Robust RLS Wiener Fixed-Lag Smoothing Algorithm in Linear Discrete-Time Stochastic Systems with Uncertain Parameters

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Abstract
This paper, by combining the robust recursive least-squares (RLS) Wiener filter and the RLS Wiener fixed-lag smoothing algorithm, proposes the robust RLS Wiener fixed-lag smoothing algorithm. In the robust estimation problem, it is assumed that the system and observation matrices include some uncertain parameters. With the observations generated by the state-space model including the uncertain parameters, the robust RLS Wiener fixed-lag smoother estimates the signal recursively as the time advances. Both the signal and the degraded signal processes are fitted to the finite order auto-regressive (AR) models. The robust RLS Wiener fixed-lag smoother uses the following information. (1) The covariance function of the state for the degraded signal. (2) The cross-covariance function of the state for the signal with the state for the degraded signal. (3) The observation matrices for the signal and the degraded signal. (4) The system matrices for the signal and the degraded signal. (5) The variance of the white observation noise. A numerical simulation example shows that the robust RLS Wiener fixed-lag smoother, proposed in this paper, is superior in estimation accuracy to the H-infinity RLS Wiener fixed-point smoother and the RLS Wiener fixed-lag smoother.

Keywords: Robust RLS Wiener Estimator; Fixed-Lag Smoother; Discrete-Time Stochastic Systems; Uncertain Parameters; Auto-Regressive Model

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1. Introduction

The robust estimation problems for the state-space model with the uncertain parameters have been investigated extensively, e.g. [1]-[21]. The robust estimation problem is solved with the linear matrix inequality (LMI) method as discussed by Wang et al. [12], [13], the H-infinity estimation method by Yang et al. [18], the method based on the regularization and penalty function by Ishihara et al. [19], etc.

In Nakamori [22], the RLS Wiener fixed-point smoother and filter are proposed. In Nakamori [23], the H-infinity RLS Wiener fixed-point smoothing and filtering algorithms are devised. In Nakamori [24], the RLS Wiener fixed-point smoother and filter are proposed, given the randomly delayed or uncertain observations. In Nakamori [25], the RLS Wiener fixed-lag smoothing algorithm is proposed. In Nakamori [26], for the purpose of estimating the signal process, the robust RLS Wiener fixed-point smoother and filter are proposed for the discrete-time stochastic systems with the uncertain parameters. In Nakamori [27], the robust RLS Wiener estimation technique for the state variables is developed.
This paper, by combining the robust RLS Wiener filter [26] and the RLS Wiener fixed-lag smoothing algorithm [25], proposes the robust RLS Wiener fixed-lag smoothing algorithm in Theorem 1. In the robust estimation problem, it is assumed that the system and observation matrices include some uncertain parameters. With the observations generated by the state-space model including the uncertain parameters, the robust RLS Wiener fixed-lag smoother estimates the signal recursively as the time advances. Both the signal and the degraded signal processes are fitted to the finite order auto-regressive (AR) models. The robust RLS Wiener fixed-lag smoother uses the following information. (1) The covariance function of the state for the degraded signal. (2) The cross-covariance function of the state for the signal with the state for the degraded signal. (3) The observation matrices for the signal and the degraded signal. (4) The system matrices for the signal and the degraded signal. (5) The variance of the white observation noise.

A numerical simulation example shows that the robust RLS Wiener fixed-lag smoother, proposed in this paper, is superior in estimation accuracy to the H-infinity RLS Wiener fixed-point smoother [23] and the RLS Wiener fixed-lag smoother [25].

In the appendix, by using MAXIMA and MATLAB, the derivation method of the coefficients, used in the robust RLS Wiener fixed-lag smoothing algorithm, is shown.

2. Robust least-squares fixed-lag smoothing problem

Let an m-dimensional observation equation and an n-dimensional state equation be described by

$$\begin{aligned} &\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \bar{H}(k)\bar{x}(k), \bar{H}(k) = H + \Delta H(k), \\ &\bar{x}(k+1) = \bar{\Phi}(k)\bar{x}(k) + \Gamma w(k), \bar{\Phi}(k) = \Phi + \Delta\Phi(k), \\ &E[v(k)v^T(s)] = R\delta_K(k-s), E[w(k)w^T(s)] = Q\delta_K(k-s) \end{aligned} \quad (1)$$

in linear discrete-time stochastic systems with uncertain parameters [20]. It is assumed that $\Delta H(k)$ and $\Delta\Phi(k)$ contain uncertain parameters respectively. Here, $v(k)$ is the white observation noise with the variance R. $w(k)$ is the white input noise with the variance Q. Their auto-covariance functions are expressed with the Kronecker delta function $\delta_K(k-s)$. The state equation, which generates $\bar{x}(k+1)$, contains the uncertain quantity $\Delta\Phi(k)$ in the system matrix $\bar{\Phi}(k)$. In addition, in the observation equation the observation matrix $\bar{H}(k)$ contains the uncertain quantity $\Delta H(k)$. Hence, $\breve{z}(k)$ is deviated from the nominal signal $z(k)$ in the state-space model (2), which does not contain the uncertain quantities. In (1), as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$, the observed value $\breve{y}(k)$ is measured. The state-space model without containing the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ in (1) is described by

$$\begin{aligned} &y(k) = z(k) + v(k), z(k) = Hx(k), \\ &x(k+1) = \Phi x(k) + \Gamma w(k). \end{aligned} \quad (2)$$

In (2), $z(k)$ represents the signal to be estimated. H an m by n observation matrix, $x(k)$ the state vector and $v(k)$ the white observation noise with the auto-covariance function given in (1). The auto-covariance function of the input noise $w(k)$ is also given in (1). It is assumed that the signal and the observation noise are zero-mean mutually independent stochastic processes. The purpose of this paper is to design the RLS Wiener fixed-lag smoother to estimate the signal $z(k)$ with the observed value $\breve{y}(k)$ without using any information on the uncertain quantities $\Delta\Phi(k)$ and $\Delta H(k)$.

Let the degraded signal $\breve{z}(k)$ be fitted to the Nth order AR model as

$$\begin{aligned} &\breve{z}(k) = -a_1\breve{z}(k-1) - a_2\breve{z}(k-2) \cdots - a_N\breve{z}(k-N) + \breve{e}(k), \\ &E[\breve{e}(k)\breve{e}^T(s)] = \breve{Q}\delta_K(k-s). \end{aligned} \quad (3)$$

Let $\breve{z}(k)$ be expressed by

$$\begin{aligned} &\breve{z}(k) = \breve{H}\breve{x}(k), \\ &\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix}, \\ &\breve{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]. \end{aligned} \quad (4)$$

It is seen that the state equation for the state vector $\breve{x}(k)$ is described by

$$\begin{bmatrix} \breve{x}_1(k+1) \\ \breve{x}_2(k+1) \\ \vdots \\ \breve{x}_{N-1}(k+1) \\ \breve{x}_N(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix} \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix} \zeta(k), \zeta(k) = \breve{e}(k+N), E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s). \tag{5}$$

Let $\breve{K}(k,s) = \breve{K}(k-s)$ represent the auto-covariance function of the state vector $\breve{x}(k)$ in wide-sense stationary stochastic systems [28]. Hence, $\breve{K}(k,s)$ has the form of

$$\breve{K}(k,s) = \begin{cases} A(k)B^T(s), 0 \le s \le k, \\ B(k)A^T(s), 0 \le k \le s, \end{cases} \tag{6}$$

$A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s)$. Here, $\breve{\Phi}$ is the system matrix for the state vector $\breve{x}(k)$. The system matrix $\breve{\Phi}$ in the state equation (5) is written as

$$\breve{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix}. \tag{7}$$

Also, by putting $K_{\breve{z}}(k,s) = K_{\breve{z}}(k-s) = E[\breve{z}(k)\breve{z}^T(s)]$, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is described by

$$\begin{aligned} \breve{K}(k,k) &= E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \times [\breve{z}^T(k) \quad \breve{z}^T(k+1) \quad \cdots \quad \breve{z}^T(k+N-2) \quad \breve{z}^T(k+N-1)]\right] \\ &= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \tag{8}$$

Using $K_{\breve{z}}(k-s)$, we have the Yule-Walker equation for the AR parameters as

$$\begin{aligned} &\hat{K}(k,k) \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_{N-1}^T \\ a_N^T \end{bmatrix} = - \begin{bmatrix} K_{\breve{z}}^T(1) \\ K_{\breve{z}}^T(2) \\ \vdots \\ K_{\breve{z}}^T(N-1) \\ K_{\breve{z}}^T(N) \end{bmatrix}, \\ &\hat{K}(k,k) = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \tag{9}$$

Let $K_{x\breve{x}}(k,s) = K_{x\breve{x}}(k-s) = E[x(k)\breve{x}^T(s)]$ represent the cross-covariance function of the state vector $x(k)$ with $\breve{x}(s)$ in wide-sense stationary stochastic systems. $K_{x\breve{x}}(k,s)$ is expressed in the form of

$$K_{x\breve{x}}(k,s) = \alpha(k)\beta^T(s), 0 \le s \le k, \tag{10}$$

$\alpha(k) = \Phi^k, \beta^T(s) = \Phi^{-s}K_{x\breve{x}}(s,s)$. Here, Φ is the system matrix for the state vector $x(k)$.

Let the fixed-lag smoothing estimate $\hat{x}(k-L,k)$ of the state vector $x(k-L)$ be expressed by

$$\hat{x}(k-L,k) = \sum_{i=1}^{k} h(k,i)\breve{y}(i) \quad (11)$$

in terms of the observed values $\{\breve{y}(i), 1 \le i \le L\}$. In (11), $h(k,i)$ donotes a time-varying impulse response function and L the fixed lag. We consider the estimation problem, which minimizes the mean-square value (MSV)

$$J = E[||x(k-L) - \hat{x}(k-L,k)||^2] \quad (12)$$

of the fixed-lag smoothing error. From an orthogonal projection lemma [28],

$$x(k-L) - \sum_{i=1}^{k} h(k,i)\breve{y}(i) \perp \breve{y}(s), 1 \le s \le k, \quad (13)$$

the impulse response function satisfies the Wiener-Hopf equation

$$E[x(k-L)\breve{y}^T(s)] = \sum_{i=1}^{k} h(k,i)E[\breve{y}(i)\breve{y}^T(s)]. \quad (14)$$

Here '⊥' denotes the notation of the orthogonality. Substituting (1) into (14), from (4) and (8), and using $E[x(k-L)\breve{y}^T(s)] = K_{x\breve{z}}(k-L,s) = K_{x\breve{x}}(k-L,s)\breve{H}^T$, we obtain

$$h(k,s)R = K_{x\breve{x}}(k-L,s)\breve{H}^T - \sum_{i=1}^{k-L} h(k,i)\breve{H}\breve{K}(i,s)\breve{H}^T. \quad (15)$$

Here, $K_{x\breve{z}}(k-L,s)$ represent the cross-covariance function of the state vector $x(k-L)$ with the degraded signal $\breve{z}(s)$, $E[x(k)\breve{z}^T(s)]$.

2.1 Auto-regressive model for the signal process

Let the signal process be modeled in terms of the Nth order AR model

$$z(k) = -a_1 z(k-1) - a_2 z(k-2) - \cdots - a_N z(k-N) + w(k). \quad (16)$$

It is seen that the observation matrix H and the state equation for the state vector $x(k)$ in (2) are given by

$$H = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0], \quad (17)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{N-1}(k+1) \\ x_N(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{N-1}(k) \\ x_N(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix} w(k), \Gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix}, E[w(k)w(s)] = Q\delta_K(k-s). \quad (18)$$

In (15), the function $K_{x\breve{x}}(k-L,s)$, $1 \le s \le k-L$, cannot be expressed in the form of (10) explicitly. The Robust RLS Wiener filtering algorithm [26] is obtained for $L=0$. The robust RLS Wiener fixed-lag smoothing estimate $\hat{z}(k-L,k)$ of $z(k-L)$ is calculated by (26). Let us consider how to get the coefficients $\bar{a}_{1,N}$, $\bar{a}_{2,N}$, $\bar{a}_{3,N}$, $\bar{a}_{L+1,N}$ in the followings.

From (16), we have

$$z(k+N-1) = -a_1 z(k+N-2) - a_2 z(k+N-3) - \cdots - a_N z(k-1) + w(k+N-1) \quad (19)$$

From (19) and the relationship $K(k,s)=E[z(k)z(s)]$, it follows that

$$K(k+N-1,s)=-a_1K(k+N-2,s)-a_2K(k+N-3,s)-\cdots-a_NK(k-1,s). \quad (20)$$

From (20), we have

$$K(k-1,s)=(K(k+N-1,s)+a_1K(k+N-2,s)+a_2K(k+N-3,s)+\cdots+a_{N-1}K(k,s))/(-a_N). \quad (21)$$

Similarly, the following equations hold.

$$K(k-2,s)=(K(k+N-2,s)+a_1K(k+N-3,s)+a_2K(k+N-4,s)+\cdots+a_{N-1}K(k-1,s))/(-a_N) \quad (22)$$

$$K(k-3,s)=(K(k+N-3,s)+a_1K(k+N-4,s)+a_2K(k+N-5,s)+\cdots+a_{N-1}K(k-2,s))/(-a_N) \quad (23)$$

....

$$K(k-(N-1),s)=(K(k+1,s)+a_1K(k,s)+a_2K(k-1,s)+\cdots+a_{N-1}K(k-(N-2),s))/(-a_N) \quad (24)$$

It is clear that $K(k-(N-1),s)$ is calculated by successive substitutions of $K(k-1,s)$ into (22), $K(k-2,s)$ into (23), and finally by substituting $K(k-(N-2),s)$ into (24).

As a result $K(k-L,s)$, $1\le L\le N-1$, are obtained as follows from the above iterative substitutions [25]

$$K(k-L,s)=\bar{a}_{1,N}K(k,s)+\bar{a}_{2,N}K(k+1,s)+\bar{a}_{3,N}K(k+2,s)+\cdots+\bar{a}_{L+1,N}K(k+L,s). \quad (25)$$

in terms of the newly introduced parameters, $\bar{a}_{i,N}$, $1\le i\le N$. In the derivations of the RLS Wiener fixed-lag smoother, the relationship (25) is available. The derivations of the parameters, $\bar{a}_{i,N}$, $1\le i\le N$, by MAXIMA and MATLAB, are summarized in the appendix. It should be noted, for the Nth order AR model, that the fixed-lag smoothing estimates $\hat{z}(k-L,k)$, $1\le L\le N-1$, can be calculated.

3. Robust RLS Wiener fixed-lag smoothing and filtering algorithms

Under the linear least-squares estimation problem of the signal $z(k)$ in section 2, Theorem 1 presents the robust RLS Wiener fixed-lag smoothing and filtering algorithms.

Theorem 1 [25], [26] Let the state equation and the observation equation, including the uncertain quantities $\Delta\Phi$ and ΔH respectively, be given by (1). Let Φ and H represent the system and observation matrices respectively for the signal $z(k)$. Let $\breve{\Phi}$ and $\breve{H}$ represent the system and observation matrices respectively for the degraded signal $\breve{z}(k)$, fitted to the AR model (3) of the order N. Let the variance $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and the cross-variance function $K_{x\breve{x}}(k,k)$ of the state vector $x(k)$ for the signal $z(k)$ with the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ be given. Let the variance of the white observation noise $v(k)$ be R. Then, the robust RLS Wiener algorithms for the fixed-lag smoothing estimate $\hat{z}(k-L,k)$ of the signal $z(k-L)$ and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (26)-(33) in linear discrete-time stochastic systems.

Robust RLS Wiener fixed-lag smoothing estimate of $z(k-L)$: $\hat{z}(k-L,k)$

$$\hat{z}(k-L,k)=(\bar{a}_{1,N}H+\bar{a}_{2,N}H\Phi+\bar{a}_{3,N}H\Phi^2+\cdots+\bar{a}_{L+1,N}H\Phi^L)\hat{x}(k,k) \quad (26)$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k)=H\hat{x}(k,k) \quad (27)$$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$$\hat{x}(k,k)=\Phi\hat{x}(k-1,k-1)+G(k)(\breve{y}(k)-\breve{H}\breve{\Phi}\hat{x}(k-1,k-1)),\quad \hat{x}(0,0)=0 \quad (28)$$

Filter gain for $\hat{x}(k,k)$ in the equation (28): $G(k)$

$$G(k) = [K_{x\breve{z}}(k,k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T] \times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}, \quad K_{x\breve{z}}(k,k) = K_{x\breve{x}}(k,k)\breve{H}^T \tag{29}$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad \hat{\breve{x}}(0,0) = 0 \tag{30}$$

Filter gain for $\hat{\breve{x}}(k,k)$ in the equation (30): $g(k)$

$$g(k) = [\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T] \times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1} \tag{31}$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T], \quad S_0(0) = 0 \tag{32}$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S(k) = \Phi S(k-1)\breve{\Phi}^T + G(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T], \quad S(0) = 0 \tag{33}$$

For the stability of the filtering and fixed-lag smoothing algorithms, the following conditions are necessary.

1. All the real parts in the eigenvalues of the matrix $\breve{\Phi} - g(k)\breve{H}\breve{\Phi}$ are negative.
2. $R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T > 0$

The fixed-lag smoothing error variance function of the signal is shown in section 4.

4. Fixed-lag smoothing error variance function of signal

In this section the existence of the fixed-lag smoothing estimate $\hat{z}(k-L,k)$ is shown. The variance function $\tilde{P}_z(k-L)$ of the fixed-lag smoothing error $z(k-L) - \hat{z}(k-L,k)$ is formulated as

$$\tilde{P}_z(k-L) = E[(z(k-L) - \hat{z}(k-L,k))(z(k-L) - \hat{z}(k-L,k))^T]. \tag{34}$$

(34) might be written as

$$\begin{aligned}&\tilde{P}_z(k-L) = K(k-L,k-L) - P_z(k-L),\\ &P_z(k-L) = E[\hat{z}(k-L,k)\hat{z}^T(k-L,k)]\\ &= (\bar{a}_{1,L}H + \bar{a}_{2,L}H\Phi + \cdots + \bar{a}_{N,L}H\Phi^{L-1)})E[\hat{x}(k,k)\hat{x}^T(k,k)](\bar{a}_{1,L}H\\ &+\bar{a}_{2,L}H\Phi + \cdots + \bar{a}_{N,L}H\Phi^{L-1)})^T\end{aligned} \tag{35}$$

Since $\tilde{P}_z(k-L) \geq 0$ and the variance $P_z(k-L)$ of the fixed-lag smoothing estimate $\hat{z}(k-L,k)$ satisfies

$$P_z(k-L) \geq 0, \tag{36}$$

it is shown that

$$0 \leq P_z(k-L) \leq K(k-L,k-L) \tag{37}$$

is valid. (37) indicates that the variance of the fixed-lag smoothing error is upper bounded by the variance of the signal and lower bounded by zero matrix. This validates the existence of the robust fixed-lag smoothing estimate $\hat{z}(k-L,k)$ of the signal $z(k-L)$.

In section 5, The estimation accuracy of the proposed RLS Wiener fixed-lag smoother is compared with the H-infinity RLS Wiener fixed-point smoother [23] and the RLS Wiener fixed-lag smoother [25] from the numerical aspect.

5. A numerical simulation example

Let a scalar observation equation and the state equation for $x(k)$ be given by

$$
\begin{aligned}
&y(k)=z(k)+v(k), z(k)=Hx(k), H=[1 \quad 0 \quad 0 \quad \cdots \quad 0],\\
&x(k+1)=\Phi x(k)+\Gamma w(k), x(k)=\begin{bmatrix} x_1(k)\\ x_2(k)\\ \vdots\\ x_{N-1}(k)\\ x_N(k)\end{bmatrix},\\
&\Phi=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1\end{bmatrix}, \Gamma=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix},\\
&a_1=-0.6135, a_2=0.1635, a_3=-1.2912, a_4=0.4335, a_5=-0.6697,\\
&a_6=0.7693, a_7=0.0800, a_8=0.6141, a_9=-0.1770, a_{10}=-0.3007, N=10,\\
&E[v(k)v(s)]=R\delta_K(k-s), E[w(k)w(s)]=Q\delta_K(k-s), Q=2.1727.
\end{aligned} \tag{38}
$$

The observation noise $v(k)$ is a zero-mean white noise process. Let us consider to estimate a vowel signal spoken by the author. Its phonetic symbol is expressed as "$/i\text{:}/$." The sampling frequency of the voice signal is 10.025[kHz]. The auto-covariance data of the signal is calculated in terms of 5,000 sampled signal data. Let the process of the signal $z(k)$ is fitted to the AR model of the order 10 in (16) for $m=1$. The 1×10 observation vector, the state equation for the state vector $x(k)$ and the system matrix Φ are given in (38). Let the state-space model containing the uncertain quantity $\Delta\Phi(k)$ be described by

$$
\begin{aligned}
&\breve{y}(k)=\breve{z}(k)+v(k), \breve{z}(k)=H(k)\breve{x}(k), \breve{x}(k)=\begin{bmatrix} \breve{x}_1(k)\\ \breve{x}_2(k)\\ \vdots\\ \breve{x}_{N-1}(k)\\ \breve{x}_N(k)\end{bmatrix},\\
&\breve{x}(k+1)=\breve{\Phi}(k)\breve{x}(k)+\Gamma w(k), \breve{\Phi}(k)=\Phi+\Delta\Phi(k),\\
&\Delta\Phi(k)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & 0 & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\\ \Delta_N(k) & \Delta_{N-1}(k) & \Delta_{N-2}(k) & \cdots & \Delta_1(k)\end{bmatrix},\\
&\Delta_1(k)=0.0220, \Delta_2(k)=0.0083, \Delta_3(k)=-0.0623, \Delta_4(k)=-0.0196,\\
&\Delta_5(k)=-0.0092, \Delta_6(k)=0.0506, \Delta_7(k)=0.0575, \Delta_8(k)=0.0147,\\
&\Delta_9(k)=-0.0258, \Delta_{10}(k)=-0.0362, N=10,
\end{aligned} \tag{39}
$$

in linear discrete-time stochastic systems. It should be noted that the uncertain quantity $\Delta\Phi(k)$ is unknown. It is a task to estimate the signal $z(k)$ recursively in terms of the observed value $\breve{y}(k)$, which is given as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$. Let $\breve{z}(k)$ be also fitted to the N-th order AR model of

$$
\begin{aligned}
&\breve{z}(k)=-\breve{a}_1\breve{z}(k-1)-\breve{a}_2\breve{z}(k-2)-\cdots-\breve{a}_N\breve{z}(k-N)+\breve{e}(k),\\
&E[\breve{e}(k)\breve{e}(s)]=\breve{Q}\delta_K(k-s), N=10.
\end{aligned} \tag{40}
$$

In this example, the state equation for $\breve{x}(k)$, given by (5), corresponds to the case of $m=1$. The relationship $\breve{K}(k,s)=\breve{K}(k-s)$ represents the auto-covariance function of the state vector $\breve{x}(k)$ in wide-sense stationary stochastic systems. $\breve{K}(k,s)$ is expressed in the form of the semi-degenerate function (6). $\breve{\Phi}$ represents the system matrix for the state vector $\breve{x}(k)$. $\breve{\Phi}$ is given by (7). Also, from $K_{\breve{z}}(k-s)=K_{\breve{z}}(s-k)=E[\breve{z}(k)\breve{z}(s)]$ for the scalar degraded signal $\breve{z}(k)$, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is expressed as

$$\breve{K}(k,k)=E\left[\begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}[\breve{z}(k)\quad \breve{z}(k+1)\quad \cdots\quad \breve{z}(k+N-2)\quad \breve{z}(k+N-1)]\right]$$

$$=\begin{bmatrix}K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0)\end{bmatrix}. \quad (41)$$

Let $K_{z\breve{z}}(k,s)=E[z(k)\breve{z}(s)]$ represent the cross-covariance function between the signal $z(k)$ and the degraded signal $\breve{z}(s)$. From (4) (38) and (39), the cross-covariance function $K_{x\breve{x}}(k,s)$ is expressed as

$$K_{x\breve{x}}(k,s)=\Phi^{k-s}K_{x\breve{x}}(s,s), 0\le s\le k,$$

$$K_{x\breve{x}}(k,k)=E\left[\begin{bmatrix}x_1(k)\\ x_2(k)\\ \vdots\\ x_{N-1}(k)\\ x_N(k)\end{bmatrix}[\breve{z}(k)\quad \breve{z}(k+1)\quad \cdots\quad \breve{z}(k+N-2)\quad \breve{z}(k+N-1)]\right]$$

$$=\begin{bmatrix}E[x_1(k)\breve{z}(k)] & E[x_1(k)\breve{z}(k+1)] & \cdots & E[x_1(k)\breve{z}(k+N-2)] & E[x_1(k)\breve{z}(k+N-1)]\\ E[x_2(k)\breve{z}(k)] & E[x_2(k)\breve{z}(k+1)] & \cdots & E[x_2(k)\breve{z}(k+N-2)] & E[x_2(k)\breve{z}(k+N-1)]\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ E[x_{N-1}(k)\breve{z}(k)] & E[x_{N-1}(k)\breve{z}(k+1)] & \cdots & E[x_{N-1}(k)\breve{z}(k+N-2)] & E[x_{N-1}(k)\breve{z}(k+N-1)]\\ E[x_N(k)\breve{z}(k)] & E[x_N(k)\breve{z}(k+1)] & \cdots & E[x_N(k)\breve{z}(k+N-2)] & E[x_N(k)\breve{z}(k+N-1)]\end{bmatrix}$$

$$=\begin{bmatrix}E[z(k)\breve{z}(k)] & E[z(k)\breve{z}(k+1) & \cdots & E[z(k)\breve{z}(k+N-2)] & E[z(k)\breve{z}(k+N-1)]\\ E[z(k+1)\breve{z}(k)] & E[z(k+1)\breve{z}(k+1) & \cdots & E[z(k+1)\breve{z}(k+N-2)] & E[z(k+1)\breve{z}(k+N-1)]\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ E[z(k+N-2)\breve{z}(k)] & E[z(k+N-2)\breve{z}(k+1)] & \cdots & E[z(k+N-2)\breve{z}(k+N-2)] & E[z(k+N-2)\breve{z}(k+N-1)]\\ E[z(k+N-1)\breve{z}(k)] & E[z(k+N-1)\breve{z}(k+1)] & \cdots & E[z(k+N-1)\breve{z}(k+N-2)] & E[z(k+N-1)\breve{z}(k+N-1)]\end{bmatrix} \quad (42)$$

$$=\begin{bmatrix}K_{z\breve{z}}(k,k) & K_{z\breve{z}}(k,k+1) & \cdots & K_{z\breve{z}}(k,k+N-2) & K_{z\breve{z}}(k,k+N-1)\\ K_{z\breve{z}}(k+1,k) & K_{z\breve{z}}(k+1,k+1) & \cdots & K_{z\breve{z}}(k+1,k+N-2) & K_{z\breve{z}}(k+1,k+N-1)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{z\breve{z}}(k+N-2,k) & K_{z\breve{z}}(k+N-2,k+1) & \cdots & K_{z\breve{z}}(k+N-2,k+N-2) & K_{z\breve{z}}(k+N-2,k+N-1)\\ K_{z\breve{z}}(k+N-1,k) & K_{z\breve{z}}(k+N-1,k+1) & \cdots & K_{z\breve{z}}(k+N-1,k+N-2) & K_{z\breve{z}}(k+N-1,k+N-1)\end{bmatrix}.$$

The AR parameters $\breve{a}_1,\breve{a}_2,\cdots,\breve{a}_{N-1},\breve{a}_N$ in (40) are calculated by the Yule-Walker equation

$$\begin{bmatrix} K_{\check{z}}(0) & K_{\check{z}}(1) & \cdots & K_{\check{z}}(N-2) & K_{\check{z}}(N-1) \\ K_{\check{z}}(1) & K_{\check{z}}(0) & \cdots & K_{\check{z}}(N-3) & K_{\check{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\check{z}}(N-2) & K_{\check{z}}(N-3) & \cdots & K_{\check{z}}(0) & K_{\check{z}}(1) \\ K_{\check{z}}(N-1) & K_{\check{z}}(N-2) & \cdots & K_{\check{z}}(1) & K_{\check{z}}(0) \end{bmatrix} \begin{bmatrix} \check{a}_1 \\ \check{a}_2 \\ \vdots \\ \check{a}_{N-1} \\ \check{a}_N \end{bmatrix} = \begin{bmatrix} -K_{\check{z}}(1) \\ -K_{\check{z}}(2) \\ \vdots \\ -K_{\check{z}}(N-1) \\ -K_{\check{z}}(N) \end{bmatrix}.$$

The coefficients $\bar{a}_{i,N}$, $1 \le i \le N$, $N = 10$, in (26) for the robust RLS Wiener fixed-lag smoothing estimate are shown in the appendix. The coefficients are listed for the fixed lag L, $0 \le L \le 9$. Substituting H, $\breve{H}$, Φ, $\breve{\Phi}$, $K_{x\breve{x}}(k,k)$, $\breve{K}(k,k) = \breve{K}(L,L)$, the coefficients and R into the robust RLS Wiener fixed-lag smoothing and filtering algorithms of Theorem 1, the fixed-lag smoothing and filtering estimates are calculated recursively. In evaluating $\breve{\Phi}$ in (7) for $m = 1$, $\breve{K}(k,k)$ in (41) and $K_{x\breve{x}}(k,k)$ in (42), 2,000 number of the signal and degraded signal data are used. Fig.1 illustrates the signal process $z(k)$ and its degraded signal $\breve{z}(k)$ by the uncertain parameters in the system matrix vs. k. In comparison with the signal process, the degraded signal is influenced by the uncertain parameters in the system matrix $\breve{\Phi}(k)$ in (39). Fig.2 illustrates the signal and the fixed-lag smoothing estimate $\hat{z}(k,k+5)$ of the signal $z(k)$ vs. k for the white Gaussian observation noise with the signal-to-noise ratio (SNR) 10 [dB]. Fig.3 illustrates the mean-square values (MSVs) of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-lag smoothing errors $z(k) - \hat{z}(k,k+L)$ vs. L, $0 \le L \le 9$, for SNR=3 [dB] and 5 [dB]. For the fixed lag, $L = 0$, the MSVs of the filtering errors $z(k) - \hat{z}(k,k)$, $1 \le k \le 1000$, are plotted. From Fig.3, it is seen that the robust RLS Wiener filter and fixed-lag smoother, proposed in this paper, are superior in estimation accuracy to the H-infinity RLS Wiener filter and fixed-point smoother [23], and the RLS Wiener filter and fixed-lag smoother [25]. Concerning the robust RLS Wiener fixed-lag smoother, proposed in this paper, the MSVs of the fixed-lag smoothing errors are smaller than those of the filtering errors and decease gradually as the fixed lag L increases. This shows that the proposed robust RLS Wiener fixed-lag smoother improves the estimation accuracy of the robust RLS Wiener filter as the lag increases. Here, the MSVs of the fixed-lag smoothing and filtering errors are evaluated by $\sum_{i=1}^{1000}(z(k) - \hat{z}(k,k+L))^2/1000$ and $\sum_{i=1}^{1000}(z(k) - \hat{z}(k,k))^2/1000$ respectively. In the above calculations, MATLABR12 is used.

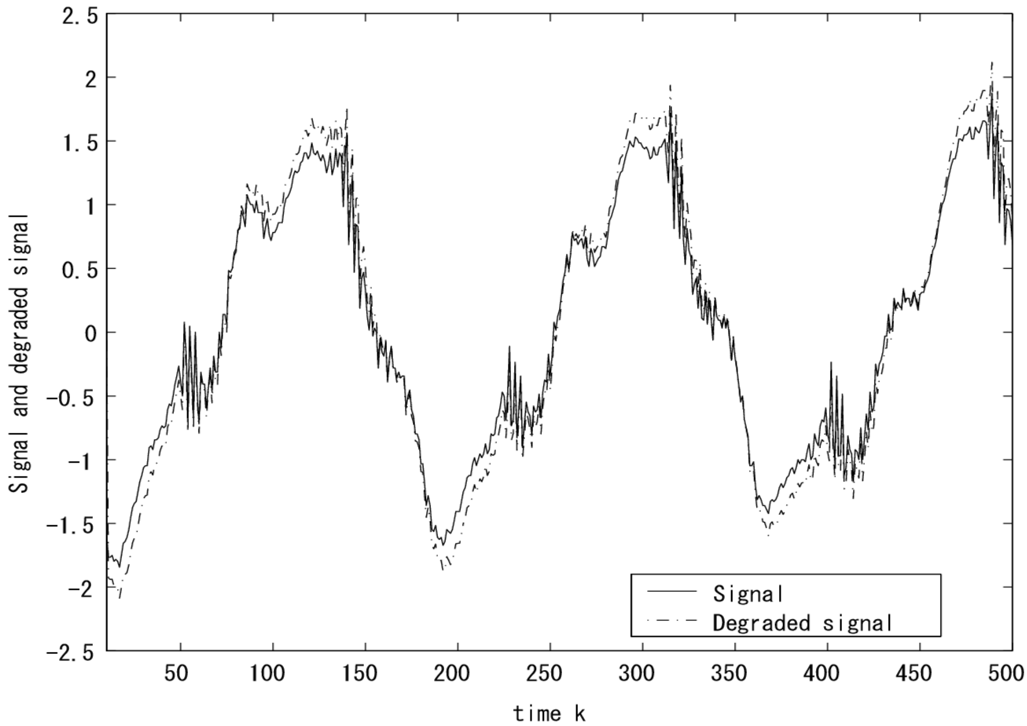

Fig. 1 Signal process $\boldsymbol{z(k)}$ and its degraded signal $\boldsymbol{\breve{z}(k)}$ by the uncertain parameters in the system matrix vs. $\boldsymbol{k}$.

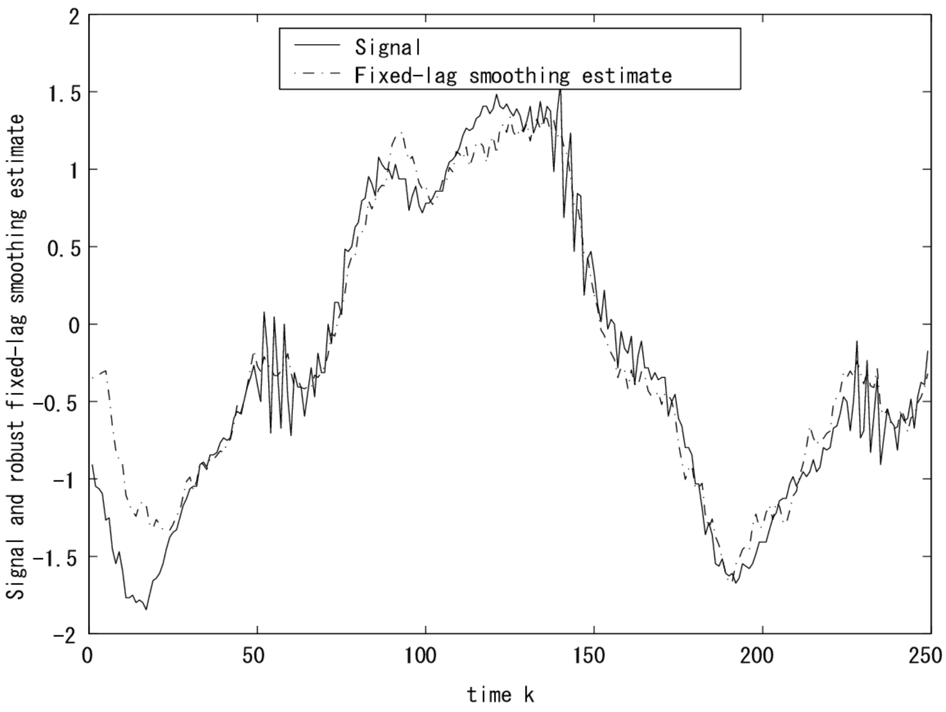

Fig. 2 Signal and the fixed-lag smoothing estimate $\hat{z}(k, k+5)$ of the signal $z(k)$ vs. k for the white Gaussian observation noise with the signal-to-noise ratio (SNR) 10 [dB].

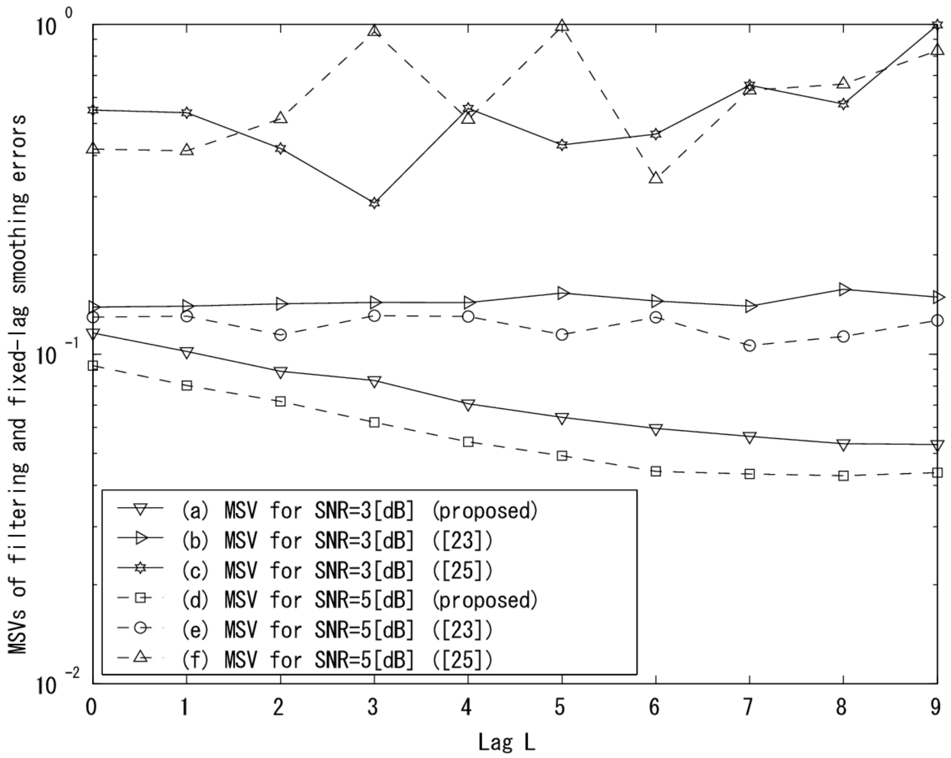

Fig. 3 Mean-square values of the filtering errors $z(k) - \hat{z}(k, k)$ and the fixed-lag smoothing errors $z(k) - \hat{z}(k, k+L)$ vs. L, $0 \le L \le 9$, by the H-infinity RLS Wiener fixed-point smoother [23] and the RLS Wiener fixed-lag smoother [25] for SNR=3 [dB] and 5 [dB].

6. Conclusions

This paper, by combining the robust RLS Wiener filter and the RLS Wiener fixed-lag smoothing algorithm, has proposed the robust RLS Wiener fixed-lag smoothing algorithm in Theorem 1. In the problem formulation of the robust estimation problem, it is assumed that the system and observation matrices include some uncertain parameters. With the observations generated by the state-space model including the uncertain parameters, the robust RLS Wiener fixed-lag smoother estimates the signal recursively as the time advances. Both the signal and the degraded signal processes are fitted to the finite order AR models. The robust RLS Wiener fixed-lag smoother uses the following information. (1) the covariance function of the state for the degraded signal. (2) The cross-covariance function of the state for the signal with the state for the degraded signal. (3) The observation matrices for the signal and the degraded signal. (4) The system matrices for the signal and the degraded signal. (5) The variance of the white observation noise.

A numerical simulation example has shown that the robust RLS Wiener fixed-lag smoother, proposed in this paper, is superior in estimation accuracy to the H-infinity RLS Wiener fixed-point smoother and the RLS Wiener fixed-lag smoother. Also, the MSV of the fixed-lag smoothing errors decreases as the fixed lag increases.

Appendix Derivation method of coefficients in (25) by MAXIMA and MATLAB

As an example, let us show the MAXIMA commands for the coefficients $\bar{a}_{i,N}, 1 \le i \le 10$, in the cases of the AR model order $N = 10$:

(%i1) km1: km1=(k9+a1*k8+a2*k7+a3*k6+a4*k5+a5*k4+a6*k3+a7*k2
+a8*k1+a9*k)/(-a10);
(%i2) km2: km2=(k8+a1*k7+a2*k6+a3*k5+a4*k4+a5*k3+a6*k2+a7*k1
+a8*k+a9*km1)/(-a10);
(%i3) ratsimp(km2);
(%i4) km3: km3=(k7+a1*k6+a2*k5+a3*k4+a4*k3+a5*k2+a6*k1+a7*k
+a8*km1+a9*km2)/(-a10);
(%i5) ratsimp(km3);

km1 for the fixed lag $L = 1$:
-1/a10*a9*k
-1/a10*a8*k1
-1/a10*a7*k2
-1/a10*a6*k3
-1/a10*a5*k4
-1/a10*a4*k5
-1/a10*a3*k6
-1/a10*a2*k7
-1/a10*a1*k8
-1/a10*k9

km2 for the fixed lag $L = 2$:after substitution of km1 into km2:
(a9 k9 + (a1 a9 - a10) k8 + (a2 a9 - a1 a10) k7
+ (a3 a9 - a10 a2) k6 + (a4 a9 - a10 a3) k5 + (a5 a9 - a10 a4) k4
+ (a6 a9 - a10 a5) k3 + (a7 a9 - a10 a6) k2 + (a8 a9 - a10 a7) k1
+ ($a9^2$ - a10 a8) k)/$a10^2$)

km3 for the fixed lag $L = 3$ after substitutions of km1 and km2 into km3:
- (($a9^2$ - a10 a8) k9 + (a1 $a9^2$ - a10 a9 - a1 a10 a8) k8
+ (a2 $a9^2$ - a1 a10 a9 - a10 a2 a8 + $a10^2$) k7
+ (a3 $a9^2$ - a10 a2 a9 - a10 a3 a8 + a1 $a10^2$) k6
+ (a4 $a9^2$ - a10 a3 a9 - a10 a4 a8 + $a10^2$ a2) k5
+ (a5 $a9^2$ - a10 a4 a9 - a10 a5 a8 + $a10^2$ a3) k4

+ (a6 $a9^2$ - a10 a5 a9 - a10 a6 a8 + $a10^2$ a4) k3
+ (a7 $a9^2$ - a10 a6 a9 - a10 a7 a8 + $a10^2$ a5) k2
+ (a8 $a9^2$ - a10 a7 a9 - a10 $a8^2$ + $a10^2$ a6) k1
+ ($a9^3$ - 2 a10 a8 a9 + $a10^2$ a7) k)/$a10^3$))

Here, a1=a_1, a2=a_2, a3=a_3, a4=a_4, a5=a_5, a6=a_6, a7=a_7, a8=a_8, a9=a_9, a10=a_{10}.

MATLAB program for the coefficients $\bar{a}_{i,N}$,$1 \le i \le 10$, in the case of the AR model order $N = 10$, by using Symbolic Math Toolbox:

```
clear variables
syms a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 k k1 k2 k3 k4 k5 k6 k7 k8 k9 km km1 km2 km3 km4 km5 km6 km7 km8
km9
km1=(k9+a1*k8+a2*k7+a3*k6+a4*k5+a5*k4+a6*k3+a7*k2+a8*k1
+a9*k)/(-a10)
expand(km1)
km2=(k8+a1*k7+a2*k6+a3*k5+a4*k4+a5*k3+a6*k2+a7*k1+a8*k
+a9*km1)/(-a10)
expand(km2)
km3=(k7+a1*k6+a2*k5+a3*k4+a4*k3+a5*k2+a6*k1+a7*k+a8*km1
+a9*km2)/(-a10)
expand(km3)
km4=(k6+a1*k5+a2*k4+a3*k3+a4*k2+a5*k1+a6*k+a7*km1+a8*km2
+a9*km3)/(-a10)
expand(km4)
km5=(k5+a1*k4+a2*k3+a3*k2+a4*k1+a5*k+a6*km1+a7*km2+a8*km3
+a9*km4)/(-a10)
expand(km5)
km6=(k4+a1*k3+a2*k2+a3*k1+a4*k+a5*km1+a6*km2+a7*km3+a8*km4
+a9*km5)/(-a10)
expand(km6)
km7=(k3+a1*k2+a2*k1+a3*k+a4*km1+a5*km2+a6*km3+a7*km4+a8*km5
+a9*km6)/(-a10)
expand(km7)
km8=(k2+a1*k1+a2*k+a3*km1+a4*km2+a5*km3+a6*km4+a7*km5+a8*km6
+a9*km7)/(-a10)
expand(km8)
km9=(k1+a1*k+a2*km1+a3*km2+a4*km3+a5*km4+a6*km5+a7*km6+a8*km7
+a9*km8)/(-a10)
expand(km9)
```

In the case of the fixed lag $L = 1$, km1 is written as follows.
$-1/a10*a9*k$
$-1/a10*a8*k1$
$-1/a10*a7*k2$
$-1/a10*a6*k3$
$-1/a10*a5*k4$
$-1/a10*a4*k5$
$-1/a10*a3*k6$
$-1/a10*a2*k7$
$-1/a10*a1*k8$
$-1/a10*k9$

Here, k=$K(0)$, k1=$K(1)$, k2=$K(2)$, k3=$K(3)$, k4=$K(4)$, k5=$K(5)$, k6=$K(6)$, k7=$K(7)$, k8=$K(8)$, k9=$K(9)$.

For the fixed lag $L = 1$, the coefficients $\bar{a}_{i,N}$, $i = 1,2,\ldots,10$, $N = 10$, in (25) are given by

$\bar{a}_{1,N}$ =−1/a10∗a9
$\bar{a}_{2,N}$ =−1/a10∗a8
$\bar{a}_{3,N}$ =−1/a10∗a7
$\bar{a}_{4,N}$ =−1/a10∗a6
$\bar{a}_{5,N}$ =−1/a10∗a5
$\bar{a}_{6,N}$ =−1/a10∗a4
$\bar{a}_{7,N}$ =−1/a10∗a3
$\bar{a}_{8,N}$=−1/a10∗a2
$\bar{a}_{9,N}$ =−1/a10∗a1
$\bar{a}_{10,N}$ =−1/a10

In a similar manner, for the fixed lag, $2 \le L \le 10$, the coefficients $\bar{a}_{i,N}$, $i = 1,2,\ldots,10$, $N = 10$, in (25) are obtained by computing the following equations for km2, km3, km4, km5, km6, km7, km8 and km9 in the case of the fixed lag $L = 2$, $L = 3$,..., $L = 9$ respectively. In the followings, the coefficients for the fixed lag, $2 \le L \le 4$, are shown as an example.

km1=(k9+a1∗k8+a2∗k7+a3∗k6+a4∗k5+a5∗k4+a6∗k3+a7∗k2+a8∗k1+a9∗k)/(−a10)
km2=(k8+a1∗k7+a2∗k6+a3∗k5+a4∗k4+a5∗k3+a6∗k2+a7∗k1+a8∗k+a9∗km1)/(−a10)
km3=(k7+a1∗k6+a2∗k5+a3∗k4+a4∗k3+a5∗k2+a6∗k1+a7∗k+a8∗km1+a9∗km2)/(−a10)
km4=(k6+a1∗k5+a2∗k4+a3∗k3+a4∗k2+a5∗k1+a6∗k+a7∗km1+a8∗km2+a9∗km3)/(−a10)
km5=(k5+a1∗k4+a2∗k3+a3∗k2+a4∗k1+a5∗k+a6∗km1+a7∗km2+a8∗km3+a9∗km4)/(−a10)
km6=(k4+a1∗k3+a2∗k2+a3∗k1+a4∗k+a5∗km1+a6∗km2+a7∗km3+a8∗km4+a9∗km5)/(−a10)
km7=(k3+a1∗k2+a2∗k1+a3∗k+a4∗km1+a5∗km2+a6∗km3+a7∗km4+a8∗km5+a9∗km6)/(−a10)
km8=(k2+a1∗k1+a2∗k+a3∗km1+a4∗km2+a5∗km3+a6∗km4+a7∗km5+a8∗km6+a9∗km7)/(−a10)
km9=(k1+a1∗k+a2∗km1+a3∗km2+a4∗km3+a5∗km4+a6∗km5+a7∗km6+a8∗km7+a9∗km8)/(−a10)

% Coefficients for the 10th order AR model in the case of the fixed lag $L = 2$:

$\bar{a}_{1,N}$ =−1/a10∗a8+1/a10^2∗a9^2
$\bar{a}_{2,N}$ =−1/a10∗a7+1/a10^2∗a9∗a8
$\bar{a}_{3,N}$ =−1/a10∗a6+1/a10^2∗a9∗a7
$\bar{a}_{4,N}$ =−1/a10∗a5+1/a10^2∗a9∗a6
$\bar{a}_{5,N}$ =−1/a10∗a4+1/a10^2∗a9∗a5
$\bar{a}_{6,N}$ =−1/a10∗a4+1/a10^2∗a9∗a5
$\bar{a}_{7,N}$ =−1/a10∗a2+1/a10^2∗a9∗a3
$\bar{a}_{8,N}$=−1/a10∗a1+1/a10^2∗a9∗a2
$\bar{a}_{9,N}$ =−1/a10+1/a10^2∗a9∗a1
$\bar{a}_{10,N}$ =+1/a10^2∗a9

% Coefficients for the 10th order AR model in the case of the fixed lag $L = 3$:

$\bar{a}_{1,N}$ =−1/a10^3∗a9^3+2/a10^2∗a8∗a9
$\bar{a}_{2,N}$ =1/a10^2∗a8^2−1/a10∗a6+1/a10^2∗a9∗a7−1/a10^3∗a9^2∗a8
$\bar{a}_{3,N}$ =−1/a10∗a5+1/a10^2∗a8∗a7+1/a10^2∗a9∗a6−1/a10^3∗a9^2∗a7
$\bar{a}_{4,N}$ =−1/a10∗a4+1/a10^2∗a8∗a6−1/a10^3∗a9^2∗a6+1/a10^2∗a9∗a5
$\bar{a}_{5,N}$ =−1/a10∗a3+1/a10^2∗a9∗a4+1/a10^2∗a8∗a5−1/a10^3∗a9^2∗a5
$\bar{a}_{6,N}$ =−1/a10∗a2+1/a10^2∗a9∗a3+1/a10^2∗a8∗a4−1/a10^3∗a9^2∗a4
$\bar{a}_{7,N}$ =−1/a10∗a1+1/a10^2∗a9∗a2+1/a10^2∗a8∗a3−1/a10^3∗a9^2∗a3
$\bar{a}_{8,N}$=−1/a10∗a7+1/a10^2∗a8∗a2+1/a10^2∗a9∗a1−1/a10^3∗a9^2∗a2−1/a10
$\bar{a}_{9,N}$ =1/a10^2∗a9+1/a10^2∗a8∗a1−1/a10^3∗a9^2∗a1
$\bar{a}_{10,N}$ =1/a10^2∗a8−1/a10^3∗a9^2

% Coefficients for the 10th order AR model in the case of the fixed lag $L = 4$:

$\bar{a}_{1,N}$ =+1/a10^4∗a9^4+1/a10^2∗a8^2−1/a10∗a6+2/a10^2∗a7∗a9−3/a10^3∗a8∗a9^2

$\bar{a}_{2,N}$ =−1/a10∗a5−1/a10^3∗a9^2∗a7+1/a10^4∗a9^3∗a8+2/a10^2∗a7∗a8+1/a10^2∗a9∗a6−2/a10^3∗a8^2∗a9

$\bar{a}_{3,N}$ = −1/a10∗a4+1/a10^2∗a7^2−2/a10^3∗a8∗a9∗a7+1/a10^2∗a8∗a6+1/a10^2∗a9∗a5+1/a10^4∗a9^3∗a7−1/a10^3∗a9^2∗a6

$\bar{a}_{4,N} =$ −1/𝑎10^3∗𝑎9^2∗𝑎5+1/𝑎10^2∗𝑎9∗𝑎4+1/𝑎10^2∗𝑎8∗𝑎5+1/𝑎10^2∗𝑎7∗𝑎6+1/𝑎10^4∗𝑎9^3∗𝑎6−1/𝑎10∗𝑎3−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎6

$\bar{a}_{5,N} =$ −1/𝑎10∗𝑎2−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎5−1/𝑎10^3∗𝑎9^2∗𝑎4+1/𝑎10^2∗𝑎8∗𝑎4+1/𝑎10^4∗𝑎9^3∗𝑎5+1/𝑎10^2∗𝑎7∗𝑎5+1/𝑎10^2∗𝑎9∗𝑎3

$\bar{a}_{6,N} =$ −1/𝑎10∗𝑎1+1/𝑎10^4∗𝑎9^3∗𝑎4−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎4+1/𝑎10^2∗𝑎8∗𝑎3+1/𝑎10^2∗𝑎9∗𝑎2−1/𝑎10^3∗𝑎9^2∗𝑎3+1/𝑎10^2∗𝑎7∗𝑎4

$\bar{a}_{7,N} =$ −1/𝑎10+1/𝑎10^2∗𝑎9∗𝑎1−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎3+1/𝑎10^4∗𝑎9^3∗𝑎3−1/𝑎10^3∗𝑎9^2∗𝑎2+1/𝑎10^2∗𝑎7∗𝑎3+1/𝑎10^2∗𝑎8∗𝑎2

$\bar{a}_{8,N}$=+1/𝑎10^2∗𝑎9−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎2−1/𝑎10^3∗𝑎9^2∗𝑎1+1/𝑎10^2∗𝑎7∗𝑎2+1/𝑎10^4∗𝑎9^3∗𝑎2+1/𝑎10^2∗𝑎8∗𝑎1

$\bar{a}_{9,N}$ =−1/𝑎10^3∗𝑎9^2+1/𝑎10^2∗𝑎8−2/𝑎10^3∗𝑎8∗𝑎9∗𝑎1+1/𝑎10^4∗𝑎9^3∗𝑎1+1/𝑎10^2∗𝑎7∗𝑎1

$\bar{a}_{10,N}$ =+1/𝑎10^2∗𝑎7+1/𝑎10^4∗𝑎9^3−2/𝑎10^3∗𝑎8∗𝑎9

The lengthy expressions of the coefficients $\bar{a}_{i,N}$,$1 \le i \le 10$, for the fixed lag, $5 \le L \le 9$, are omitted here.

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