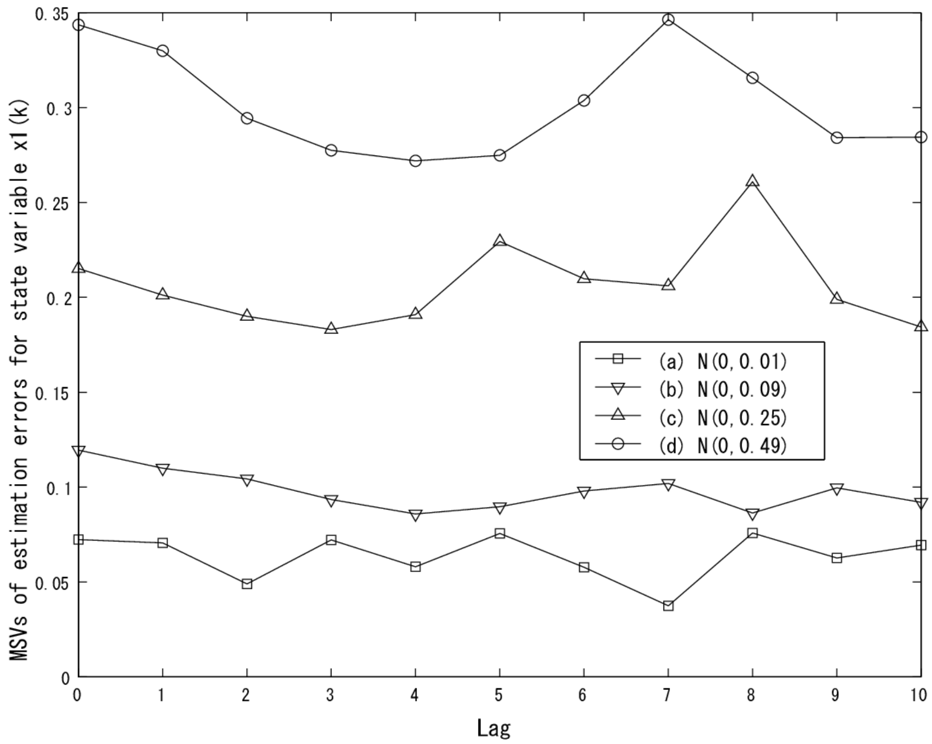

Fig. 6 MSVs of the filtering errors $x_1(k)-\hat{x}_1(k,k)$ and the fixed-point smoothing errors $x_1(k)-\hat{x}_1(k,k+Lag)$, $1 \le k \le 2000$, vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$ in the case of the AR model orders $M=5$ and $N=5$.

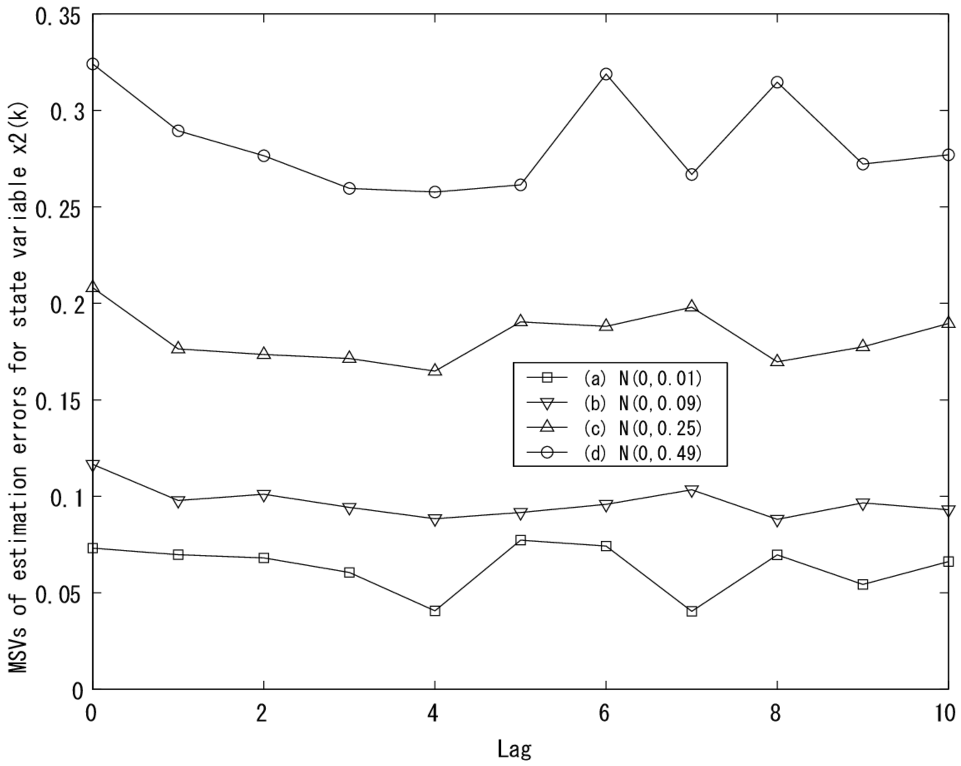

Fig. 7 MSVs of the filtering errors $x_2(k)-\hat{x}_2(k,k)$ and the fixed-point smoothing errors $x_2(k)-\hat{x}_2(k,k+Lag)$, $1 \le k \le 2000$, vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$ in the case of the AR model orders $M=5$ and $N=5$.

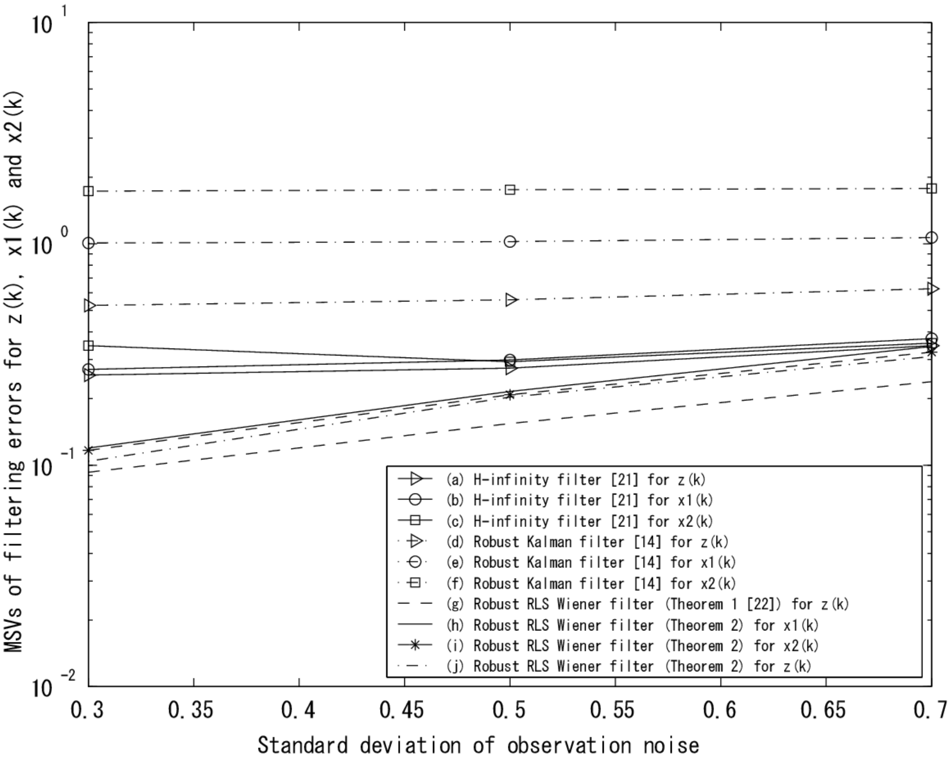

Fig. 8 MSVs of the filtering errors $\boldsymbol{z(k)-\hat{z}(k,k)}$ by the $\boldsymbol{H_\infty}$ filter in [21], the robust Kalman filter [14] and the robust RLS Wiener filter in Theorem 1 [22], the MSVs of the filtering errors $\boldsymbol{x_1(k)-\hat{x}_1(k,k)}$ and $\boldsymbol{x_2(k)-\hat{x}_2(k,k)}$ by the $\boldsymbol{H_\infty}$ filter in [21], the robust Kalman filter [14] and the robust RLS Wiener filter in Theorem 2, and the MSV of the filtering errors $\boldsymbol{z(k)-\hat{z}_{TH2}(k,k)}$ by the robust RLS Wiener filter in Theorem 2 vs. the standard deviation of the observation noise.

The AR parameters $\breve{a}_i, 1 \le i \le N$, in (7) for the degraded signal $\breve{z}(k)$ are calculated by the Yule-Walker equation (12). The AR parameters $\underline{a}_i, 1 \le i \le M$, in (2) for the signal $z(k)$ are calculated by the Yule-Walker equation (5). By substituting $\underline{H}$, $\breve{H}$, $\underline{\Phi}$, $\breve{\Phi}$, $K_{\underline{x}\breve{x}}(k,k)$, $\breve{K}(k,k)=\breve{K}(L,L)$ and R into the robust RLS Wiener estimation algorithms of Theorem 1 [22], the filtering and fixed-point smoothing estimates of the signal $z(k)$ are calculated recursively. In the simulation, the AR model orders for the sequences of the signal $z(k)$ and the degraded signal $\breve{z}(k)$ are $M=5$ and $N=5$ respectively. Fig.1 illustrates the signal $z(k)$ and the degraded signal $\breve{z}(k)$ vs. k, $1 \le k \le 200$, for the white Gaussian observation noise $N(0,0.3^2)$. It is clear that the sequence of the degraded signal is different from that of the signal. Fig.2 illustrates the signal $z(k)$, the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,k+3)$ of the signal $z(k)$ vs. k, $1 \le k \le 200$, for the white Gaussian observation noise $N(0,0.3^2)$. It is shown that the filtering and fixed-point smoothing estimates of $z(k)$ have the values near those of the signal along with the time k. By substituting the robust filtering estimate $\hat{z}(k,k)$, calculated by Theorem 1 [22], the observation matrix H, the system matrix Φ, and the variance $K_x(k,k)$ of the state vector $x(k)$ into the RLS Wiener filtering and fixed-point smoothing algorithms of Theorem 2, the filtering estimate $\hat{z}_{TH2}(k,k)$ and the fixed-point smoothing estimate $\hat{z}_{TH2}(k,L)$ of the signal $z(k)$, and the filtering estimates $\hat{x}_1(k,k)$, $\hat{x}_2(k,k)$ and the fixed-point smoothing estimate $\hat{x}_1(k,L)$, $\hat{x}_2(k,L)$ of the state variables $x_1(k)$ and $x_2(k)$ are calculated, respectively. Fig.3 illustrates the state variable $x_1(k)$ and its filtering estimate $\hat{x}_1(k,k)$ vs. k, $1 \le k \le 200$, for the white Gaussian observation noise $N(0,0.3^2)$. It is shown that the filtering estimate $\hat{x}_1(k,k)$ has the values near those of the state variable $x_1(k)$ along the time k. Fig.4 illustrates the state variable $x_2(k)$ and its filtering estimate $\hat{x}_2(k,k)$ vs. k, $1 \le k \le 200$, for the white Gaussian observation noise $N(0,0.3^2)$. It is also shown that the filtering estimate $\hat{x}_2(k,k)$ has the values near those of the state variable $x_2(k)$ along time k. Fig.5 illustrates the mean-square values (MSVs) of the filtering errors $z(k)-\hat{z}(k,k)$ and the fixed-point smoothing errors $z(k)-\hat{z}(k,k+Lag)$ by the RLS Wiener filter and fixed-point smoother of Theorem 1 [22] and the MSVs of the filtering errors $z(k)-\hat{z}_{TH2}(k,k)$ and the fixed-point smoothing errors $z(k)-\hat{z}_{TH2}(k,k+Lag)$ by the RLS Wiener filter and fixed-point smoother of Theorem 2 vs.

Lag, $0 \leq Lag \leq 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. For $Lag = 0$, the MSVs are shown for the filtering errors. The MSVs of the fixed-point smoothing errors are greater than the MSVs of the filtering errors for each observation noise. The MSVs by the robust RLS Wiener filter and fixed-point smoother of Theorem 1 [22] are smaller than those by the RLS Wiener filter and fixed-point smoother of Theorem 2 for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. Fig.6 illustrates the MSVs of the filtering errors $x_1(k) - \hat{x}_1(k,k)$ and the fixed-point smoothing errors $x_1(k) - \hat{x}_1(k,k+Lag)$, vs. Lag, $0 \leq Lag \leq 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. The MSVs have a tendency to be small as Lag increases for $1 \leq Lag \leq 3$ in the cases of the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. Fig.7 illustrates the MSVs of the filtering errors $x_2(k) - \hat{x}_2(k,k)$ and the fixed-point smoothing errors $x_2(k) - \hat{x}_2(k,k+Lag)$, vs. Lag, $0 \leq Lag \leq 10$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. The MSVs tend to be small as Lag increases for $1 \leq Lag \leq 4$ in the case of each observation noise. Fig.8 illustrates the MSVs of the filtering errors $z(k) - \hat{z}(k,k)$ by the H_∞ RLS Wiener filter in [21], the robust Kalman filter [14] and the robust RLS Wiener filter in Theorem 1 [22], the MSVs of the filtering errors $x_1(k) - \hat{x}_1(k,k)$ and $x_2(k) - \hat{x}_2(k,k)$ by the H_∞ RLS Wiener filter in [21], the robust Kalman filter [14] and the robust RLS Wiener filter in Theorem 2, and the MSV of the filtering errors $z(k) - \hat{z}_{TH2}(k,k)$ by the robust RLS Wiener filter in Theorem 2 vs. the standard deviation of the observation noise. From Fig. 8, concerning the estimation of the signal $z(k)$, the estimation accuracy is feasible in the order of the robust RLS Wiener filter in Theorem 1 [22], the robust RLS Wiener filter in Theorem 2, the H_∞ RLS Wiener filter in [21] and the robust Kalman filter [14]. Concerning the estimations of the state variables $x_1(k)$ and $x_2(k)$, the estimation accuracy is feasible in the order of the robust RLS Wiener filter in Theorem 2, the H_∞ RLS Wiener filter in [21] and the robust Kalman filter [14].

Here, in the simulation of section 5, the value of γ in [17] is $\gamma = 0.9$, and the parameters used in the Riccati-type equations of the robust Kalman filter [14] are $\varepsilon_k = 0.1$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $H_1 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $H_2 = [3 \quad -3]$. Here, the MSVs of the filtering and fixed-point smoothing errors are evaluated as follows:
$\sum_{i=1}^{2000}(z(k) - \hat{z}(k,k))^2/2000$, $\sum_{i=1}^{2000}(z(k) - \hat{z}(k,k+Lag))^2/2000$,
$\sum_{i=1}^{2000}(z(k) - \hat{z}_{TH2}(k,k))^2/2000$, $\sum_{i=1}^{2000}(z(k) - \hat{z}_{TH2}(k,k+Lag))^2/2000$,
$\sum_{i=1}^{2000}(x_i(k) - \hat{x}_i(k,k))^2/2000$, $\sum_{i=1}^{2000}(x_i(k) - \hat{x}_i(k,k+Lag))^2/2000$, $i = 1,\ 2$.

6. Conclusions

This paper, based on the robust RLS Wiener filter and fixed-point smoother [22] for the signal estimation, has proposed the estimation technique for the signal and the state variables in linear discrete-time stochastic systems with uncertain parameters. It is assumed that the signal and degraded signal processes are fitted to the finite order AR models. By fitting the signal process to the AR model, the system matrix for the signal is transformed to the controllable canonical form. Then the robust RLS Wiener filter and fixed-point smoother [22] are applicable to the estimation of the signal even if the original system matrix is not in the controllable canonical form. With respect to the state estimation, to begin with, the robust RLS Wiener filtering estimate of the signal is calculated the robust RLS Wiener filter in Theorem 1 [22]. Then, by replacing the robust filtering estimate of the signal with the observed value in the RLS Wiener filtering and fixed-point smoothing algorithms [25], the robust filtering and fixed-point smoothing estimates of the signal and the state variables are calculated.

The simulation example has shown that the proposed robust estimation technique for the signal and the state variables is feasible. Concerning the estimation of the signal $z(k)$, the estimation accuracy is feasible in the order of the robust RLS Wiener filter in Theorem 1 [22], the robust RLS Wiener filter in Theorem 2, the H_∞ RLS Wiener filter in [21] and the robust Kalman filter [14]. Concerning the estimations of the state variables $x_1(k)$ and $x_2(k)$, the estimation accuracy is feasible in the order of the robust RLS Wiener filter in Theorem 2, the H_∞ RLS Wiener filter in [21] and the robust Kalman filter [14].

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