New Exact Solutions for Modified Burgers Vortex

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Abstract: Exact solutions of Navier-Stokes equation, presenting a new asymmetric vortex, in framework of modified by Shivamoggi Burgers vortex [8], has been obtained both for steady and for specific unsteady cases. The steady vortex is expressed by the function of parabolic cylinder, unsteady one is expressed by Hermit polynomials.

Kay words: Asymmetric, Burgers vortex, Hydrodynamics, Exact solution, Turbulence

1 Introduction

The standard whirlwind of Burgers is axisymmetric. In cylindrical system of co-ordinates (r, θ, z) it is defined as

$v_r = -\gamma r,$

$v_\theta = \omega r_0^2\,[1- \exp(-r^2/r_0^2)]/r,$

$v_z = 2\gamma z$ (1)

Also represents a whirlwind with a converging stream of substance to its center where γ characterizes a converging stream, and ω and r_0 - circulation and the size of a trunk of a whirlwind. Rotation in the field of a whirlwind trunk almost solid-state and on the big distances a profile of rotary speed falls down under the hyperbolic law.

In the Cartesian system of co-ordinates (x, y, z) the standard whirlwind of Burgers will be presented in a kind

$v_x = -\gamma x - \omega r_0^2 y\,[1 \exp(-r^2/r_0^2)]/r^2,$

$v_y = -\gamma y + \omega r_0^2 x\,[1 \exp(-r^2/r_0^2)]/r^2,$

$v_z = 2\gamma z,$ (2)

where $r^2 \equiv x^2 + y^2$.

This whirlwind in works [1,2] has been used by as a trap for dust particles in the course of planetesimals formation.

Liquid turbulent flows show on a profuseness of the stretched whirlwinds of an average and a vast scale. Process of a stretching of a whirlwind is connected with energy transport in various scales of turbulence, and also with processes of disintegration and whirlwinds recombination, process - not quite understood now. In the literature there are the works devoted to generalization of Burgers vortex with no axisymmetric stream lines.

Authors of work [3] investigated the solution for a whirlwind of Burgers, and have found the closed form of steady private solutions. Unstable 2D by solutions of a whirlwind of Burgers have been simulated spatial structure of rough turbulent layers ([4,5]). The axisymmetric whirlwind of Burgers is widely used in problems of modelling of thin structure of turbulence of a homogeneous incompressible liquid ([6,7]).

The author of work [8] considered the modified whirlwind of Burgers which describes convection lines of a whirlwind round axis Z and extended along axis Y. Exact solutions have been found in a special stationary case when the stream parameter γ is constant.

In the present work possibility of new exact solutions, as for stationary, and a specific non-stationary case is shown.

2. Generalized Burgers Vortex

The field of speeds of the modified whirlwind of Burgers considered in work [8], in the Cartesian system of co-ordinates has been presented in a kind

$$\mathbf{v} = (-\gamma(t)\,x,\ \gamma(t)\,y,\ W(x,t)), \quad (3)$$

which describes convection lines of a whirlwind round axis Z and stretched along axis Y. Lines of speed of a stream are identical in the planes parallel *XY* to a plane. A rotor of speed field (3) has only component Y:

$$\boldsymbol{\omega} = \nabla\times\mathbf{v} = (0,\ -\partial W/\partial x,\ 0) \quad (4)$$

therefore, vortical stream lines are extended along the axis Y.

Using (3) and (4), the equation of Navier-Stocks we will present in a kind

$$\partial\boldsymbol{\omega}/\partial t + (\mathbf{v}\cdot\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\,\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}$$

or

$$\partial\Omega/\partial t - \gamma x\,\partial\Omega/\partial x = \gamma\,\Omega + \nu\partial^2\Omega/\partial x^2 \quad (5)$$

where v is kinematic viscosity of substance, and

$\Omega \equiv \partial W/\partial X$.

Introduction of dimensionless independent variables

$$\xi = x\sqrt{\gamma(t)/\nu}, \qquad \tau = \int\gamma(t)\,dt, \quad (6)$$

The equation (5) is led to a look [8]

$$\frac{\partial\Omega}{\partial\tau} + \frac{\dot{\gamma}}{2\gamma^2}\xi\frac{\partial\Omega}{\partial\xi} = \frac{\partial(\xi\Omega)}{\partial\xi} + \frac{\partial^2\Omega}{\partial\xi^2}. \quad (7)$$

Let's notice that the factor of the second member in the left part (7) depends on τ through the relation (6). Generally this equation does not suppose exact solutions, except for special cases. In work [8] the case γ = constant at which (7) supposes the exact solution in the form of polynomials of Hermit has been considered.

Other case at which exact solution there exists, is

$d\gamma/dt = 2A\gamma^2$,

which solution looks like

$\gamma(t) = -1 / (2At + B)$,

where A and B constants chosen so that, $\gamma(t) > 0$. It is necessary, that the co-ordinates defined by parities (6), were real. We will notice that, at A = 0 case considered in work [8] γ = constant is received. The equation (7) takes the form now

$\partial\Omega / \partial\tau = \Omega + \alpha\xi\partial\Omega / \partial\xi + \partial^2\Omega / \partial\xi^2$,   (8)

where $\alpha = 1 - 2A > 0$. The solution of this equation should satisfy to a condition

$|\xi| \to \infty, \Omega \to 0$.

3 Exact Solutions

If to assume $\partial\Omega / \partial\tau = 0$ the exact solution of the equation (8) looks like function of the parabolic cylinder:

$\Omega(\xi) = C\, exp\{- \alpha\xi^2/4\} \cdot D_{1/\alpha-1}(\xi)$.

Therefore a z - component of speed will be expressed as

$$W(\xi) = C\sqrt{\frac{\gamma(t)}{\nu}}\int_0^{\xi} e^{-\alpha\xi^2/4} D_{1/\alpha-1} d\xi,$$

where C is arbitrary constant.

Graphs of function of the parabolic cylinder, $D_{1/\alpha-1}(\xi)$, for values of parameter α = l and α = 1/3 are presented on Figure 1.

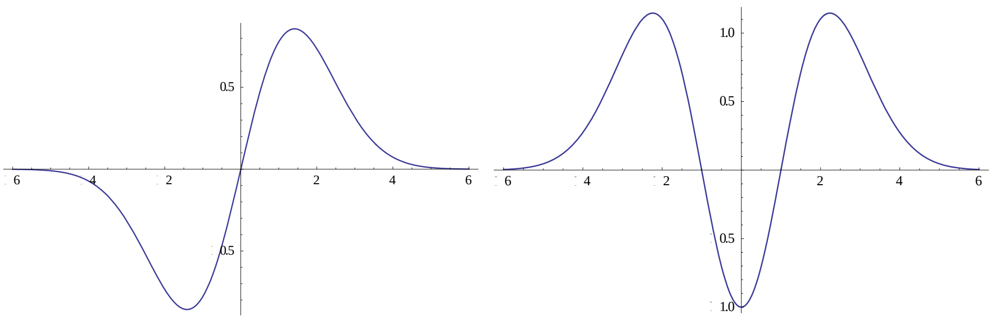

Fig. 1 Function of parabolic cylinder $D_1(\xi)$ and $D_2(\xi)$ - down

The equation (8) supposes also the exact non-stationary solution with divided variables. Representing the solution in a kind

$\Omega(\xi, \tau) = h(\xi)\, e^{-\lambda\tau}$,   (9) Where λ - a constant, from (8) for *h* we receive the equation

$$\frac{d^2h}{d\xi^2}+\alpha\xi\frac{dh}{d\xi}+h=-\lambda h. \qquad (10)$$

That the solution (9) was limited, we will demand

$\lambda_n = (n + 1)\alpha - 1, \qquad n = 0,1,2... \qquad (11)$

That gives the solution in the form of polynomials of Hermit

$h_n(\xi) = (-1)^n exp\{-\alpha\xi^2/4\} H_n(\xi\sqrt{\alpha}/2), \qquad (12)$

Where

$H_0(\xi) = 1$, $H_1(\xi) = \xi$, $H_2(\xi) = \xi^2 - 1$...

Detailed properties of the received solutions and their application to protoplanetary disks will be given in the subsequent works.

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