The Lagrangean Hydrodynamic Representation of Dirac Equation

Piero Chiarelli

National Council of Research of Italy, Area of Pisa, 56124 Pisa, Moruzzi 1, Italy

Interdepartmental Center "E. Piaggio" University of Pisa

Abstract

This work derives the Lagrangean hydrodynamic representation of the Dirac field that, by using the minimum action principle in the non-Euclidean generalization, can possibly lead to the formulation of the Einstein equation as a function of the fermion field. The paper shows that the bi-spinor field $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is equivalent to the mass densities $|\psi_{\pm 1}|^2, |\psi_{\pm 2}|^2$, where $\psi_{\pm} = \frac{\psi_1 \pm \psi_2}{\sqrt{2}} = \begin{pmatrix} \psi_{\pm 1} \\ \psi_{\pm 2} \end{pmatrix} = \begin{pmatrix} |\psi_{\pm 1}| exp[i \frac{S_{\pm 1}}{\hbar}] \\ |\psi_{\pm 2}| exp[i \frac{S_{\pm 2}}{\hbar}] \end{pmatrix}$, that move with momenta $p_{\pm 1\mu} = -\partial_\mu S_{\pm 1}$ $p_{\pm 2\mu} = -\partial_\mu S_{\pm 2}$ being subject to the non-local interaction of the theory defined quantum potential. The expression of the quantum potential as well as of the hydrodynamic energy-impulse tensor is explicitly derived as a function of the charged fermion field. The generalization to the non-Minkowskian space-time with the definition of the gravitational equation for the classical Dirac field as well as its quantization is outlined.

Keywords: Hydrodynamic Form of Dirac Equation, Energy-Impulse Tensor of Charged Fermion Field, Gravity of Classical Charged Fermion Field

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1. Introduction

One of the intriguing problems of the theoretical physics is the integration of the general relativity with the quantum mechanics. The Einstein gravitation has a fully classical ambit, the quantum mechanics mainly concerns the small atomic or sub-atomic scale, and the fundamental interactions.

Many are the unexplained aspects of the matter on cosmological scale [1]. Even if the general relativity has opened some understanding about the cosmological dynamics [2-4], the complete explanation of generation of matter and its motion and distribution in the universe needs the integration of the cosmological physics with the quantum one [5-11]. Nevertheless, difficulties arise when one attempts to apply, to the force of gravity, the standard recipe of quantum field theories [12-13].

Recently, the author has shown [14] that by using the quantum hydrodynamic formalism [15-19] is possible to achieve a non-contradictory coupling of the boson field of the Klein-Gordon equation with the gravitational one via the derivation of the energy-impulse tensor [14] by using the hydrodynamic representation of the quantum mechanics.

Moreover, the work shows that the result is independent by the initial hydrodynamic model and that it can be expressed in term of the standard quantum formalism [14].

A first outcome of the resulting quantum-gravitational model shows that the energy of the quantum potential gives an important contribution to the space-time curvature and in the definition of the cosmological constant [14] leading to a value that agrees with the order of magnitude of the measured one.

Another measurable output of the quantum-gravitational theory is the detailed description of the gravitational field of antimatter. Many and discordant are the hypotheses on the gravitational features of the antimatter [20-25]. The hydrodynamic model allows the explicit calculation about the Newtonian weak gravity interaction between matter and antimatter [25].

Moreover, the theory shows that at the Planck scale, due to the repulsive force of the (non-local) quantum potential (that originates the uncertainty relations) the formation of a black hole with a mass smaller than that one of the Planck mass is forbidden [26].

The objective of this work is to derive the Lagrangean hydrodynamic representation. of the Dirac field that, by using the minimum action principle, can possibly lead, in the non-Euclidean generalization, to the formulation of the Einstein equation for the fermion field.

2. Results and Discussion

The section is organized as follows: in the sub-section 2.1, the hydrodynamic formulation of the Dirac equation is carried out; in sub-section 2.2, the Lagrangean quantum motion equations for the Dirac field are derived; in sub-section 2.3 the hydrodynamic energy-impulse tensor density (EITD) of the field is calculated and, in sub-section 2.4, the generalization to the non-Euclidean space-time as well as the analytical procedure for the definition of the gravitational equation for the Dirac field is outlined.

2.1. The hydrodynamic formulation of Dirac equation

The Dirac equation

$$\left(i\hbar\gamma^{\mu}\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)-mc\right)\Psi=0 \qquad (2.1.1)$$

where

$$\gamma^{\mu}=\left(\gamma^{0},\gamma^{i}\right)=\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\right), \qquad (2.1.2)$$

where $\sigma^{\mu}=(\sigma_0,\sigma_i)$ are the 4-D extended Pauli matrices [16], where

$$\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{2.1.3}$$

$$\overline{\Psi}=\Psi^{\dagger}\gamma^{0}, \tag{2.1.4}$$

As a function of the components ψ_k of the bi-spinor $\Psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ reads

$$\frac{i\hbar}{mc}\sigma^{0}\left(\partial_0+\frac{ie}{\hbar}A_0\right)\psi_1+\frac{i\hbar}{mc}\sigma_i\left(\partial_i+\frac{ie}{\hbar}A_i\right)\psi_2=\psi_1. \tag{2.1.5}$$

$$\frac{i\hbar}{mc}\sigma^{0}\left(\partial_0+\frac{ie}{\hbar}A_0\right)\psi_2+\frac{i\hbar}{mc}\sigma_i\left(\partial_i+\frac{ie}{\hbar}A_i\right)\psi_1=-\psi_2 \tag{2.1.6}$$

Moreover, by considering the linear combinations of (2.1.5- 6)

$$\frac{i\hbar}{mc}\sigma^{\mu}\left(\partial_{\mu}+\frac{ie}{\hbar}A_0\right)\psi_{+}=\psi_{-} \tag{2.1.7}$$

$$\frac{i\hbar}{mc}\tilde{\sigma}^{\mu}\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)\psi_{-}=\psi_{+} \tag{2.1.8}$$

where $\psi_{\pm}=\frac{\psi_1\pm\psi_2}{\sqrt{2}}$ and by substituting (2.1.7) in (2.1.8) it follows that

$$\tilde{\sigma}^{\nu}\sigma^{\mu}\left(\partial_{\nu}+\frac{ie}{\hbar}A_{\nu}\right)\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)\psi_{\pm}=-\frac{m^2c^2}{\hbar^2}\psi_{\pm}, \tag{2.1.9}$$

where $\tilde{\sigma}^{\mu}=(\tilde{\sigma}_0,\tilde{\sigma}_i)=(\sigma_0,-\sigma_i)=\sigma_{\mu}$. By splitting the matrix $\tilde{\sigma}^{\nu}\sigma^{\mu}$ in its symmetric and antisymmetric part

$$\tilde{\sigma}^{\nu}\sigma^{\mu}=g^{\mu\nu}+\alpha^{\mu\nu} \tag{2.1.10}$$

equation (2.1.9) leads to

$$\left[\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)\left(\partial^{\mu}+\frac{ie}{\hbar}A^{\mu}\right)+\left(\frac{m^2c^2}{\hbar^2}+\alpha^{\mu\nu}\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)\left(\partial_{\nu}+\frac{ie}{\hbar}A_{\nu}\right)\right)\right]\psi_{\pm}=0 \tag{2.1.11}$$

and, as shown in appendix A, to

$$\left[\left(\partial_{\mu}+\frac{ie}{\hbar}A_{\mu}\right)\left(\partial^{\mu}+\frac{ie}{\hbar}A^{\mu}\right)+\left(\frac{m^2c^2}{\hbar^2}+\frac{ie}{2\hbar}\alpha^{\mu\nu}F_{\mu\nu}\right)\right]\psi_{\pm}=0 \tag{2.1.12}$$

where

$$\alpha^{\mu\nu} = \begin{pmatrix} 0 & \tilde{\sigma}^0\sigma^1 & \tilde{\sigma}^0\sigma^2 & \tilde{\sigma}^0\sigma^3 \\ \tilde{\sigma}^1\sigma^0 & 0 & \tilde{\sigma}^1\sigma^2 & \tilde{\sigma}^1\sigma^3 \\ \tilde{\sigma}^2\sigma^0 & \tilde{\sigma}^2\sigma^1 & 0 & \tilde{\sigma}^2\sigma^3 \\ \tilde{\sigma}^3\sigma^0 & \tilde{\sigma}^3\sigma^1 & \tilde{\sigma}^3\sigma^2 & 0 \end{pmatrix}, \qquad (2.1.13)$$

where $F_{\mu\nu}$ is the electromagnetic (em) tensor [27]. The analougus of equation (2.1.12) in bi-spinor form is given by Birula et al. [16] and reads

$$\left(\partial_\mu - \frac{e}{i\hbar}A_\mu\right)\left(\partial^\mu - \frac{e}{i\hbar}A^\mu\right)\Psi = -\left(\frac{m^2c^2}{\hbar^2} + \frac{e}{\hbar}\sigma^{\mu\nu}F_{\mu\nu}\right)\Psi \qquad (2.1.14)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $A_\mu = (\frac{\phi}{c}, -A_i)$. (2.1.15)

Furthermore, by using the hydrodynamic notation

$$\psi_\pm \equiv \psi_{\pm i} = \begin{pmatrix} \psi_{\pm 1} \\ \psi_{\pm 2} \end{pmatrix} = \begin{pmatrix} |\psi_{\pm 1}| exp\left[i\frac{S_{\pm 1}}{\hbar}\right] \\ |\psi_{\pm 2}| exp\left[i\frac{S_{\pm 2}}{\hbar}\right] \end{pmatrix} \qquad (2.1.16)$$

and equating the real and imaginary part of the spinor KGE (2.1.12), it follows both the quantum hydrodynamic Hamilton-Jacobi motion equation [14,16] (see appendix B) that reads

$$\left(\partial_\mu S_{\pm i} + eA_\mu\right)\left(\partial^\mu S_{\pm i} + eA^\mu\right) = \left(m^2c^2 - mV_{qu\pm i}\right), \qquad (2.1.17)$$

where the spinorial italic index i=1,2 ; where $S_{\pm i} = \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}$ and where the quantum potential reads

$$V_{qu\pm i} = \begin{pmatrix} V_{qu\pm 1} \\ V_{qu\pm 2} \end{pmatrix} = -\frac{\hbar^2}{m}\begin{pmatrix} \frac{\partial_\mu\partial^\mu |\psi_{\pm 1}|}{|\psi_{\pm 1}|} \\ \frac{\partial_\mu\partial^\mu |\psi_{\pm 2}|}{|\psi_{\pm 2}|} \end{pmatrix} - Re\{\frac{ie\hbar}{2m}\begin{pmatrix} \alpha^{\lambda\kappa}{}_{1j}\frac{|\psi_{\pm j}|}{|\psi_{\pm 1}|} \\ \alpha^{\lambda\kappa}{}_{2j}\frac{|\psi_{\pm j}|}{|\psi_{\pm 2}|} \end{pmatrix} F_{\lambda\kappa}\}, \qquad (2.1.18)$$

and the current equation for the spinors $\psi_{\pm i}$ that, by using the identity $S_{\pm i} = \frac{\hbar}{2i} ln[\frac{\psi_{\pm i}}{\psi_{\pm i}*}]$ derived by (2.1.16), reads

$$-\tilde{\sigma}^\mu\sigma^\nu\partial_\mu\left(\frac{|\psi_\pm|^2}{m}\left(\partial_\nu S + eA_\nu\right)\right) = \partial_\mu J^{(\pm)\mu}, \qquad (2.1.19)$$

that, being the hydrodynamic spinorial KGEs (2.1.11-2) coupled each other (through the em tensor) leads to conservation of the overall current

$$\frac{\partial J^{(+)\mu}+J^{(-)\mu}}{\partial q^{\mu}}=\frac{\partial J^{\mu}}{\partial q^{\mu}}=0 \tag{2.1.20}$$

where, after simple manipualtions, the current J^{μ} reads

$$\begin{aligned} J^{\mu} &= J^{(+)\mu}+J^{(-)\mu} \\ &= \frac{\hbar}{2im}\left[\begin{array}{l} \psi_+{}^{\dagger}\tilde{\sigma}^{\mu}\sigma^{\nu}\left(\partial_{\nu}-\frac{e}{i\hbar}A_{\nu}\right)\psi_+ - \psi_+\tilde{\sigma}^{\mu}\sigma^{\nu}\left(\partial_{\nu}+\frac{e}{i\hbar}A_{\nu}\right)\psi_+{}^{\dagger} \\ +(\psi_-{}^{\dagger}\sigma^{\mu}\tilde{\sigma}^{\nu}\left(\partial_{\nu}-\frac{e}{i\hbar}A_{\nu}\right)\psi_- - \psi_-\sigma^{\mu}\tilde{\sigma}^{\nu}\left(\partial_{\nu}+\frac{e}{i\hbar}A_{\nu}\right)\psi_-{}^{\dagger}) \end{array}\right] \end{aligned} \tag{2.1.21}$$

that by using (2.1.7-8), leads to

$$\begin{aligned} J^{\mu} &= -\frac{\hbar}{2im}\left[\begin{array}{l} \psi_+{}^{\dagger}\tilde{\sigma}^{\mu}\psi_- - \psi_+\tilde{\sigma}^{\mu}\psi_-{}^{\dagger} \\ +\psi_-{}^{\dagger}\tilde{\sigma}^{\mu}\psi_+ + \psi_-\tilde{\sigma}^{\mu}\psi_+{}^{\dagger} \end{array}\right] \\ &= -\frac{\hbar}{4im}\left[\begin{array}{l} \left(\psi^1+\psi^2\right)^{\dagger}\tilde{\sigma}^{\mu}\left(\psi^1-\psi^2\right)+\left(\psi^1+\psi^2\right)\tilde{\sigma}^{\mu}\left(\psi^1-\psi^2\right)^{\dagger} \\ +\left(\psi^1-\psi^2\right)^{\dagger}\tilde{\sigma}^{\mu}\left(\psi^1+\psi^2\right)+\left(\psi^1-\psi^2\right)\tilde{\sigma}^{\mu}\left(\psi^1+\psi^2\right)^{\dagger} \end{array}\right] \\ &= \frac{\hbar}{4im}\left[\begin{array}{l} -\psi^{1\dagger}\tilde{\sigma}^{\mu}\psi^1+\psi^{2\dagger}\tilde{\sigma}^{\mu}\psi^2-\psi^1\tilde{\sigma}^{\mu}\psi^{1\dagger}+\psi^2\tilde{\sigma}^{\mu}\psi^{2\dagger} \\ -\psi^{1\dagger}\tilde{\sigma}^{\mu}\psi^1+\psi^{2\dagger}\tilde{\sigma}^{\mu}\psi^2-\psi^1\tilde{\sigma}^{\mu}\psi^{1\dagger}+\psi^2\tilde{\sigma}^{\mu}\psi^{2\dagger} \end{array}\right] \\ &= \frac{\hbar}{2im}\left[-\psi^{1\dagger}\tilde{\sigma}^{\mu}\psi^1+\psi^{2\dagger}\tilde{\sigma}^{\mu}\psi^2-\psi^1\tilde{\sigma}^{\mu}\psi^{1\dagger}+\psi^2\tilde{\sigma}^{\mu}\psi^{2\dagger}\right] \\ &= -\frac{\hbar}{2im}\left[\overline{\Psi}\gamma^{\mu}\Psi+\overline{\Psi *}\gamma^{\mu}\Psi *\right]=J^{(+)\mu}+J^{(-)\mu}=-\frac{\hbar}{im}\overline{\Psi}\gamma^{\mu}\Psi \end{aligned} \tag{2.1.22}$$

in agreement with the standard output.

The hydrodynamic Hamilton-Jacobi equation applied to eigenstates

Bu using the hydrodynamic notations

$$\partial_{\mu}S_{\pm i}=\partial_{\mu}\begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}=-\begin{pmatrix} p_{\pm 1_{\mu}} \\ p_{\pm 2_{\mu}} \end{pmatrix}=-p_{\pm i_{\mu}} \tag{2.1.22}$$

the Hamilton-Jacobi equation (2.1.17) that, as a function of the spinors components, reads

$$\left(\partial_{\mu}\begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}+eA_{\mu}\right)\left(\partial^{\mu}\begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}+eA^{\mu}\right)=m^2c^2\left(\mathbf{1}-\frac{1}{mc^2}\begin{pmatrix} V_{qu\pm 1} \\ V_{qu\pm 2} \end{pmatrix}\right) \tag{2.1.23}$$

for the k-th eigenstate leads to the relation

$$\left(\partial_\mu \begin{pmatrix} S_{\pm1(k)} \\ S_{\pm2(k)} \end{pmatrix} + eA_\mu\right)\left(\partial^\mu \begin{pmatrix} S_{\pm1(k)} \\ S_{\pm2(k)} \end{pmatrix} + eA^\mu\right) = -\left(\begin{pmatrix} p_{\pm1(k)_\mu} \\ p_{\pm2(k)_\mu} \end{pmatrix} - eA_\mu\right)\left(\begin{pmatrix} p_{\pm1(k)}{}^\mu \\ p_{\pm2(k)}{}^\mu \end{pmatrix} - eA^\mu\right) \tag{2.1.24}$$
$$= m^2c^2\left(\mathbf{1} - \frac{1}{mc^2}\begin{pmatrix} V_{qu\pm1(k)} \\ V_{qu\pm2(k)} \end{pmatrix}\right)$$

from where, by using the notation

$$p^{(\pm)}{}_{\pm i(k)_{\mu\pm}} = \pm p_{\pm i(k)_\mu}, \tag{2.1.25}$$

where the superscript $(\pm)$ stands for positive and negative energy states, respectively, it follows that

$$p_{\pm i(k)_\mu} = \left(m\gamma\, \dot{q}_\mu \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\mu\right) \tag{2.1.26}$$

$$E_{\pm i(k)_\mu} - e\phi = m\gamma c^2 \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \tag{2.1.27}$$

$$\pi_{\pm i(k)_\mu} = \left(m\gamma\, \dot{q}_\mu \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}}\right), \tag{2.1.28}$$

where $\pi_{\pm(k)_\mu}$ represents the "mechanical moment". From (2.1.22, 2.1.26) it follows that the Lagrangean

$$L_{\pm i(k)} = \frac{dS_{\pm i(k)}}{dt} = -p_{\pm i(k)_\mu} \dot{q}^\mu \tag{2.1.29}$$

for the k-eigenstates reads

$$L_{\pm i(k)} = -\frac{mc^2}{\gamma}\sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\mu \dot{q}^\mu \tag{2.1.30}$$

2.2. The Lagrangean motion equations for generic superposition of Dirac field

Given the generic fermion field

$$\psi_{\pm i} = \begin{pmatrix} \psi_{\pm1} \\ \psi_{\pm2} \end{pmatrix} = \begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right] \end{pmatrix} = \begin{pmatrix} \sum_k a_{\pm k}\, |\psi_{\pm1(k)}|\, exp[\frac{iS_{\pm1(k)}}{\hbar}] \\ \sum_k a_{\pm k}\, |\psi_{\pm2(k)}|\, exp[\frac{iS_{\pm2(k)}}{\hbar}] \end{pmatrix} \tag{2.2.1}$$

it follows that

$$S_{\pm i} = \frac{\hbar}{2i} ln[\frac{\psi_{\pm i}}{\psi_{\pm i}*}] = \frac{\hbar}{2i} \begin{pmatrix} ln[\sum_k a_k |\psi_{\pm i(k)}| exp(\frac{iS_{\pm i(k)}}{\hbar})] \\ -ln[\sum_k a*_k |\psi_{\pm i(k)}| exp(-\frac{iS_{\pm i(k)}}{\hbar})] \end{pmatrix} \qquad (2.2.2)$$

where for the k-th eigenstates

$$S_{\pm i(k)} = \begin{pmatrix} S_{\pm 1(k)} \\ S_{\pm 2(k)} \end{pmatrix} = \frac{\hbar}{2i} \begin{pmatrix} ln \frac{\psi_{\pm 1(k)}}{\psi_{\pm 1(k)}*} \\ ln \frac{\psi_{\pm 2(k)}}{\psi_{\pm 2(k)}*} \end{pmatrix}. \qquad (2.2.3)$$

Moreover, from (2.2.1) the hydrodynamic momentum $p_{\pm i\mu}$, the Lagrangean function $L_{\pm i}$ and the energy $E_{\pm i}$, respectively, read

$$p_{\pm i\mu} = -\partial_\mu S_{\pm i} = -\frac{1}{2} \frac{\sum_k a_{\pm k} |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu ln |\psi_{\pm i(k)}| - p_{\pm i(k)\mu} \right)}{\sum_k a_k |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}]}$$
$$+ \frac{1}{2} \frac{\sum_k a*_{\pm k} |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \partial_\mu ln |\psi_{\pm i(k)}| + p_{\pm i(k)\mu} \right)}{\sum_k a*_{\pm k} |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}]} , \qquad (2.2.4)$$

$$L_{\pm i} = \frac{dS_{\pm i}}{dt} = \frac{\partial S_{\pm i}}{\partial t} + \frac{\partial S_{\pm i}}{\partial q_i} \dot{q}_i = -p_{\pm i\mu} \dot{q}^\mu$$
$$= \frac{1}{2} \frac{\sum_k a_{\pm k} |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \dot{q}^\mu \partial_\mu ln |\psi_{\pm i(k)}| + L_{\pm i(k)} \right)}{\sum_k a_{\pm k} |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}]} \quad . \quad . \qquad (2.2.5)$$
$$- \frac{1}{2} \frac{\sum_k a*_{\pm k} |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \dot{q}^\mu \partial_\mu ln |\psi_{\pm i(k)}| - L_{\pm i(k)} \right)}{\sum_k a*_{\pm k} |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}]}$$

$$E_{\pm i} = -c\partial_0 S_{\pm i} = -\frac{1}{2}\frac{\sum_k a_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{iS_{\pm i(k)}}{\hbar}]\left(\frac{\hbar c}{i}\partial_0 \ln|\psi_{\pm i(k)}| - E_{\pm i(k)}\right)}{\sum_k a_{\pm k}\,|\psi_{(k)\pm i}|\,exp[\frac{iS_{(k)\pm i}}{\hbar}]} + \frac{1}{2}\frac{\sum_k a^*_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{-iS_{\pm(k)}}{\hbar}]\left(\frac{\hbar c}{i}\partial_0 \ln|\psi_{(k)\pm i}| + E_{\pm i(k)}\right)}{\sum_k a^*_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{-iS_{\pm i(k)}}{\hbar}]}, \quad (2.2.6)$$

Furthermore, by using the identity $L_{\pm i(k)} = -p_{\pm i_{(k)\mu}}\dot{q}^{\mu}$, it follows that

$$-\frac{\partial L_{\pm i}}{\partial \dot{q}^{\mu}} = -\frac{1}{2}\frac{\sum_k a_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{iS_{\pm i(k)}}{\hbar}]\left(\frac{\hbar}{i}\partial_\mu \ln|\psi_{\pm i(k)}| + \frac{\partial L_{\pm i(k)}}{\partial \dot{q}^{\mu}}\right)}{\sum_k a_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{iS_{\pm i(k)}}{\hbar}]} + \frac{1}{2}\frac{\sum_k a^*_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{-iS_{\pm i(k)}}{\hbar}]\left(\frac{\hbar}{i}\partial_\mu \ln|\psi_{\pm i(k)}| - \frac{\partial L_{\pm i(k)}}{\partial \dot{q}^{\mu}}\right)}{\sum_k a^*_{\pm k}\,|\psi_{\pm i(k)}|\,exp[\frac{-iS_{\pm i(k)}}{\hbar}]}, \quad (2.2.7)$$

$$= p_{\pm i\mu}$$

and that

$$\dot{p}_{\pm i\mu} = \frac{d}{dt}\left(-\frac{\partial S_{\pm i}}{\partial q^{\mu}}\right) = -\frac{\partial}{\partial q^{\mu}}\frac{dS_{\pm i}}{dt} = -\frac{\partial L_{\pm i}}{\partial q^{\mu}}, \quad (2.2.8)$$

Equation (2.2.7-8) can be further simplified for a system that does not depend explicitly by time (see ref. [14]). In this case, for the k-th eigenstates, is straightforward to obtain that

$$p_{\pm i(k)_\mu} = -\frac{\partial L_{\pm i(k)}}{\partial \dot{q}^{\mu}}, \quad (2.2.9)$$

$$\dot{p}_{\pm i(k)_\mu} = -\frac{\partial L_{\pm i(k)}}{\partial q^{\mu}}. \quad (2.2.10)$$

Moreover, given that the Lagrangean function for positive and negative energy states $L^{(\pm)}{}_{\pm(k)}$ reads $L^{(\pm)}{}_{\pm(k)} = \pm L_{\pm(k)}$, equations (2.2.9-10) lead to motion equation (both positive and negative energy eigenstates)

$$\frac{d}{dt}\left(-\frac{\partial L_{\pm i(k)}}{\partial \dot{q}^{\mu}}\right) = -\frac{\partial L_{\pm i(k)}}{\partial q^{\mu}} \quad (2.2.11)$$

By introducing (2.1.30) in (2.2.11) it is possible to obtain the hydrodynamic equation of motion for the free KGEs (2.1.9) or (2.1.12) whose stationary solutions (satisfying the irrotational condition [14,16]) (i.e., eigenstates) and their linear superpositions are the solutions of the fermion field.

2.3. The hydrodynamic energy-impulse tensor of the Dirac field

Given the EITD for the k-th eigenstate of the fermion field interacting with the em one

$$T_{\pm i(k)\mu}{}^{\nu}{}_{F-em} = |\psi_{\pm i(k)}|^2 \, \mathrm{T}_{\pm i(k)\mu}{}^{\nu} = -|\psi_{\pm i(k)}|^2 \left(\dot{q}_\mu \frac{\partial L_{\pm i(k)}}{\partial \dot{q}_\nu} - L_{\pm i(k)} \delta_\mu{}^\nu \right) \qquad (2.3.1)$$

the energy-impulse tensor $\mathrm{T}_{\pm i\mu}{}^{\nu}$ for the generic quantum state reads

$$\mathrm{T}_{\pm i\mu}{}^{\nu} = -\left(\dot{q}_\mu \frac{\partial L_{\pm i}}{\partial \dot{q}_\nu} - L_{\pm i} \delta_\mu{}^\nu \right) = -\dot{q}_\mu p_{\pm i}{}^{\nu} + L_{\pm i} \delta_\mu{}^\nu$$

$$= -\frac{1}{2} \frac{\sum_k a_k |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \left(\dot{q}_\mu \partial^\nu |\psi_{\pm i(k)}| + \dot{q}^\alpha \partial_\alpha |\psi_{\pm i(k)}| \delta_\mu{}^\nu \right) - \mathrm{T}_{\pm k\mu}{}^{\nu} \right)}{\sum_k a_k |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}]} \qquad (2.3.2),$$

$$+\frac{1}{2} \frac{\sum_k a^*_k |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}] \left(\frac{\hbar}{i} \left(\dot{q}_\mu \partial^\nu |\psi_{\pm i(k)}| + \dot{q}^\alpha \partial_\alpha |\psi_{\pm i(k)}| \delta_\mu{}^\nu \right) + \mathrm{T}_{\pm k\mu}{}^{\nu} \right)}{\sum_k a^*_k |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}]}$$

leading, for systems that do not explicitly depend by time [14], to

$$T_{\pm i\mu}{}^{\nu}{}_{F-em} = |\psi_{\pm i}|^2 \, \mathrm{T}_{\pm i\mu}{}^{\nu} = |\psi_{\pm i}|^2 \left(\frac{1}{2} \frac{\sum_k a_k |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}] \mathrm{T}_{\pm i(k)\mu}{}^{\nu}}{\sum_k a_k |\psi_{\pm i(k)}| exp[\frac{iS_{\pm i(k)}}{\hbar}]} + \frac{1}{2} \frac{\sum_k a^*_k |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}] \mathrm{T}_{\pm i(k)\mu}{}^{\nu}}{\sum_k a^*_k |\psi_{\pm i(k)}| exp[\frac{-iS_{\pm i(k)}}{\hbar}]} \right) \qquad (2.3.3)$$

where

$$T_{\pm i(k)\mu}{}^{\nu}{}_{F-em} = |\psi_{\pm i(k)}|^2 \, \mathrm{T}_{\pm i(k)\mu}{}^{\nu}$$

$$= |\psi_{\pm i(k)}|^2 \, m\gamma \left[\sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \right]^{-1} \qquad (2.3.4)$$

$$\left(\left(\dot{q}_\mu \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\mu \right) \dot{q}^\nu \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} - \left(\dot{q}_\alpha \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + eA_\alpha \right) \dot{q}^\alpha \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} \delta_\mu{}^\nu \right)$$

$$T_{\pm i(k)\mu}{}^{\nu}{}_{F-em} = |\psi_{\pm i(k)}|^2 \frac{mc^2}{\gamma} \begin{pmatrix} \left(u_\mu \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + \frac{eA_\mu}{mc} \right) u^\nu \\ - \left(u_\alpha \sqrt{1 - \frac{V_{qu\pm i(k)}}{mc^2}} + \frac{eA_\alpha}{mc} \right) u^\alpha \delta_\mu{}^\nu \end{pmatrix} \quad (2.3.5)$$

Moreover, by using the expression of the action as a function of the field

$$S_\pm = \frac{\hbar}{2i} ln[\frac{\psi_\pm}{\psi_\pm *}] = \frac{\hbar}{2i} \left(ln \frac{[\sum_k a_k |\psi_{\pm i(k)}| exp(\frac{iS_{\pm i(k)}}{\hbar})]}{\sum_k a*_k |\psi_{\pm i(k)}| exp(-\frac{iS_{\pm i(k)}}{\hbar})]} \right) \quad (2.3.6)$$

$$S_{\pm i(k)} = \frac{\hbar}{2i} ln[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)} *}] = \begin{pmatrix} S_{\pm 1(k)} \\ S_{\pm 2(k)} \end{pmatrix} = \frac{\hbar}{2i} \begin{pmatrix} ln[\frac{\psi_{\pm 1(k)}}{\psi_{\pm 1(k)} *}] \\ ln[\frac{\psi_{\pm 2(k)}}{\psi_{\pm 2(k)} *}] \end{pmatrix} \quad (2.3.7)$$

and the identity

$$-p_{\pm i_\mu} = \partial_\mu S_{\pm i} = \partial_\mu \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix} = - \begin{pmatrix} p_{\pm 1_\mu} \\ p_{\pm 2_\mu} \end{pmatrix}, \quad (2.3.8)$$

with the help of the identity

$$\begin{aligned} T_{\pm i(k)\mu}{}^{\nu}{}_{F-em} &= -|\psi_{\pm i(k)}|^2 \left(-p_{\pm i(k)}{}^{\nu} \dot{q}_\mu +_{\pm i(k)}{}^{\alpha} \dot{q}_\alpha \delta_\mu{}^\nu \right) \\ &= |\psi_{\pm i(k)}|^2 c^2 \left[\frac{\partial S_{\pm i(k)}}{\partial t} + e\phi \right]^{-1} \begin{pmatrix} p_{\pm i(k)\mu} \left(p_{\pm i(k)}{}^{\nu} - eA^\nu \right) \\ -p_{\pm i(k)\alpha} \left(p_{\pm i(k)}{}^{\alpha} - eA^\alpha \right) \delta_\mu{}^\nu \end{pmatrix} \end{aligned} \quad (2.3.9)$$

(see (2.3.1-2)) it is possible to obtain the expression of the EITD that is independent by the hydrodynamic formalism as follows

$$T_{\pm i(k)\mu}{}^{\nu}{}_{F-em} = |\psi_{\pm k}|^2 \frac{4ic^2}{\hbar} \left(\frac{\partial}{\partial t} ln[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)} *}] + \frac{4ie}{\hbar} \phi \right)^{-1} \begin{pmatrix} \left(\frac{\partial ln[_{\pm i(k)}]}{\partial q^\mu} - \frac{4ie}{\hbar} A_\mu \right) \frac{\partial ln[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)} *}]}{\partial q_\nu} \\ - \left(\frac{\partial ln[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)} *}]}{\partial q^\alpha} - \frac{4ie}{\hbar} A_\alpha \right) \frac{\partial ln[\frac{\psi_{\pm i(k)}}{\psi_{\pm i(k)} *}]}{\partial q_\alpha} \delta_\mu{}^\nu \end{pmatrix} \quad (2.3.10)$$

It is noteworthy to observe that result (2.3.10) takes only into account irrotational states of the hydrodynamic treatment that belong to the quantum mechanics.

2.4. The Gravitational Equation of the Fermion Field

The motivation of the present work consists in defining an analytical procedure to derive the gravitational equation for the classical Dirac field.

This objective is planned to be achieved by assuming that the hydrodynamic representation of the Dirac bi-spinor field, that is depicted by mass densities $|\psi_{\pm 1}|^2$ and $|\psi_{\pm 2}|^2$ moving with momenta $p_{\pm\mu} = \begin{pmatrix} p_{\pm 1\mu} \\ p_{\pm 2\mu} \end{pmatrix} = -\partial_\mu S_{\pm} = -\partial_\mu \begin{pmatrix} S_{\pm 1} \\ S_{\pm 2} \end{pmatrix}$ under the action of the quantum potential (2.1.18), corresponds to a physical reality whose Einstein equation actually describes its gravitational effect. This postulate can be justified by the empirical existence of the quantum potential given by the Bohm-Aharonov effect [15].

By assuming the physics covariance, the generalization of the Dirac equation to the curved space-time can be defined. Then, it is possible to obtain the metric of the space-time, from the gravity equation derived by applying the minimum action principle to the overall hydrodynamic action of both the Dirac and the gravitational field. This procedure can be fulfilled in a similar way as done for the scalar boson field [14] where the obtained gravitational equation leads to a cosmological constant value, due to the zero-point vacuum energy, that agrees with the astronomical observations [14].

The resulting gravity equation will contain the explicit coupling with the Dirac (classical) field (overcoming the limit of the semiclassical approximation [14]). Once the system of Dirac equation, coupled to the gravitational one, is defined in the curved space-time, then the quantization procedure can be applied both in near-Minkowski space-time (when the particle mass densities are very far prom the Plankian one, as in the standard model) and in very high curved space-time where particles with Plankian mass densities have to be described by high energy QFT.

Conclusions

This work derives the Lagrangean hydrodynamic representation of the Dirac field. The paper shows that the evolution of the bi-spinor field $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is equivalent to the motion of the mass densities $|\psi_{\pm 1}|^2$ and $|\psi_{\pm 2}|^2$, where $\psi_{\pm} = \frac{\psi_1 \pm \psi_2}{\sqrt{2}} = |\psi_{\pm}| exp[i \frac{S_{\pm}}{\hbar}] = \begin{pmatrix} \psi_{\pm 1} \\ \psi_{\pm 2} \end{pmatrix}$, that move with momenta $p_{\pm 1\mu} = -\partial_\mu S_{\pm 1}$ $p_{\pm 2\mu} = -\partial_\mu S_{\pm 2}$ and that are subject to the non-local interaction of the theory derived quantum potential. The expression of the quantum potential as well as of the hydrodynamic energy-impulse tensor is explicitly derived as a function of the charged fermion field regardless the hydrodynamic formalism.

Data Availability

Theoretical work without use of empirical data.

Conflicts of Interest

There are not conflict of interest.

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Supplementary Material

Appendix A

From the identity

$$\tilde{\sigma}^{\nu}\sigma^{\mu} = g^{\mu\nu} + \alpha^{\mu\nu} \tag{A.1}$$

it follows that

$$\alpha^{\mu\nu} = \begin{pmatrix} \tilde{\sigma}^0\sigma^0 & \tilde{\sigma}^0\sigma^1 & \tilde{\sigma}^0\sigma^2 & \tilde{\sigma}^0\sigma^3 \\ \tilde{\sigma}^1\sigma^0 & \tilde{\sigma}^1\sigma^1 & \tilde{\sigma}^1\sigma^2 & \tilde{\sigma}^1\sigma^3 \\ \tilde{\sigma}^2\sigma^0 & \tilde{\sigma}^2\sigma^1 & \tilde{\sigma}^2\sigma^2 & \tilde{\sigma}^2\sigma^3 \\ \tilde{\sigma}^3\sigma^0 & \tilde{\sigma}^3\sigma^1 & \tilde{\sigma}^3\sigma^2 & \tilde{\sigma}^3\sigma^3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \tilde{\sigma}^0\sigma^1 & \tilde{\sigma}^0\sigma^2 & \tilde{\sigma}^0\sigma^3 \\ \tilde{\sigma}^1\sigma^0 & 0 & \tilde{\sigma}^1\sigma^2 & \tilde{\sigma}^1\sigma^3 \\ \tilde{\sigma}^2\sigma^0 & \tilde{\sigma}^2\sigma^1 & 0 & \tilde{\sigma}^2\sigma^3 \\ \tilde{\sigma}^3\sigma^0 & \tilde{\sigma}^3\sigma^1 & \tilde{\sigma}^3\sigma^2 & 0 \end{pmatrix} \tag{A.2}$$

Moreover, given that, for the properties of $\tilde{\sigma}^{\mu}\sigma^{\nu}$, $\alpha^{\mu\nu}$ is antisymmetric, it follows that

$$\alpha^{\mu\nu} A_{\mu} A_{\nu} = 0, \tag{A.3}$$

that

$$\alpha^{\mu\nu} \partial_{\mu} \partial_{\nu} = 0, \tag{A.4}$$

that

$$\alpha^{\mu\nu} \left(A_{\nu} \partial_{\mu} + A_{\mu} \partial_{\nu} \right) = 0 \tag{A.6}$$

and, finally, that

$$\begin{aligned} \alpha^{\mu\nu} \left(\partial_{\mu} + \frac{ie}{\hbar} A_{\mu} \right) \left(\partial_{\nu} + \frac{ie}{\hbar} A_{\nu} \right) &= \frac{ie}{\hbar} \alpha^{\mu\nu} \partial_{\mu} A_{\nu} \\ &= \frac{ie}{2\hbar} \left(\alpha^{\mu\nu} \partial_{\mu} A_{\nu} + \alpha^{\nu\mu} \partial_{\nu} A_{\mu} \right) = \frac{ie}{2\hbar} \alpha^{\mu\nu} \left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \right) \\ &= \frac{ie}{2\hbar} \alpha^{\mu\nu} F_{\mu\nu} \end{aligned} \tag{A.7}$$

where $F_{\mu\nu}$ is the em tensor. For obtaining the identity (A.4-6), the continuity and derivable properties of $\psi_{\pm}$ have also been used.

Appendix B

By using the identity (2.16), the equation (2.12) reads

$$\left(\partial_\mu\partial^\mu + \frac{ie}{\hbar}\left(A_\mu\partial^\mu + \partial_\mu A^\mu\right) - \frac{e^2}{\hbar^2}A_\mu A^\mu\right)\begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\end{pmatrix}$$

$$= -\left(\frac{m^2c^2}{\hbar^2} + \frac{ie}{2\hbar}\alpha^{\mu\nu}F_{\mu\nu}\right)\begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\end{pmatrix}$$

(B.1)

that, by developing the derivatives, leads to

$$\left(\begin{array}{l}
\begin{pmatrix} exp\left[i\frac{S_{\pm1}}{\hbar}\right]\partial_\mu\left(\partial^\mu|\psi_{\pm1}| + \frac{i}{\hbar}|\psi_{\pm1}|\partial^\mu S_{\pm1}\right) \\ exp\left[i\frac{S_{\pm2}}{\hbar}\right]\partial_\mu\left(\partial^\mu|\psi_{\pm2}| + \frac{i}{\hbar}|\psi_{\pm2}|\partial^\mu S_{\pm2}\right)\end{pmatrix} + \begin{pmatrix} \frac{i}{\hbar}\partial^\mu S_{\pm1}\, exp\left[i\frac{S_{\pm1}}{\hbar}\right]\left(\partial^\mu|\psi_{\pm1}| + \frac{i}{\hbar}|\psi_{\pm1}|\partial^\mu S_{\pm1}\right) \\ \frac{i}{\hbar}\partial^\mu S_{\pm2}\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\left(\partial^\mu|\psi_{\pm2}| + \frac{i}{\hbar}|\psi_{\pm2}|\partial^\mu S_{\pm2}\right)\end{pmatrix} \\
+\frac{ie}{\hbar}\left(\begin{array}{l}
A_\mu\begin{pmatrix} exp\left[i\frac{S_{\pm1}}{\hbar}\right]\left(\partial^\mu|\psi_{\pm1}| + \frac{i}{\hbar}|\psi_{\pm1}|\partial^\mu S_{\pm1}\right) \\ exp\left[i\frac{S_{\pm2}}{\hbar}\right]\left(\partial^\mu|\psi_{\pm2}| + \frac{i}{\hbar}|\psi_{\pm2}|\partial^\mu S_{\pm2}\right)\end{pmatrix} \\
+A^\mu\begin{pmatrix} exp\left[i\frac{S_{\pm1}}{\hbar}\right]\left(\partial_\mu|\psi_{\pm1}| + \frac{i}{\hbar}|\psi_{\pm1}|\partial_\mu S_{\pm1}\right) \\ exp\left[i\frac{S_{\pm2}}{\hbar}\right]\left(\partial_\mu|\psi_{\pm2}| + \frac{i}{\hbar}|\psi_{\pm2}|\partial_\mu S_{\pm2}\right)\end{pmatrix} \\
+\begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\end{pmatrix}\partial_\mu A^\mu
\end{array}\right) - \frac{e^2}{\hbar^2}A_\mu A^\mu\begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\end{pmatrix}
\end{array}\right)$$

$$= -\left(\frac{m^2c^2}{\hbar^2} + \frac{ie}{2\hbar}\alpha^{\mu\nu}F_{\mu\nu}\right)\begin{pmatrix} |\psi_{\pm1}|\, exp\left[i\frac{S_{\pm1}}{\hbar}\right] \\ |\psi_{\pm2}|\, exp\left[i\frac{S_{\pm2}}{\hbar}\right]\end{pmatrix}$$

(B.2)

and, by dividing each component by $exp\left[i\frac{S_{\pm i}}{\hbar}\right]$, where i =1,2 , to

$$\begin{pmatrix}\left(\partial_\mu\partial^\mu|\psi_{\pm1}|+\frac{i}{\hbar}\partial_\mu\partial^\mu S_{\pm1}\right)\\ \left(\partial_\mu\partial^\mu|\psi_{\pm2}|+\frac{i}{\hbar}\partial_\mu\partial^\mu S_{\pm2}\right)\end{pmatrix}+\begin{pmatrix}\left(\partial^\mu|\psi_{\pm1}|+\frac{i}{\hbar}|\psi_{\pm1}|\partial^\mu S_{\pm1}\right)\frac{i}{\hbar}\partial_\mu S_{\pm1}\\ \left(\partial^\mu|\psi_{\pm2}|+|\psi_{\pm2}|\frac{i}{\hbar}\partial^\mu S_{\pm2}\right)\frac{i}{\hbar}\partial_\mu S_{\pm2}\end{pmatrix}$$

$$+\frac{ie}{\hbar}\left(A_\mu\begin{pmatrix}|\psi_{\pm1}|\left(\partial^\mu|\psi_{\pm1}|+\frac{i}{\hbar}\partial^\mu S_{\pm1}\right)\\ |\psi_{\pm2}|\left(\partial^\mu|\psi_{\pm2}|+\frac{i}{\hbar}\partial^\mu S_{\pm2}\right)\end{pmatrix}+A^\mu\begin{pmatrix}|\psi_{\pm1}|\left(\partial_\mu|\psi_{\pm1}|+\frac{i}{\hbar}\partial_\mu S_{\pm1}\right)\\ |\psi_{\pm2}|\left(\partial_\mu|\psi_{\pm2}|+\frac{i}{\hbar}\partial_\mu S_{\pm2}\right)\end{pmatrix}\right). \tag{B.3}$$

$$-\frac{e^2}{\hbar^2}A_\mu A^\mu\begin{pmatrix}|\psi_{\pm1}|\\ |\psi_{\pm2}|\end{pmatrix}=-\left(\frac{m^2c^2}{\hbar^2}+\frac{ie}{2\hbar}\alpha^{\mu\nu}F_{\mu\nu}\right)\begin{pmatrix}|\psi_{\pm1}|\\ |\psi_{\pm2}|\end{pmatrix}$$

Moreover, by grouping the terms and equating the real part of the equation, it follows that

$$\begin{pmatrix}\partial_\mu\partial^\mu|\psi_{\pm1}|\\ \partial_\mu\partial^\mu|\psi_{\pm2}|\end{pmatrix}-\frac{1}{\hbar^2}\begin{pmatrix}\partial^\mu S_{\pm1}\partial_\mu S_{\pm1}\\ \partial^\mu S_{\pm2}\partial_\mu S_{\pm2}\end{pmatrix}$$

$$-\frac{e}{\hbar^2}\left(A_\mu\begin{pmatrix}|\psi_{\pm1}|\partial^\mu S_{\pm1}\\ |\psi_{\pm2}|\partial^\mu S_{\pm2}\end{pmatrix}+A^\mu\begin{pmatrix}|\psi_{\pm1}|\partial_\mu S_{\pm1}\\ |\psi_{\pm2}|\partial_\mu S_{\pm2}\end{pmatrix}\right) \tag{B.4}$$

$$-\frac{e^2}{\hbar^2}A_\mu A^\mu\begin{pmatrix}|\psi_{\pm1}|\\ |\psi_{\pm2}|\end{pmatrix}=-\left(\frac{m^2c^2}{\hbar^2}+Re\left\{\frac{ie}{2\hbar}\begin{pmatrix}\alpha^{\mu\nu}{}_{1j}\frac{|\psi_{\pm j}|}{|\psi_{\pm1}|}\\ \alpha^{\mu\nu}{}_{2j}\frac{|\psi_{\pm j}|}{|\psi_{\pm2}|}\end{pmatrix}F_{\mu\nu}\right\}\right)\begin{pmatrix}|\psi_{\pm1}|\\ |\psi_{\pm2}|\end{pmatrix}$$

and, by dividing each component by $|\psi_{\pm i}|$, where i =1,2 , that

$$\begin{pmatrix}\begin{pmatrix}\partial^\mu S_{\pm1}\partial_\mu S_{\pm1}\\ \partial^\mu S_{\pm2}\partial_\mu S_{\pm2}\end{pmatrix}\\ +e\left(A_\mu\begin{pmatrix}\partial^\mu S_{\pm1}\\ \partial^\mu S_{\pm2}\end{pmatrix}+A^\mu\begin{pmatrix}\partial_\mu S_{\pm1}\\ \partial_\mu S_{\pm2}\end{pmatrix}\right)\\ +e^2A_\mu A^\mu\end{pmatrix}$$

$$=\left(m^2c^2+Re\left\{\frac{ie\hbar}{2}\begin{pmatrix}\alpha^{\mu\nu}{}_{1j}\frac{|\psi_{\pm j}|}{|\psi_{\pm1}|}\\ \alpha^{\mu\nu}{}_{2j}\frac{|\psi_{\pm j}|}{|\psi_{\pm2}|}\end{pmatrix}F_{\mu\nu}\right\}+\hbar^2\begin{pmatrix}\frac{\partial_\mu\partial^\mu|\psi_{\pm1}|}{|\psi_{\pm1}|}\\ \frac{\partial_\mu\partial^\mu|\psi_{\pm2}|}{|\psi_{\pm2}|}\end{pmatrix}\right), \tag{B.5}$$

leading to the final expression

$$\left(\begin{pmatrix}\partial^{\mu}S_{\pm 1}\\ \partial^{\mu}S_{\pm 2}\end{pmatrix}+eA^{\mu}\right)\left(\begin{pmatrix}\partial_{\mu}S_{\pm 1}\\ \partial_{\mu}S_{\pm 2}\end{pmatrix}+eA_{\mu}\right)=\left(m^{2}c^{2}+m\begin{pmatrix}V_{qu\pm 1}\\ V_{qu\pm 2}\end{pmatrix}\right) \tag{B.6}$$

where the quantum potential of the Dirac KGE reads

$$V_{qu\pm i}=\begin{pmatrix}V_{qu\pm 1}\\ V_{qu\pm 2}\end{pmatrix}=-\frac{\hbar^{2}}{m}\begin{pmatrix}\dfrac{\partial_{\mu}\partial^{\mu}|\psi_{\pm 1}|}{|\psi_{\pm 1}|}\\ \dfrac{\partial_{\mu}\partial^{\mu}|\psi_{\pm 2}|}{|\psi_{\pm 2}|}\end{pmatrix}-Re\left\{\frac{ie\hbar}{2m}\begin{pmatrix}\alpha^{\mu\nu}{}_{1j}\dfrac{|\psi_{\pm j}|}{|\psi_{\pm 1}|}\\ \alpha^{\mu\nu}{}_{2j}\dfrac{|\psi_{\pm j}|}{|\psi_{\pm 2}|}\end{pmatrix}F_{\mu\nu}\right\} \tag{B.7}$$

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