Capillary Waves Generated by a Rock in a Stream

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Abstract

This paper is concerned with capillary waves generated by a rock, or fishing line in a stream. It is found mathematicaly that the angle between the capillary wave envelope and the direction of a point source is dependent of the velocity, surface tension, and density of the fluid:

This angle increases with increasing the surface tension, but decreases with increasing the square of velocity and the density of fluid. These theoretical outcomes are consistent with the detailed behaviors of capillary waves observed in the natural running streams. To enhance the mathematical analyses on capillary waves generated by a fishing line with constant speed, the relevant non-dimensional parameters $\pi_1=Ut/r$ and

$\pi_2=r^3\rho/(t^2\sigma)$ have derived based on the Buckingham π-Theorem, where U the relative stream velocity, t the time, r the distance from the point source of wave generation, σ the surface tension, and ρ the density of the fluid.

It has been confirmed by the present proposed approach that the angle between the gravity wave envelope and the direction of a duck or ship moving on the calm water surface is constant of 19.28°,which agrees with the result obtained by "Thomson, W., 1887 On ship waves, Institution of Mechanical Engineers, Minutes of Proceedings, 409-434" and "Adam, J.A., 2003 Mathematics in nature: modelling patterns in the natural world, Princeton University Press, Princeton, 161-172".

Keywords: Capillary Wave; Gravity Wave; Surface Tension; Kelvin Wave; Group Velocity

1. Introduction

In historical report on waves, Russell[8] wrote:'One observation which I have made is curious. It is that in the case of oscillating waves of the second order, I have found that the wave motion of propagation of the whole group is different from the apparent each wave motion of translation along the surface.'

At that time, Stokes was working on the measurement of sea waves for the Britsch Meteorological Council. In particular, he was asked to determine the origin of the strong swells some times observed in fine weather. Stokes immediately explained these swells by wave propagation from far distant storms.

Stokes[9] reported to Airy:'I have lately perceived a result of theory which I believe is new-that the group velocity of waves on water is, if the water be deep, only half of the velocity of the individual waves.

Osborne Reynolds[7] reported his own observations of wave groups produced by throwing a stone into a pond, by the interference of sea waves, or by the motion of a ship. Like Russell, he noted that groups of waves in deep water traveled slower than the individual waves of which they were made.

To explain this result, he first noted that the velocity of a wave group obviously represented the velocity of propagation of energy. He then showed that the energy velocity differed from the phase velocity. For instant, the waves produced by wind in a corn field obviously do not propagate any energy. In a more complex case of a sine wave on deep water, the particles of water move on along circles with almost constant velocity, so that no kinetic energy is transmitted by the wave.

On the other hand, the potential energy is transmitted at the phase velocity c. Since the potential energy of such waves is just half of their total energy, the speed of energy propagation c_g is half the phase velocity c, as is well known today.

The concept of group velocity could plausibly have emerged in the field of physics, where dispersion was first known, namely optics and acoustics(Lamb [4]).

While Thomson[10], who had another name, Lord Kelvin was sailing on his personal yacht, he observed a gentle but no less interesting phenomenon: A fishing line hanging from the slowly-crusing yatcht casues very short capillary waves (wavelength $\lambda<1.72$ cm) or 'ripples' directly in front of the line, and the much longer gravity waves(wavelength $\lambda>1.72$ cm) in its wake.

The whole pattern is steady with respect to the line, so that the celerity of both kinds of waves is equal to the velocity of the line's progression through the water.

Thomson only reasoned on free waves, but did not try to the process through which the fishing line causes the waves. Rayleigh[5] accomplished this much difficult task. Using a favorite strategy, he first changes the problem of progressive waves into an equivalent steady-wave problem by selecting the suitable system of reference bound to the perturbing cause or fishing line.

Then, he computes the distribution of surface pressure that corresponds to a sine wave in the restricted two-dimensional problem. Like Rayleigh, Thomson has superposed the disturbances produced by pressures constantly applied on straight horizontal lines passing through a fixed point of the water surface, to be identified as the location of the boat. Had he followed Rayleigh even further, he could have obtained the wave pattern geometrically, by tracing the envelops of the component wave crests(Lamb [3]).

Of all the forms of water waves, generated by a rock or fishing line, are perhaps most beautiful and intriguing phenomena among those observed in our daily lives(Adam [1,2]). This prompts the authors to investigate in more detailed aspects on these waves, in terms of much more simplified and straight forward method of mathematical analyses than Thomson[11] and Adam[1.2].

The main purpose of the present paper is to confirm mathematically that the ship waves or Kelvin waves are independent of ship speed and/or gravity, but capirally waves depend on the speed of the source point generating the waves, surface tension and the density of fluid. The theoretical results have been argued by referring the relevant observation, to undermine the current knowledge on physics of waves on water surface.

2. Individual Wave Speed and the Group Speed

Let us examine very short waves or ripples such that h/λ is large, so that the speed of an individual wave crest propagating in one dimension may be expressed by

$$c=\lambda\nu=\{[(g\lambda/(2\pi) + 2\pi\sigma/(\rho\lambda)]\tanh(2\pi h/\lambda)\}^{1/2}, \qquad (2.1)$$

where λ is wavelength, ν wave frequency, ρ fluid density, h fluid depth, σ surface tension, and g gravitational acceleration. In another formulation, the speed c of an individual wave of wavelength λ is given by $c=\omega/k$, and the speed c_g of a group of wave is given by $c_g=d\omega/dk$, because (2.1) can be expressed in terms of the angular frequency $\omega(=2\pi\nu)$ and wavenumber $k(=2\pi/\lambda)$ as

$$\omega=ck=k[(g/k + \sigma k/\rho)\tanh(kh)]^{1/2}. \qquad (2.2)$$

If we now examine very short waves or ripples such that h/λ is large, the wave speed reduces to now approximately,

$$c=[2\pi\sigma/(\rho\lambda)]^{1/2}. \qquad (2.3)$$

Since in this approximation, c is inversely proportional to the square root of λ, shorter waves travel faster than longer waves.

Using the expression of the speed of a group of wave $c_g=d\omega/dk$, we can derive an important relationship among c, λ, and c_g:

$$c_g=c-\lambda dc/d\lambda. \qquad (2.4)$$

It is worth noting in (2.4), and it follows if $dc/d\lambda > 0$, $c_g< c$, whereas if $dc/d\lambda< 0$, $c_g > c$. In the first case, such as for surface gravity waves(wavelength $\lambda> 1.72$ cm), the speed of individual wave of a given wavelength is greater than the speed of a group of them.

A typical example of the other situation is found in capillary waves (wavelength $\lambda< 1.72$ cm), for which the group speed exceeds the wave speed(Apendix 2).

In fact, for gravity waves, the wave speed is exactly twice the group speed $c=2c_g$, while for capillary waves it is two-thirds of the group speed $c=2c_g/3$. In optics, these situations are referred to as normal and anomalous dispersion, respectively.

Using (2.4), it is easy to show that individual ripple gets overtaken by groups of this type of capillary wave.

If an individual ripple has radius r at time t, it is part of a group whose speed on average r/t, but the ripple's speed is only 2/3 of the group speed, and is represented by the derivative dr/dt, so that now

$$dr/dt=(2/3)r/t, \quad (2.5)$$

Referring (2.1), we have for long wave $c\approx(gh)^{1/2}$, i.e., the wave speed is independent of wavelength, and so there is no dispersion or c is independent of λ: Waves travel with the same speed regardless of wavelength. It follows from (2.4) that $c=c_g$.

In fact 'Tsunami' fall into this category of shallow water waves, but if we focus on shorter waves in deep waves: Then,consider the case when effect of gravity is dominant comparing with effect of surface tension, and we have $c\approx[g\lambda/(2\pi)]^{1/2}$, the longer the wavelength, the faster it moves.

This time, the group speed is half the wave speed, $c_g=c/2$, as is shown using (2.4).

It is known that there is a simple model explaining the reason why ocean waves tend to line up parallel to the beach contour, even if they are approaching to the beach obliquely far out the sea.

To simplify the discussion, picking up a particular wave having a constant wavelength. Then, far out the beach, the wave must be in the category of so called deep wave($\lambda \ll h$) and so $c\propto\lambda^{1/2}$. On the other hand, nearer the beach, the wave is in shallow water($\lambda\gg h$), and so $c\propto h^{1/2}$, which is smaller than $\lambda^{1/2}$.

Hence, the part of the wave front near the beach slows down compared to that of the wave far out the beach, and this results in the wave front line tends to be aligned to being parallel to the beach contour.

3. Buckingham π-Theorem

In this section, surface capillary wave pattern generated by a fishing line with constant speed, or a rock protruding to the air from the channel bed in the stream, is discussed based on Buckingham π-theorem(Zierep & Nakagawa [13]) in order to obtain the relevant non-dimensional parameters consisting of 5 independent governing variables, to be used in the theoretical analyses.

Let us consider that the surface wave pattern generated by a fishing line moving with a constant velocity U, is governed by the distance r from the point source of wave generation to the present position, the time t, the velocity U, the surface tension σ , and the fluid density ρ as summarized in Table 1.

Table 1 The relevant physical quantities.

Cassification	Symbol	Variable	Dimension	Number
Geometrical Quantity	r	distance	L	B_G
Kinematical Quantity	t	time	T	B_K
	U	velocity	LT^{-1}	A_1
Dynamical Quantity	σ	surface tension	MT^{-2}	B_D
	ρ	density	ML^{-3}	A_2

Referring Table 1, we have the two π ratios,

$\pi_1 = (B_G)^{x_1}(B_K)^{y_1}(B_D)^{z_1}(A_1),$ (3.1)

$\pi_2 = (B_G)^{x_2}(B_K)^{y_2}(B_D)^{z_2}(A_2).$ (3.2)

Substituting the symbols and dimensions in (3.1), we have

$\pi_1 = (B_G)^{x_1}(B_K)^{y_1}(B_D)^{z_1}(A_1) = \pi_1 = (r)^{x_1}(t)^{y_1}(\sigma)^{z_1}(U)$

or

$(M^0L^0T^0) = (L)^{x_1}(T)^{y_1}(MT^{-2})^{z_1}(LT^{-1}).$

The method of undetermined coefficient(s) together with the above relation provides us

$x_1=-1,\ y_1=1,\ z_1=0,$

and thus we obtain

$\pi_1=Ut/r.$ (3.3)

Similarly to the above, we get the second noo-dimensional parameter ,using (3.2)

$\pi_2=r^3t^{-2}\sigma^{-1}\rho.$ (3.4)

Let us now examine how the distance r from the point source of wave generation changes dependeing on the time t, surface tension σ, density ρ and velocity U, in (3.3) and (3.4). It may be convenient to write (3.3) and

(3.4), respectively,

$r=Ut/\pi_{1,}$ (3.5)

and

$r=\pi_2^{1/3}(\sigma/\rho)^{1/3}t^{2/3}$. (3.6)

Since it is realized that the first non-dimensional parameter,(3.5) results in trivial outcome, we concentrate (3.6) for the further consideration.

4. Theoretical Analyses

At first, it may be adequate to rewrite (3.6) as

$r=\varphi(\sigma/\rho)^{1/3}t^{2/3}$, (4.1)

where $\varphi = \pi_2^{1/3}$ is constant. Then, we consider the case on the still water surface, in which a single fish line moving with a constant velocity U continues to generate waves, or a stationary rock under the stream of a constant velocity U is generating waves. Define the origin of the coordinate x at the present point and at the time t, as depicted in Fig. 1, and then let be back to the point at t=0: The distance from the present point to a point source generating waves is

$x_0(t)=Ut$. (4.2)

Now, it is possible to express each of the circles on the water surface delineated by the wave crest at the arbitrary time t as

$[x - x_0(t)]^2 + y^2 = r^2(t)$. (4,3)

By definition, the envelope of a series of circular waves is derived by solving the following system of simultaneous equations,

$F=[x - x_0(t)]^2 + y^2 - r^2(t) =0$, (4.4)

and

$\partial F/\partial t=0$. (4.5)

Subsitution of (4.1) and (4.2) in (4.4) gives us

$F=[x - Ut]^2 + y^2 - \varphi^2(\sigma/\rho)^{2/3}t^{4/3} = 0$. (4.6)

Following (4.5), differenciating (4.6) with respect to the time t, we get

$x=Ut-(2/3)(\varphi^2/U)\ (\sigma/\rho)^{2/3}t^{1/3}$. (4.7)

Then, substituting (4.7) in (4.4), we obtain

$y=\pm\{\varphi^2(\sigma/\rho)^{2/3}t^{4/3}-(4/9)(1/U^2)\varphi^4(\sigma/\rho)^{4/3}t^{2/3}\}^{1/2}$, (4.8)

where φ is so called group parameter, which means that we can draw curves corresponding to for each of the value. Then, we seek for the points ,where they stay on the envelope, so that we must find the time t, at which x and y take the local maxima, respectively. For this purpose, differentiating x with respect to the time t, we find such time,

$t=(\surd 2/3)^3(\varphi/U)^3(\sigma/\rho)$. (4.9)

Substituting t of (4.9) in (4.7) and (4.8), we get their local maxima for x and y, respectively,

$x_{max}=-(4\surd 2/27)(\varphi^3/U^2)(\sigma/\rho)$. (4.10)

and

$y_{max}=\mp(4/81)(\varphi^6/U^4)(\sigma/\rho)^2$. (4.11)

It may be interesting to examine the ratio of y_{max}/x_{max}, which provides us the envelope of the capillary waves generated by a fish line moving with a constant velocity or a rock on the river bed in the stream extruding through the water surface to the air. Normally, the envelope is consisting of the left and right branches having a certain angle θ with respect to the primary direction of the fish line or the stream, and shows almost symmetry fashion, as depicted in Fig. 1.

Eqs.(4.10) and (4.11) immediately guves us

$y_{max}/x_{max}=\pm(1/3)(1/\surd 2)(\varphi^3/U^2)(\sigma/\rho)$. (4.12)

This relation indicates clealy that the ratio (4.12) is not constant, but dependent on the velocity U, the surface tension σ, and the fluid density ρ. The angle θ between the envelope and the primary direction or stream direction is expressed by

$\Theta=\tan^{-1}y_{max}/x_{max}$. (4.13)

Referring (4.12), it is realized that θ increses with increasing the surface tension σ, but decreases with increasing the square velocity U^2 and the fluid density ρ. It is worth noting here that the angle between the gravity wave envelope and the direction of ship moving on the calm water is constant of 19.28°, which agrees with the value obtained by Thomson[11] and Adam[1].

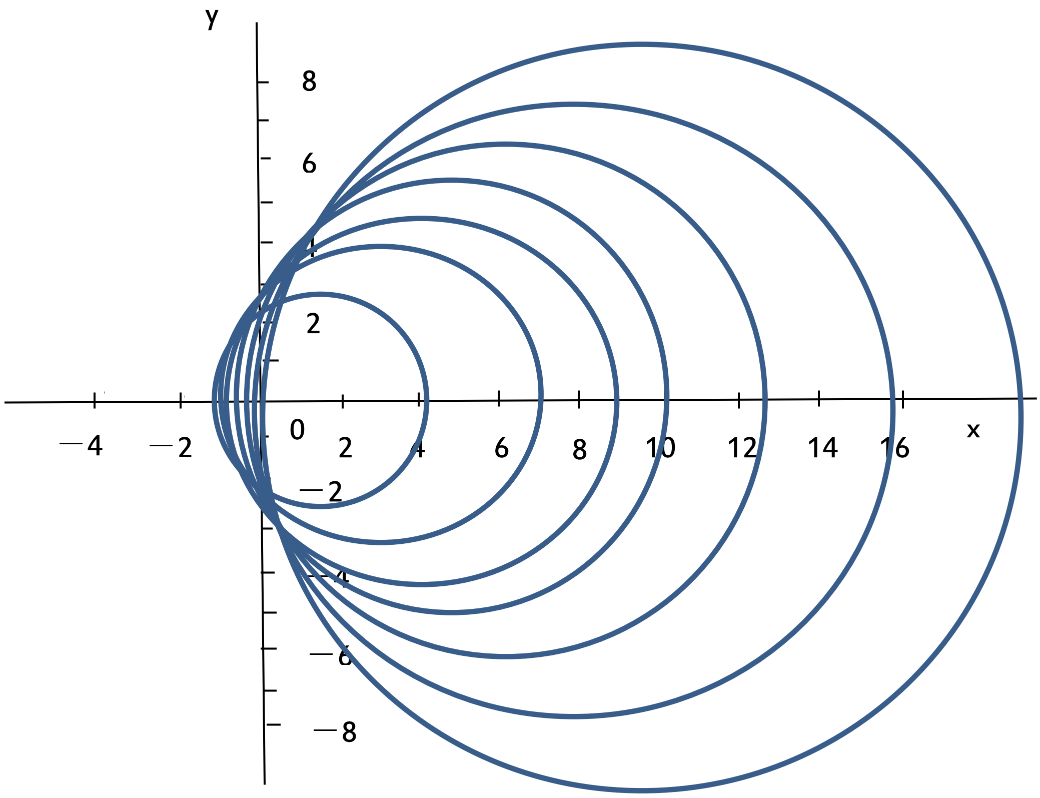

Fig. 1 An envelope of capillary waves generated by a moving point source, or stationary point source in the water stream with a constant speed.

5. Discussion

In this section, the theoretical results ontained in the previous section will be discussed with the relevant observation to undermine the knowledge on waves generated on the water surface.. Fig. 2 shows an example of capillary waves upstream of a rock in the water channel in the Castle garden, Komatsu,Japan.

(a)

(b)

Fig. 2 Capillary waves (ripples) upstream of a rock in the water channel in the castle garden, Komatsu, Japan.

Flow is from upper left to below right. (b) is an enlarged view of (a). Note that there co-exist gravity waves havin much larger wavelength than 1.72cm around the rock. In Fig. 2(b), it is notable that angle of the wave crest with respect to the direction of stream decreases with increasing the distance from the rock in the transverse direction, for the local velocity of the stream might increses accordingly. It is evident that the wavelength of capillary wave decreses with increasing the distance from the rock, for in case of capillary wave, the speed increases as the wavelength decreases(Appendix 2). This is the reason why the capillary wave with the minimum wavelength stays at the top front of capillary waves. It is, however, considered that in front of the rock, both of the capillary and gravity waves appear being co-exsisting in such a way the envelope of gravity waves is formed closer positions from the rock, while capillary waves are arranged in front of the gravity wave envelop in a way that waves taking longer wavelength stay near the gravity wave enelop but those with shorter wavelength take at more remote position from the envelop. It is quite noteworthy that the envelope of capillary wave crests varies depending on the stream velocy in such way that the angle with respect the stream direction decreases with increasing the velocity, as being found theoretically in the previous section(see equation 4.12). On one hand, the envelope of the gravity waves does not be altered the angle with respect to the flow direction irrespective of the change of the stream velocity:

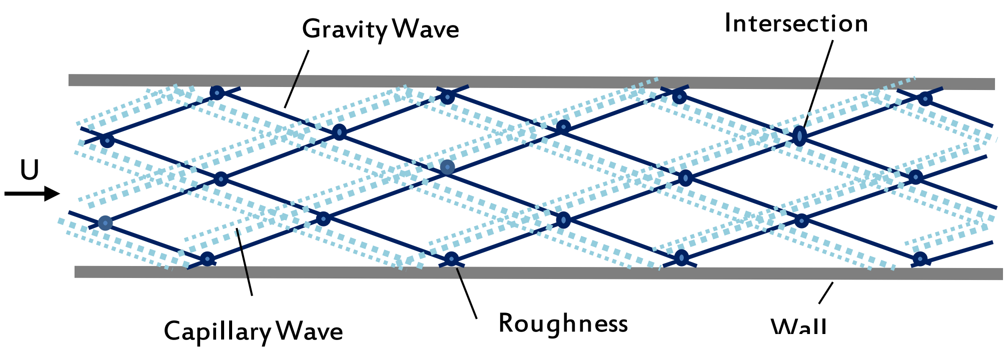

Fig. 3 Skech of interactions among gravity waves and capillary waves on the water surface in a channel.

Flow direction is from left to right, and flow velocity is constant. Everyday, the present authors are observing the surface of running water in an irrigation channel with two walls: when we looks at the water surface from the upstream to the downstream, there are numeral parallel lines of wave crests generated from the left wall having an angle of ca. 20° to clockwise didirection with respect to the line of water edge, while from the right wall having an angle of ca. 20° to counter clockwise, so that eah the crest line from the left wall interferes with that from the right wall and vice versa, as shown in Fig. 3. Furthermore, in front of each the envelop due to the gavity wave, several crest lines of capillary wave have been noticed. In the case of flow in the irrigation channel or riverl, each of the roughness on the wall surfaces play a role as point source of wave generation, so that these points emit gravity, capillary and/or their combined waves continuously depending on the flow and roughness conditions. Such a water surface ,therefore, looks like gallery on gavity waves, capillary waves and their complicated interactions as depicted in Figure 3, so this results in delicate artistic pattern woven by the waves on water surface, which have attracted people in this globe since the second half of 19 century. The similar approach has been done to analize the gravity wave patterns, or so called Kelvin waves, due to a single point source such as ship or duck in still water pond(Nakagawa & Chanson[6]). Contrary to intuition, in this case the value of this ratio y_{max}/x_{max} is absolutely constant, so is independent of the ship speed U, and the gravitational acceleration g. That is, when a body such as ship or duck moves with a constant velocity on calm water surface with a constant velocity, the envelope delineates two slant lines to the right and left,espectively, with the same angle of 19.28° with respect to the moving direction of the body. Note that Thomson[11] derived the same value in terms of a different and much more complicated mathematical analyses. See Lamb[3,4] for the more details. Fig. 4 depicts an envelope of surface gravity waves produced by a moving point source such as a ship or duck in the still water.

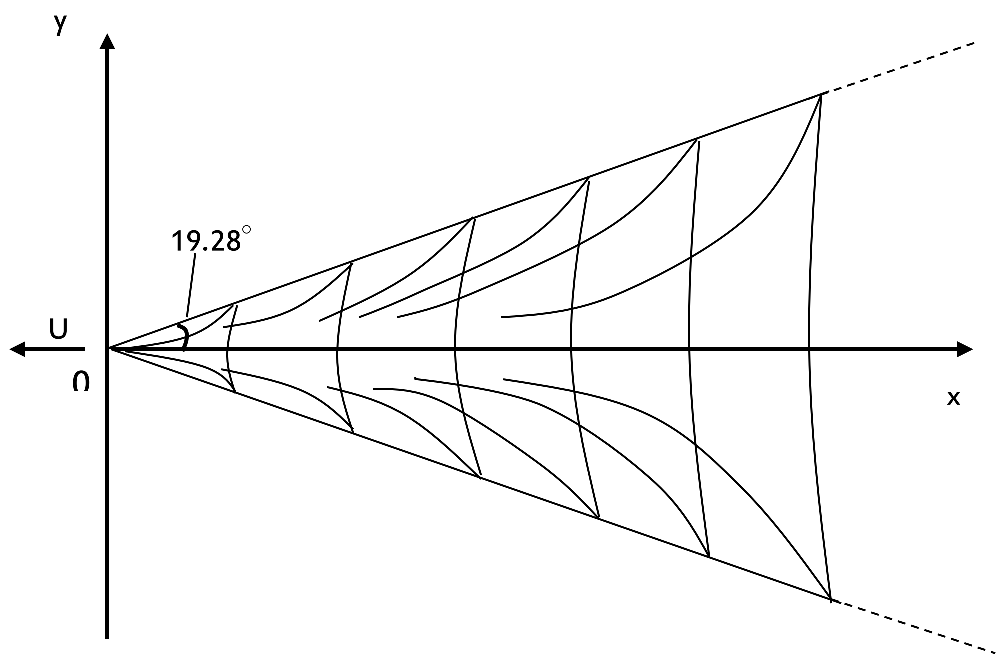

Fig. 4 Gravity wave patterns generated by a moving duck or ship with constant velocity U.

Fig. 5 shows the gravity wave patterns produced by the movement of ducks on the water surface in the Kakehashi river near Komatsutenmangu-shrine, which is located on the right bank at about 3km upstream from the river mouth. These photos had been taken during the flood tide in the afternoon on 19 January 2020, so it was observed that on the river surface weak tidal bores with the heig ht 5cm approximately, were propagating to the upstream slowly. Non the less, the water surface was reasonably flat and thus the wave patterns due to the motion of ducks were clearly delineated on the water surface as seen these photos. The bed slope of the Kakehashi-river in this reach is below 1:5000, so that the flow velocity towards the river mouth is normally slow and the flow direction is sometimes even reversed by the flood tide. Hence, at the moment these photos were taken, it is considered that the flow velocity in the river is almost null.

(a)

(b)

Fig. 5 Gravity wave patterns produced by the movement of ducks on the water surface in the Kakehashi- river near Komatsutenmangu shrine.

6. Conclusions

In this section, new knowledge and insights obtained through the present study have been summarized: It is found mathematicaly that the angle between the capillary wave envelope and the direction of a point source is dependent of the velocity, surface tension, and density of the fluid: The angle increases with increasing the surface tension, but decreases with increasing the squre velocity and the density of fluid. These theoretical outcomes are consistent with the detailed behaviors of capillary waves observed in the natural running streams.

To enhance the mathematical analyses on surface capillary waves generated by a fishing line with constant speed, two non-dimensional parameters $\pi_1=Ut/r$ and $\pi_2=r^3\rho/(t^2\sigma)$ have derived based on the Buckingham π-Theorem, where U the relative stream velocity, t the time, r the distance from the source point of wave generation to the present position, σ the surface tension, and ρ the density of the fluid.

It has been confirmed by the present proposed mathematical analyses that the angle between the gravity wave envelope and the direction of a duck or ship moving on the calm water surface is constant of 19.28° which coincides with the result obtained by Thomson[11] and Adam[1]..

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Appendix 1: Surface waves intersections:

Walking by the Kiessee in Göttinge, Germany one morning, we noticed a single duck sitting peacefully about 10m from me. As it heard the sound of our approaching footsteps, it scrambled to "walk on water", flapping its wings to achieve lift as raced out across the watery runway. Each time its webbed foot touched the water surface, waves were generated.

Long before it finally became airborne, a line of these waves started to interfere with each other and produce fascinating intersection patterns. Suddenly, an idea of the following thought experiment comes to our mind: Assume that two pebbles are thrown into a calm water surface of pond one after another, thus they act as distinct "point" sources of waves.

The points of intersection are a distance $|\overset{0}{F}P|=r_1(t)$ away from the center of circle 1, and a distance $|FP|=r_2(r)$ away from the center of circle 2. Furthermore, assuming that the speed of the both waves is constant, we have the following condition,

$dr_1/dt = dr_2/dt.$ (A.1)

Integrating (A.1) with respect to the time t, we obtain

$r_1 - r_2 = \text{constant}.$ (A.2)

Surprisingly, (A.2) is no more than the definition that the points of intersection follow a hyperbolic path. The resulting intersections for the two sets of waves

are drawn in Fig. 6.

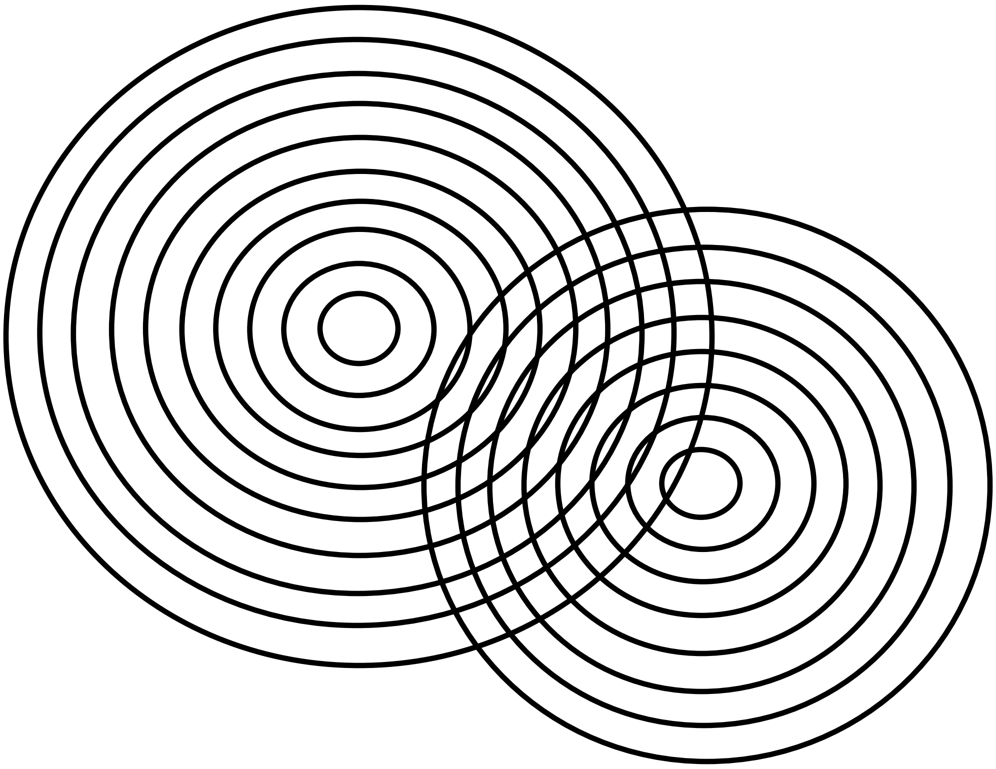

Fig. 6 Interacting waves generated by the two point sources dropped on calm water surface.

Appendix 2: Critical wavelength and the minimum speed of surface water wave

The critical wavelength can be obtained by differentiating

$c^2 \approx g\lambda/(2\pi) + 2\pi\sigma/(\rho\lambda)$, (B.1)

with respect to the wavelength λ as

$\lambda_c = 2\pi[\sigma/(g\rho)]^{1/2} = 1.72$ cm. (B.2)

For wavelengths less than or greater than this, the dominant restoring forces tend to be surface tension and gravity, respectively. Substituting the minimum wavelength λ_c into (B.1), we get the minimum speed of wave to be

c_{min} = 23cm/s. (B.3)

This means that any breeze or gust of wind with speed less than 23cm/s may not generate any propagating waves, other than a transient disturbance. Wind speeds above 23cm/s will, in principal, generate two sets of waves, with wavelengths on each side of 1.72cm, i.e., one set with λ <1.72cm(capillary waves) and one set with λ>1.72cm(gravity waves).

There is no limit to the maximum speed of water waves if their wavelength is small enough principally. It might be thought that a similar conclusion, i.e. no speed limitation of wave, applicable to long waves as well, for the maximum speed is expressed by $(gh)^{1/2}$.