Finite Difference Approximation for Caputo-Fabrizio Time Fractional Derivative on Non-Uniform Mesh and Some Applications

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This study presents the numerical approximation method for the nonsingular fractional derivative known as the Caputo-Fabrizio fractional derivative on uniform and nonuniform mesh. It has been found that the accuracy of the utilized fractional derivative can be improved on nonuniform mesh compare to the uniform one. In contrast to the previous literatures, the accuracy of nonsingular fractional derivative is highly weighted by the fractional order and the memory kernel.

Keywords: Finite difference; Caputo-Fabrizio fractional derivative; numerical algorithm; nonuniform mesh; stability

1 Introduction

Over these past years, application of fractional calculus, in different field, has been extensively studied [1, 2, 3, 4]. Recent developments of fractional derivative also attracted many researchers due to its potential applications in modelling real world phenomena [5, 6, 7, 8, 9]. Although, fractional calculus is in the verge of history, many literature still remain a theory due to the lack of feasible mathematical analysis that could encompasses great systems, from tiny body to the largest, random to discrete.

Fractional calculus committees are still pointing out that the fractional derivative, as well as fractional differential equation, have many potential applications. In fact, fractional derivative is a very useful tool in modelling nonlinear systems that can be encountered in different areas of science and engineering. However, nonlinear equation is sometimes complicated to deal if handled analytically. To remedy this problem, many researches uses the numerical methods as an alternative way since one can establish the boundaries of equation. Several studies have been conducted to test the effeciency of the numerical approximation connected to fractional derivative [10, 11, 12].

One of the recently introduced fractional derivative is the Caputo-Fabrizio fractional derivative [13] given by the definition

Definition 1.1. *Let $f \in H^1(a,b), b > a, \alpha \in (0,1)$ then the Caputo-Fabrizio fractional derivative is defined as*

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t f'(s)\exp\left\{-\alpha\frac{t-s}{1-\alpha}\right\}ds \tag{1}$$

Where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. It can also be applied to the function that does not belong to $H^1(a,b)$, such as the Caputo-Fabrizio (CF) fractional derivative can be reformulated as

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha}\int_0^t (f(t)-f(s))\exp\left\{-\alpha\frac{t-s}{1-\alpha}\right\}ds \tag{2}$$

In view of definition 1.1, the integral term has no singular behavior, which makes it as one of its interesting feature compare to the previous version of the fractional derivative, which is the Caputo derivative. In the work of Zhang et. al [14], numerical approximation of the fractional derivative with singularity were studied. L1 method has also been investigated on their work but, however, accuracy must be improved especially when the temporal variable approaches to zero.

The truncation error estimate of the singular derivative has found to be fractional order dependent on the uniform mesh. So, accuracy can be improved, naturally, upon considering a non-uniform mesh [14]. In the context of non-singular derivative, like the CF derivative, can also acquire more accurate estimates.

(Continuation...)

2 Numerical method on non-uniform mesh for CF fractional derivative

For an integer N, the interval $[0,T]$ can be subdivided into subintervals with $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_N = T$. Let the time increment be denoted as $h_n = t_n - t_{n-1},\ \ 1 \le n \le N$ and let

$$h_{\max} = \max_{1\le i\le N} h_i, \quad h_{\min} = \min_{1\le i\le N} h_i$$

Definition 2.1. *Suppose N is finite grid size and a sequence of mesh is finite. Then the mesh points are quasi-uniform if there exist a constant $\Gamma \neq 0$ such that*

$$\frac{h_{\max}}{h_{\min}} \le \Gamma$$

Definition 2.1 characterize the time increment and it must hold $h_{\max} \le \Gamma T/N$. We can deduce that when $\Gamma = 1$, we can have the uniform mesh with $h_{\max} = T/N$. Now, we present the following result for any temporal meshes.

Theorem 2.1. *Let the fractional order of CF fractional derivative be $\alpha \in (0,1)$ and a function $f(t) \in \mathcal{L}^2[0,T]$, it holds*

$$\frac{M(\alpha)}{1-\alpha}\int_0^{t_n} f'(s)\exp\left[-\frac{\alpha}{1-\alpha}(t_n - s)\right]ds = \frac{M(\alpha)}{(1-\alpha)}\sum_{j=1}^{n-1}\frac{f(t_j)-f(t_{j-1})}{h_j} \times \int_0^{t_n}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds + \mathcal{O}^n \tag{3}$$

where

$$|\mathcal{O}^n| = \frac{M(\alpha)}{(1-\alpha)}\left(\frac{h_{\max}^2}{8} + \frac{h_n(1-\alpha)}{2\alpha}\right)\max_{0\le t\le t_n}|f''(t)|\exp\left(-\frac{\alpha}{1-\alpha}h_n\right)$$

Proof. We can write the integral of the form

$$\int_0^{t_n} f'(s)\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds = \int_0^{t_{n-1}} f'(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds + \int_{t_{n-1}}^{t_n} f'(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds \tag{4}$$

□

Now, we can actually evaluate the two integral in the right-hand side of equation (4), separately. We start in the first term and by integration by parts, we can have

$$\begin{aligned}\int_0^{t_{n-1}} f'(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds &= \Big[f(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}\Big]_0^{t_{n-1}} - \frac{\alpha}{1-\alpha}\int_0^{t_{n-1}} f(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds \\ &= \Big[f(t_{n-1})e^{-\frac{\alpha}{1-\alpha}(h_n)} - f(0)e^{-\frac{\alpha}{1-\alpha}(t_n)}\Big] \\ &\quad - \frac{\alpha}{1-\alpha}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} f(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds \end{aligned} \tag{5}$$

By linear interpolation of the function $f(s)$, we have

$$\begin{aligned}\int_0^{t_{n-1}} f(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds &= \Big[f(t_{n-1})e^{-\frac{\alpha}{1-\alpha}(h_n)} - f(0)e^{-\frac{\alpha}{1-\alpha}(t_n)}\Big] \\ &\quad - \frac{\alpha}{1-\alpha}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)f(t_{j-1})-(t_{j-1}-s)f(t_j)}{h_j}e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds - \mathcal{O}_1^n \end{aligned} \tag{6}$$

we can then deduce that

$$\mathcal{O}_1^n = \frac{\alpha}{1-\alpha}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\frac{1}{2}f''(\epsilon_j)(s-t_j)(s-t_{j-1})e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds, \quad t_{j-1} < \epsilon_j < t_j$$

Further manipulation on the second term of equation (6), we can have the following equalities

$$\frac{\alpha}{1-\alpha}\int_{t_{j-1}}^{t_j}(t_j-s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds = -h_j e^{-\frac{\alpha}{1-\alpha}(t_n-t_{j-1})} + \int_{t_{j-1}}^{t_j} e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds$$

$$\frac{\alpha}{1-\alpha}\int_{t_{j-1}}^{t_j}(t_{j-1}-s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds = -h_j e^{-\frac{\alpha}{1-\alpha}(t_n-t_j)} + \int_{t_{j-1}}^{t_j} e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds$$

substituting these expressions to equation (6) to get

$$\begin{aligned}\int_0^{t_{n-1}} f'(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds &= \Big[f(t_{n-1})e^{-\frac{\alpha}{1-\alpha}(h_n)} - f(0)e^{-\frac{\alpha}{1-\alpha}(t_n)}\Big] \\ &\quad + \sum_{j=1}^{n-1} f(t_{j-1})e^{-\frac{\alpha}{1-\alpha}(t_n-t_{j-1})} - \sum_{j=1}^{n-1} f(t_j)e^{-\frac{\alpha}{1-\alpha}(t_n-t_j)} \\ &\quad + \sum_{j=1}^{n-1}\frac{f(t_j)-f(t_{j-1})}{h_j}\int_{t_{j-1}}^{t_j}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds - \mathcal{O}_1^n \end{aligned} \tag{7}$$

one can show that the first four term on the right-hand side of equation (7) cancels each other by using induction method. Hence, we have

$$\int_0^{t_{n-1}} f'(s)e^{-\frac{\alpha}{1-\alpha}(t_n-s)}ds = \sum_{j=1}^{n-1}\frac{f(t_j)-f(t_{j-1})}{h_j}\int_{t_{j-1}}^{t_j}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds - \mathcal{O}_1^n \tag{8}$$

hence,

$$\begin{aligned}|\mathcal{O}_1^n| \leq & \frac{\alpha}{1-\alpha}\max_{0\leq t\leq t_{n-1}}|f''(t)|\sum_{j=1}^{n-1}\frac{h_j^2}{8}\int_{t_{j-1}}^{t_j}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds\\ \leq & \frac{\alpha}{1-\alpha}\frac{h^2}{8}\max_{0\leq t\leq t_{n-1}}|f''(t)|\int_0^{t_{n-1}}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds\\ \leq & \frac{h_{\max}^2}{8}\max_{0\leq t\leq t_{n-1}}|f''(t)|\exp\left[-\frac{\alpha}{1-\alpha}h_n\right] \end{aligned}\tag{9}$$

and by Taylor expansion of the first order derivative

$$\left|f'(\tau)-\frac{f(t_n)-f(t_{n-1})}{h_n}\right|\leq\frac{h_n}{2}\max_{t_{n-1}\leq t\leq t_n}|f''(t)|$$

We can have the error in the interval $[t_{n-1},t_n]$ of the second term on the right-hand side of equation (4). Then the over-all interval satisfies the absolute error function of the form

$$|\mathcal{O}^n|=\left(\frac{h_{\max}^2}{8}+\frac{h_n(1-\alpha)}{2\alpha}\right)\max_{0\leq t\leq t_n}|f''(t)|\exp\left(-\frac{\alpha}{1-\alpha}h_n\right)\tag{10}$$

which satisfies to our required error estimate. □

In this case, the numerical error estimate of the CF fractional derivative is of exponentially weighted order, with the weigth term $\frac{\alpha}{1-\alpha}$. The error estimate has found to be of order $\mathcal{O}=\mathcal{O}\left(\frac{1-\alpha}{\alpha}N^{-1}\exp\left[-\frac{\alpha}{1-\alpha}N^{-1}\right]\right)$ which vanishes for large N. Thus, minimal error can be acquired easily using this method. In this study, we are also interested in the non-uniform mesh as found in [14] defined as

$$h_n=(N-n+1)\nu,\quad 1\leq n\leq N\tag{11}$$

where $\nu=\frac{2T}{N(N+1)}$. We aim to improve the error estimate upon considering the non-uniform mesh.

Theorem 2.2. *Let the fractional order be $\alpha\in(0,1)$ and a function $f(t)\in\mathcal{L}^2[0,T]$. Then for non-uniform mesh, the CF fractional derivative approximation holds*

$$|(\mathcal{O}_1)^n|\leq\left(\frac{1-\alpha}{\alpha}\right)^2\max_{0\leq t\leq t_{n-1}}|f''(t)|\left[\frac{\alpha}{1-\alpha}\frac{T}{N+1}\right]^3\exp\left[-\frac{\alpha}{1-\alpha}\frac{T}{N+1}\right]\tag{12}$$

for error estimate for the interval $[0,t_{n-1}]$. Meanwhile, the trunction function at the interval $[0,t_n]$ satisfies

$$|\mathcal{O}^n|\leq\left\{\frac{\alpha}{1-\alpha}\left[\frac{T}{(N+1)}\right]^3+\frac{1-\alpha}{\alpha}\left[\frac{T}{(N+1)}\right]\right\}\max_{0\leq t\leq t_n}|f''(t)|\exp\left[-\frac{\alpha}{1-\alpha}\left(\frac{2T}{N+1}\right)\right]\tag{13}$$

Proof. The error estimate presented in Theorem 2.1 in the interval $[0,T]$ on the nonuniform mesh satisfies

$$\begin{aligned}|(\mathcal{O}_1)^n|\leq & \frac{\alpha}{1-\alpha}\max_{0\leq t\leq t_{n-1}}|f''(t)|\sum_{j=1}^{n-1}\frac{h_j^2}{8}\int_{t_{j-1}}^{t_j}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-s)\right]ds\\ \leq & \frac{\alpha}{1-\alpha}\max_{0\leq t\leq t_{n-1}}|f''(t)|\sum_{j=1}^{n-1}\frac{h_j^3}{8}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-t_j)\right]\end{aligned}\tag{14}$$

from the given definition of the time increment, we can have

$$
\begin{aligned}
t_n - t_j =& \frac{1}{2}(h_j - h_{j+1})(n-j) \\
=& \frac{n-j}{2}((N-n+1)\nu + (N-j)\nu) \\
=& \frac{\nu(n-j)}{2}((2N+1-j-n)
\end{aligned}
$$

we can now get

$$
\begin{aligned}
\sum_{j=1}^{n-1} h_j^3 \exp\left[-\frac{\alpha}{1-\alpha}(t_n-t_j)\right] &= \nu^3 \sum_{j=1}^{n-1}(N-j+1)^3 \exp\left[-\frac{\alpha}{1-\alpha}\left(\frac{\nu(n-j)}{2}((2N+1-j-n)\right)\right] \\
&\leq \nu^3 \sum_{j=1}^{n-1}(N-j+1)^3 \exp\left[-\frac{\alpha}{1-\alpha}\left(\frac{\nu(n-j)}{2}((N-j+1)\right)\right] \\
&\leq \nu^3 N^3 \sum_{j=1}^{n-1} \exp\left[-\frac{\alpha}{1-\alpha}\left(\frac{\nu N n}{2}\right)\right] \exp\left[\frac{\alpha}{1-\alpha}\left(\frac{\nu N k}{2}\right)\right] \\
&\leq \nu^3 N^3 \exp\left[-\frac{\alpha}{1-\alpha}(\nu N)\right] \\
&\leq \frac{8T^3}{(N+1)^3} \exp\left[-\frac{\alpha}{1-\alpha}\left(\frac{2T}{N+1}\right)\right]
\end{aligned}
$$

which is the truncation error we wanted as presented in equation (12) when the above inequalities were substituted in equation (14). Now, if the mesh h_n is monotonically decreasing, such that $h_n \leq h_1$ for all $n \geq 1$. Then the truncation error at the interval $[t_{n-1}, t_n]$ can be obtained same method in Theorem 2.1 that satisfies

$$
\begin{aligned}
|(\mathcal{O}_2)^n| &\leq \frac{h_n(1-\alpha)}{2\alpha} \max_{t_{n-1}\leq t\leq t_n} |f''(t)| \exp\left[-\frac{\alpha}{1-\alpha} h_n\right] \\
&\leq \frac{h_1(1-\alpha)}{2\alpha} \max_{t_{n-1}\leq t\leq t_n} |f''(t)| \exp\left[-\frac{\alpha}{1-\alpha} h_1\right] \\
&\leq \frac{1-\alpha}{\alpha} \frac{T}{N+1} \max_{t_{n-1}\leq t\leq t_n} |f''(t)| \exp\left[-\frac{\alpha}{1-\alpha} \frac{2T}{(N+1)}\right] \qquad (15)
\end{aligned}
$$

Combining inequalities (12) and (15), we can have the estimated error for the interval $[0, t_n]$. □

The above theorems shows that the CF fractional derivative on the nonuniform mesh is having a third order truncation function which is really improves the accuracy of the numerical approximation. In addition, it can also be seen that the memory kernel of the fractional derivative is a great impact on the error estimate. In order to verify the accuracy of the presented numerical approximation, we give some numerical examples in the next section.

3 Numerical experiment

Consider a linear diffusion-wave equation whose exact solution is given by

$$x(t) = 3.5\sin(2.3\pi t)\exp(-3.23t)$$

We compute the fractional derivative of order α on the nonuniform and uniform mesh and we showed the errors by considering thegrid sizes to be $N = [20, 50, 70, 130, 220, 550, 670, 730]$. It is clear, in figure (1), that for non-uniform mesh, the error estimate have a third order convergence rate while the uniform mesh generates a second order accuracy. This new method of approximation for nonsingular fractional derivative improves the accuracy on non-uniform mesh.

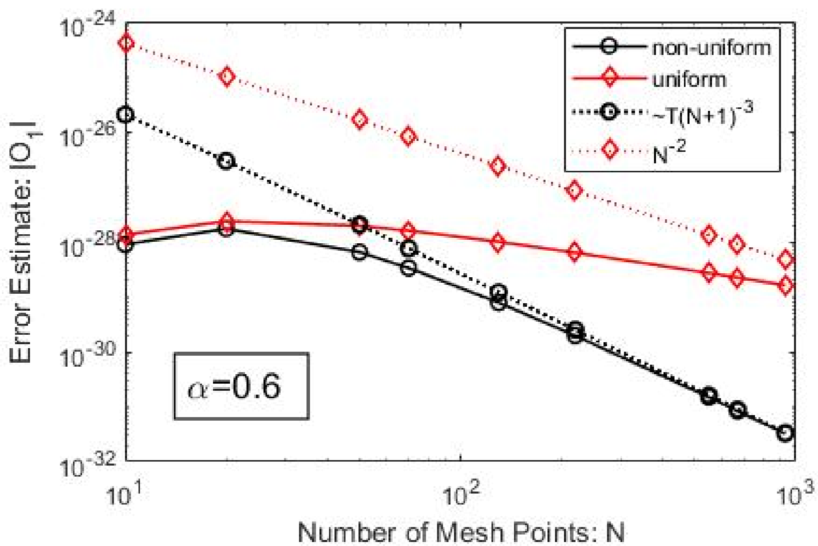

Figure 1 – Estimate errors for the approximation fractional derivative of the given diffusion-wave equation using CF fractional derivative with order $\alpha = 0.6$

Example: Consider the transcendental function of second order

$$f(t) = \exp\left(-1.7\frac{t^2}{\tau^2}\right)$$

where τ is interpreted as the relaxation time of the function. We wish to obtain its fractional derivative using numerical approximation introduced in this study and investigate the absolute error with respect to the fractional order α. We let $\tau = 1.7$ and considered the interval $[0, 1]$. We also presented the table below to see the error estimate between the uniform and nonuniform mesh.

α	N	h_n	Error(uniform)	Error(nonuniform)	ratio(%)
$1/2$	10	0.10000	0.0248830	0.0013721	5.5140
	20	0.05000	0.0124849	0.0002059	1.6493
	50	0.02000	0.0049990	0.0000148	0.2957
	70	0.01429	0.0035711	0.0000055	0.1543
	90	0.01111	0.0027776	0.0000026	0.0945
$3/4$	10	0.10000	0.0157424	0.0034319	21.8001
	20	0.05000	0.0086071	0.0005616	6.5252
	50	0.02000	0.0036258	0.0000426	1.1762
	70	0.01429	0.0026151	0.0000161	0.6145
	90	0.01111	0.0020449	0.0000077	0.3767

Table 1: Error estimate for a given function using the uniform and nonuniform mesh.

We investigate the error convergence order of a given sample function with different values of the N. Table 1 presents the error estimate for both uniform and non-uniform mesh, as well as the ratio between the two method and the values of fractional order. Evidently, the nonuniform mesh shows lesser error estimate compare to the uniform mesh, thus, accuracy can be improved if nonuniform mesh is considered in Caputo-Fabrizio fractional derivative. In addition, it is also evident that the error estimate is dependent on the fractional order value. However, the efficiency of uniform mesh becomes negligible compare to nonuniform mesh for larger value of N, as can be seen in the ratio between the two method.

4 Conclusion

We have shown that the nonuniform mesh for nonsingular fractional derivative improves the accuracy of the numerical approximation compared to the uniform mesh. To support the presented numerical scheme, we provide some examples and interestingly shows the given claimed error estimate. In addition, the presented numerical schemes can also be applied to a fractional differential equation.

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Conflict of interests

The author declares that this study have no competing interests regarding the publication.

Nomenclature

CF-Caputo-Fabrizio
α-fractional order of derivative

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