

Various bounds of group of Autocentral Automorphisms

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Abstract

In this paper, we find various bounds for the group of all autocentral automorphisms of a finite group G . We consider the cases, where the group of all autocentral automorphisms coincides with its upper bound, that is, the group of all central automorphisms and also where it coincides with its lower bound, that is, the group of all inner automorphisms.

Keywords: Automorphism, Central automorphism, Autocentral automorphism.

Introduction

Let G be a group. Let G' and $Z(G)$ denote the commutator subgroup and the center of G , respectively and G^* , $L(G)$ denote the autocommutator and autocenter of G , respectively [7]. Autocommutator and autocenter have already been studied in [7]. Let $\text{Inn}(G)$ and $\text{Aut}(G)$ denote the group of all inner automorphisms and the group of all automorphisms of G , respectively. It is well known that G^* and $L(G)$ are characteristic subgroups of G . An automorphism α of a group G is called central automorphism, if $g^{-1}\alpha(g) \in Z(G)$, for all $g \in G$. The set of all central automorphisms of G is a normal subgroup of $\text{Aut}(G)$ and is denoted by $\text{Aut}_c(G)$. An automorphism α of a group G is called autocentral automorphism, if $g^{-1}\alpha(g) \in L(G)$, for all $g \in G$. Also the set of all autocentral automorphisms of G is a normal subgroup of $\text{Aut}(G)$ contained in $\text{Aut}_c(G)$ and is denoted by $\text{Var}(G)$ [11].

The group of central automorphisms of a finite group is of great importance in the investigation of automorphism group of a group. In this article, we study a different approach to central automorphisms which we call as autocentral automorphisms. In an abelian group, every automorphism of the group is central and also if the group of automorphisms $\text{Aut}(G)$ is abelian, then every automorphism is central.

In the literature there are many examples of the groups for which the group of automorphisms $\text{Aut}(G)$ is same as the group of central automorphisms $\text{Aut}_c(G)$ but $\text{Aut}(G)$ is nonabelian. In this paper we deal with the aspects of the general problem of how the structure of a group influences the group of autocentral automorphisms..

In this direction, in 1975, Malone [9] in his paper gave an example of a nonabelian p -group with nonabelian automorphism group in which all the automorphisms are central.

In 2001, Curran and McCaughan [5] in his paper considered the case in which the group of central automorphisms

$Aut_c(G)$ coincides with $Inn(G)$, the group of all inner automorphisms of G . He proved that for a finite group G , $Aut_c(G) = Inn(G)$ iff $G' = Z(G)$ and $Z(G)$ is cyclic.

In 2004 [6], the authors considered the case when $Aut_c(G)$ is of minimal order, that is, the case when $Aut_c(G) = Z(Inn(G))$. He proved that if G is a finite nonabelian p -group and if $Aut_c(G) = Z(Inn(G))$, then $Z(G) \leq G'$ and furthermore $Aut_c(G) = Z(Inn(G))$ iff $Hom(G/G', Z(G)) \cong Z(G/Z(G))$.

In 2002, Jamali and Mousavi [8] in their paper gave a result which deals with the occurrence of elementary abelian groups in the central automorphisms groups of finite PN p -group of class 2 (p -odd). The authors proved that if G is finite PN p -group of class 2 (p -odd), then $Aut_c(G)$ is elementary abelian iff $\Omega_1(Z(G)) = \phi(G)$ and $exp(G/G') = p$ or $exp(Z(G)) = p$.

In 2007, Attar [2] in his paper 'on central automorphisms that fix the centre elementwise' proved that if G is finite p -group, then $C_{Aut_c(G)}(Z(G)) = Inn(G)$ iff G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic.

In 2009, Yadav [14] gave the necessary and sufficient condition for a group G for which $Aut_c(G)$ coincides with $C_{Aut_c(G)}(Z(G))$. He proved in his paper 'on central automorphisms fixing the centre elementwise' that if G is a finite p -group of nilpotency class 2, then $Aut_c(G) = C_{Aut_c(G)}(Z(G))$ iff $rank(G/G') = rank(Z(G))$ and exponents of G' and $Z(G)$ are same and $(G/Z(G))/\bar{M} \cong (G/G')/\bar{N}$ where M and N are defined in the paper. Further he raised a question in his paper 'to find the necessary and sufficient condition for a finite p -group G of arbitrary nilpotent class such that $Aut_c(G) = C_{Aut_c(G)}(Z(G))'$.

In 2010, Attar [3] in his paper the author, answered the question raised by Yadav. He gave necessary and sufficient condition on G such that $Aut_c(G) = C_{Aut_c(G)}(Z(G))$. He proved in his paper that if G is a nonabelian finite p -group, then $Aut_c(G) = C_{Aut_c(G)}(Z(G))$ iff $Z(G) \leq G'$ or $Z(G) \leq \phi(G)$ with some other conditions on the invariants which can be found in his paper.

In 2010, Moghaddam et al. [11] gave a new concept of central automorphism. In his paper 'Some properties of central automorphisms, of a group' he investigated the properties of $Aut_c(G)$.

This article mainly deals with the group of autocentral automorphisms. All other notations are standard and follows from the books [12, 13]

Let $Var_{Z(G)}(G)$ denote the group of all autocentral automorphisms which fixes $Z(G)$ element wise. We follow the definitions for G^* , $L(G)$ and $Var(G)$ as in [11]:

$$\begin{aligned} L(G) &= \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in Aut(G)\}, \\ G^* &= \{[g, \alpha] \mid g \in G, \alpha \in Aut(G)\}, \\ Var(G) &= \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in L(G), \text{ for all } g \in G\}. \end{aligned}$$

where $[g, \alpha] = g^{-1}\alpha(g)$. We can easily check that $G' \leq G^*$ and $L(G) \leq Z(G)$.

In [10] the terms $K_n(G)$ and $L_n(G)$ have been defined inductively as follows:

$$\begin{aligned} K_0(G) &= 1, K(G) = K_1(G) = G^* \text{ and } K_n(G) = \{[g, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in Aut(G)\} \\ L_0(G) &= 1, L_1(G) = L(G) \text{ and } \frac{L_n(G)}{L_{n-1}(G)} = L\left(\frac{G}{L_{n-1}(G)}\right), \text{ for all } n \geq 2 \end{aligned}$$

We have the relation $\gamma_{i+1}(G) \leq K_i(G), L_i(G) \leq Z_i(G)$, for all $i \geq 1$, where $\gamma_i(G)$ and $Z_i(G)$ denote the terms of lower central series and upper central series of G , respectively. Moghaddam [10] define autonilpotent group, a group G is called autonilpotent of class n if n is the least positive integer such that $L_n(G) = G$. In an autonilpotent group of class n , we have $K_n(G) = \langle 1 \rangle$, but $K_n(G) = \langle 1 \rangle$ need not imply the group is autonilpotent. Thus an autonilpotent group of class n is also nilpotent group of class at the most n . In fact if G is a group such that $K_n(G) = \langle 1 \rangle$, then G is nilpotent of class at most n .

Moghaddam and Safa [11] showed that $Var(G) \cong Hom(G/L(G), L(G))$, where $Hom(A, B)$ stands for set of all automorphisms from a group A to a group B . $Hom(A, B)$ is a group under the binary operation of product of two maps, if B is abelian. They also proved that, if G is a purely non-abelian (a group having no non-trivial abelian direct factor) finite group, then $Var(G) \cong Hom(G, L(G))$.

Curran and McCaughan [5] gave the necessary and sufficient condition for a group G in which $Aut_c(G) = Inn(G)$. They proved that, if G is a non-abelian p -group with $Inn(G)$ contained in $Aut_c(G)$, then $Aut_c(G) = Inn(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In the present paper, we deal with the similar situation with $Var(G)$. We intend to find the conditions which give $Var(G) \cong Inn(G), Var(G) = Aut_c(G),$ and $Var(G) = Var_{Z(G)}(G)$.

1 Preliminaries

Lemma 1.1. *Let G be a group in which $K_2(G) = \langle 1 \rangle$. Then*

1. $G^* \leq L(G)$,

2. $exp(G/L(G)) = exp(G^*)$.

Proof. Let $[g, \alpha] \in G^*$, where $g \in G, \alpha \in Aut(G)$. Now $K_2(G) = \langle 1 \rangle$ implies $[g, \alpha, \beta] = 1$, for all $\beta \in Aut(G)$.

Thus, $g^{-1}\alpha(g) \in L(G)$, for all $g \in G$ and for all $\alpha \in Aut(G)$. Therefore (1) holds.

To prove (2), suppose $exp(G/L(G)) = n$ and $exp(G^*) = m$. Therefore, $(g^{-1}\alpha(g))^m = 1$, for all $g \in G$ and $\alpha \in Aut(G)$.

Since $g^{-1}\alpha(g) \in L(G) \leq Z(G)$, we have

$$\begin{aligned} g^{-m}\alpha(g)^m &= (g^{-1}\alpha(g))^m = 1, \text{ for all } g \in G \text{ and } \alpha \in Aut(G) \\ \alpha(g^m) &= g^m, \text{ for all } g \in G \text{ and for all } \alpha \in Aut(G) \\ g^m &\in L(G), \text{ for all } g \in G \\ (gL(G))^m &= L(G), \text{ for all } g \in G \\ \text{Therefore, } exp(G/L(G)) &\leq exp(G^*). \end{aligned}$$

Now $exp(G/L(G))=n$, therefore,

$$\begin{aligned} x^n &\in L(G), \text{ for all } x \in G \\ \alpha(x^n) &= x^n, \text{ for all } x \in G, \alpha \in Aut(G) \\ (x^{-1}\alpha(x))^n &= 1, \text{ for all } x \in G, \alpha \in Aut(G) \end{aligned}$$

Therefore, $exp(G^*) \leq exp(G/L(G))$.

Thus, $exp(G^*) = exp(G/L(G))$.

□

Remark 1.2. In fact, converse of part(1) is also true. That is, if $G^* \leq L(G)$, then $K_2(G) = \langle 1 \rangle$.

Remark 1.3. We also have $exp(G/Z(G)) = exp(G') \leq exp(G^*) = exp(G/L(G)) \leq exp(L(G))$, if G is a group such that $K_2(G) = \langle 1 \rangle$.

Lemma 1.4. If G is a group and n is the least positive integer such that $K_n(G) = \langle 1 \rangle$, Then $L(G)$ intersects with every non- trivial characteristic subgroup of G .

Proof. Let $\langle 1 \rangle \neq N$ be a characteristic subgroup of G . Since $K_n(G) = \langle 1 \rangle$ and $K_{n-1}(G) \neq \langle 1 \rangle$. We have $K_{n-1}(G) \leq L(G)$. Therefore $L(G)$ is a non-trivial subgroup of G . If $\alpha(x) = x$, for all $x \in N$ and for all $\alpha \in Aut(G)$, then $N \leq L(G)$. Suppose there exist x in N such that $\alpha(x) \neq x$ for some $\alpha \in Aut(G)$. Therefore $N \cap K(G) \neq \langle 1 \rangle$. Let i be the least positive integer such that $N \cap K_i(G) \neq \langle 1 \rangle$ but $N \cap K_j = \langle 1 \rangle$, for all $j, i < j \leq n (i \geq 1)$. Then it is easy to check that $N \cap K_i(G) \leq L(G)$. Therefore $N \cap L(G) \neq 1$.

□

Lemma 1.5. If G is a finite group in which $K_n(G) = \langle 1 \rangle$, for some natural number n , then $L(G) \cap M \neq \langle 1 \rangle$, for all maximal subgroups M of G .

Proof. Since $K_n(G) = \langle 1 \rangle$, therefore G is nilpotent. It follows that all maximal subgroups of G are normal. Thus, we get $G' \leq M$, and by above lemma $L(G) \cap G' \neq \langle 1 \rangle$, therefore, $L(G) \cap M \neq \langle 1 \rangle$.

□

Remark 1.6. If G is a finite group in which $K_n(G) = \langle 1 \rangle$, for some natural number n , then $L(G) \cap Fr(G) \neq \langle 1 \rangle$, where $Fr(G)$ denote the Frattini subgroup of G .

We use the following standard results throughout our paper.

Lemma 1.7. Let A, B be two finite abelian p - groups with $exp(A) \leq exp(B)$. Then $Hom(A, B)$ is isomorphic to A if and only if B is cyclic.

Proof. If B is cyclic, then result is easy to prove by taking the direct product decomposition of A as cyclic subgroups, and using the facts that $Hom(A_1 \times A_2, B) \cong Hom(A_1, B) \times Hom(A_2, B)$, $Hom(C_m, C_n) \cong C_d$, where $d = (m, n)$. Conversely, suppose $Hom(A, B) \cong A$, but B is not cyclic Then B can be written as $C_m \times D$, where $exp(B) = m$. Then, we have $Hom(A, B) \cong Hom(A, C_m) \times Hom(A, D) \cong A \times Hom(A, D)$, where $Hom(A, D)$ is non-trivial p -group. A contradiction.

□

Corollary 1.8. Let A, B be two finite abelian groups such that $exp(A)$ divides $exp(B)$. Then $Hom(A, B)$ is isomorphic to A if and only if B is cyclic.

2 Main Results

In this section we aim to present our main results.

Theorem 2.1. *If G is a group, then $Var(G) \cong Hom(G/L(G), L(G))$ and if G is a finite group in which $G' \leq L(G)$ and $L(G)$ is cyclic, then $[Var(G) : Inn(G)] = [Z(G) : L(G)]$.*

Proof. The first part of the above theorem has already proved in [11]. But we are giving the proof for our convenience to be used later in proving some results.

Define a map

$$\begin{aligned} \lambda : Var(G) &\rightarrow Hom(G/L(G), L(G)) \\ \phi &\rightarrow \bar{\phi} \end{aligned}$$

where $\bar{\phi} : G/L(G) \rightarrow L(G)$ is given by $\bar{\phi}(gL(G)) = g^{-1}\phi(g)$, for all $g \in G$. $\bar{\phi}$ is well defined map and it is an injective homomorphism. Now α is surjective, for $\psi \in Hom(G/L(G), L(G))$. Define a map

$$\begin{aligned} \phi : G &\rightarrow G \\ g &\rightarrow g\psi(gL(G)) \end{aligned}$$

Then $\phi \in Var(G)$ and $\alpha(\phi) = \bar{\phi} = \psi$.

Now if $L(G)$ is cyclic, by [2.3, 2.8], $exp(G/L(G)) \leq exp(L(G), Hom(G/L(G), L(G))) \cong G/L(G)$. Since $G' \leq L(G)$, we have $Inn(G) \leq Var(G)$. We also know $Inn(G) \cong G/Z(G)$. Collecting all the facts, we have

$$\begin{aligned} [Var(G) : Inn(G)] &= |G/L(G)|/|G/Z(G)| \\ &= [Z(G) : L(G)] \end{aligned}$$

□

Corollary 2.2. *If G is finite group in which $G' \leq L(G)$ with $Z(G) = L(G)$ is cyclic, then $Var(G) = Inn(G)$.*

In [11] Moghaddam proved that, If G is a finite purely non- abelian group, then $Var(G) \cong Hom(G, L(G))$. We observe that if G is a purely finite non- abelian group in which $K_2(G) = \langle 1 \rangle$, then $Var(G) \cong Hom(G, L(G)) \cong Hom(G/G^*, L(G))$.

Lemma 2.3. *If G is a finite group in which $K_2(G) = \langle 1 \rangle$ and $L(G)$ is cyclic, then $exp(G/G^*) \leq exp(L(G))$.*

Proof. It is easy to verify. □

Theorem 2.4. *If G is a finite purely non- abelian group in which $G' \leq L(G)$ and $L(G)$ is cyclic, then $[Var(G) : Inn(G)] = [Z(G) : L(G)] = [Z(G) : G^*]$. Therefore, $G^* = L(G)$.*

Proof. Since

$$\begin{aligned} G/L(G) &\cong \text{Hom}(G/L(G), L(G)) \cong \text{Var}(G) \cong \text{Hom}(G/G^*, L(G)) \cong G/G^*, \text{ and} \\ G/Z(G) &\cong \text{Inn}(G). \text{ Therefore,} \\ [\text{Var}(G) : \text{Inn}(G)] &= [Z(G) : L(G)] = [Z(G) : G^*]. \end{aligned}$$

□

Remark 2.5. If G is a group in which $G' \leq L(G)$ with $Z(G)=L(G)$ is cyclic, then $\text{Inn}(G) = \text{Var}(G)$. But then $\text{Var}(G) = \text{Aut}_c(G)$. Thus we have $\text{Inn}(G) = \text{Aut}_c(G)$. Here G need not be a p -group. But if G is also a p -group, then $G' = Z(G)$. Thus we have an important deduction, if G is a p -group in which $K_2(G) = \langle 1 \rangle$ with $Z(G)=L(G)$ is cyclic, then $G' = Z(G) = L(G) = G^*$ and $\text{Var}(G) = \text{Inn}(G) = \text{Aut}_c(G)$

Let $C_{\text{Aut}_c(G)}(Z(G))$ denote the group of all central automorphisms fixing $Z(G)$ element wise Attar [2] proved that, if G is a finite p -group, then $C_{\text{Aut}_c(G)}(Z(G)) = \text{Inn}(G)$ if and only if G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic. In our next theorem we prove the analogous result.

Theorem 2.6. If G is a finite non-abelian group in which $\exp(G')/\exp(L(G))$, then $\text{Var}_{Z(G)}(G) \cong \text{Inn}(G)$ if and only if G is nilpotent of class 2, and $L(G)$ is cyclic.

Proof. To prove this theorem, we first prove the following lemma,

Lemma 2.7. If G is a finite non-abelian group, then $\text{Var}_{Z(G)}(G) \cong \text{Hom}(G/Z(G), L(G))$.

Proof. We proceed on the same lines as in theorem 2. Define a map

$$\begin{aligned} \lambda : \text{Var}_{Z(G)}(G) &\rightarrow \text{Hom}(G/Z(G), L(G)) \\ \phi &\rightarrow \bar{\phi} \end{aligned}$$

where $\bar{\phi} : G/Z(G) \rightarrow L(G)$ is given by $\bar{\phi}(gZ(G)) = g^{-1}\phi(g)$, for all $g \in G$. $\bar{\phi}$ is well defined map, as every automorphism in $\text{Var}_{Z(G)}(G)$ acts trivially on $Z(G)$. It is clear that λ is a well defined map. We prove that λ is an isomorphism. Let $\phi_1, \phi_2 \in \text{Var}_{Z(G)}(G)$ and $g \in G$. Then

$$\begin{aligned} \overline{\phi_1\phi_2}(gZ(G)) &= g^{-1}(\phi_1\phi_2)(g) = g^{-1}(\phi_1(\phi_2(g))) \\ &= g^{-1}(\phi_1(gg^{-1}\phi_2(g))) = g^{-1}\phi_1(g)\phi_1(g^{-1}\phi_2(g)) \\ &= g^{-1}\phi_1(g)g^{-1}\phi_2(g) = \bar{\phi}_1(gZ(G))\bar{\phi}_2(gZ(G)). \end{aligned}$$

Now it is easy to see that λ is injective. The homomorphism λ is surjective also, for let $\psi \in \text{Hom}(G/Z(G), L(G))$. Define a map

$$\begin{aligned} \phi &: G \rightarrow G \\ g &\rightarrow g\psi(gZ(G)). \end{aligned}$$

We shall prove that $\phi \in \text{Var}_{Z(G)}(G)$ and it is clear that $\bar{\phi} = \psi$. Now to prove ϕ is an autocentral automorphism fixing $Z(G)$ element wise, it is sufficient to prove ϕ is monomorphism, as G is finite, by the definition of ϕ , $g^{-1}\phi(g) = \psi(gZ(G)) \in L(G)$, and for $g \in Z(G)$, $g^{-1}\phi(g) = \psi(gZ(G)) = 1$, hence $\phi(g) = g$, thus fixes each element of $Z(G)$. Let $g_1, g_2 \in G$. We have,

$$\begin{aligned}\phi(g_1g_2) &= g_1g_2\psi(g_1g_2Z(G)), \\ &= g_1\psi(g_1Z(G))g_2\psi(g_2Z(G)), \\ &= \phi(g_1)\phi(g_2).\end{aligned}$$

Let $g \in \ker(\phi)$. Then $\phi(g) = 1$. This gives $g^{-1} = \psi(gZ(G)) \in L(G)$, and therefore $g \in L(G) \leq Z(G)$. Thus, $\psi(gZ(G)) = 1$, and which follows that $1 = \phi(g) = g$. Hence proved. \square

Proof of theorem. Suppose first that $\text{Var}_{Z(G)}(G) \cong \text{Inn}(G)$. Therefore, $\text{Hom}(G/Z(G), L(G)) \cong G/Z(G)$. since $\text{Hom}(G/Z(G), L(G))$ is abelian, therefore $G/Z(G)$ is abelian. Hence G is nilpotent of class 2. Therefore $\exp(G/Z(G)) = \exp(G')/\exp(L(G))$, thus by cor. 1.8, $L(G)$ is cyclic.

Conversely, suppose that G is nilpotent of class 2 and $L(G)$ is cyclic and since $\exp(G/Z(G)) = \exp(G')/\exp(L(G))$, therefore, $\text{Hom}(G/Z(G), L(G)) \cong G/Z(G)$ and hence, $\text{Var}_{Z(G)}(G) \cong \text{Inn}(G)$. \square

Corollary 2.8. *If G is finite group in which $K_2(G) = \langle 1 \rangle$ and $L(G)$ is cyclic, then $\text{Var}_{Z(G)}(G) = \text{Inn}(G)$.*

Theorem 2.9. *Let G be a finite group such that $L(G) \leq G'$. Then*

$$\text{Var}(G) \cong \text{Hom}(G, L(G)) \cong \text{Hom}(G/G', L(G)).$$

Proof. Define a map

$$\begin{aligned}\phi : \text{Var}(G) &\rightarrow \text{Hom}(G, L(G)) \text{ by} \\ \phi(\alpha) &= \phi_\alpha, \text{ where} \\ \phi_\alpha : G &\rightarrow L(G) \\ x &\rightarrow x^{-1}\alpha(x)\end{aligned}$$

It is easy to check that ϕ is a monomorphism. Next, we prove that ϕ is onto map. Let $f \in \text{Hom}(G, L(G))$. Define

$$\begin{aligned}\sigma : G &\rightarrow G \text{ by} \\ \sigma(x) &= xf(x)\end{aligned}$$

. Clearly σ is a homomorphism. To prove σ is automorphism, it is sufficient to prove σ is injective, as G is finite group. Let $x \in Ker(\sigma)$. Then

$$xf(x) = 1, \text{ gives } f(x) = x^{-1} \in L(G), \text{ implies } x \in L(G) \leq G'$$

Therefore, $f(x) = 1$, implies $x = 1$. Thus σ is an automorphism. And, clearly $\phi(\sigma) = f$. □

Theorem 2.10. *If G is a finite non-abelian p -group such that $G' \leq L(G)$, then $Var(G) = Var_{Z(G)}(G)$ if and only if $Z(G) = G^{p^n}L(G)$, where $exp(L(G)) = p^n$.*

Proof. Suppose $Z(G) = G^{p^n}L(G)$. Let $x \in Z(G)$. Then $x = a^{p^n}b$, where $a \in G, b \in L(G)$. Now for any $f \in Hom(G/L(G), L(G))$, we have $f(xL(G)) = f(a^{p^n}bL(G)) = f(a^{p^n}L(G))f(bL(G)) = (f(aL(G)))^{p^n} = 1$. Therefore $Z(G)/L(G) \leq Ker(f)$. Therefore, the autocentral automorphism induced by f which is $\sigma_f : G \rightarrow G, \sigma_f(x) = xf(xL(G)) = x1 = x$ fixes $Z(G)$.

Conversely, suppose that $Var(G) = Var_{Z(G)}(G)$. Since $G' \leq L(G) \leq Z(G)$, therefore, G is nilpotent of class 2 and $G/L(G)$ and $G/Z(G)$ both are abelian. Also, $exp(G/Z(G)) = exp(G') \leq exp(L(G)) = p^n$. Thus, we have $G^{p^n} \leq Z(G)$, also $L(G) \leq Z(G)$. We get $G^{p^n}L(G) \leq Z(G)$. It is sufficient to prove $Z(G) \leq G^{p^n}L(G)$. Suppose there exist $x \in z(G)$ such that $x \notin G^{p^n}L(G)$. let $G/L(G) = \langle \bar{x}_1 \rangle \times \langle \bar{x}_2 \rangle \times \dots \times \langle \bar{x}_r \rangle$ where x_i 's are chosen in such a way that there exist s_1, s_2, \dots, s_r integers such that $xL(G) = (x_1)^{p^{s_1}}L(G)(x_2)^{p^{s_2}}L(G)\dots(x_r)^{p^{s_r}}L(G)$. Since $x \notin G^{p^n}L(G)$, therefore there exist at least one integer in $\{1, 2, \dots, r\}$ say, i such that $s_i < n$. Choose $z \in L(G)$ such that $o(z) = min(|x_iL(G)|, p^n)$. Define

$$f : G/L(G) \rightarrow L(G) \text{ by } x_iL(G) \rightarrow z$$

and trivial on other components. Then, $f(xL(G)) = f(x_i^{p^{s_i}}L(G)) = z^{p^{s_i}}$. Thus, $\sigma_f(x) = xf(xL(G)) = xz$. Since, $x \in Z(G)$, therefore σ_f does not fix $Z(G)$. A contradiction. □

Lemma 2.11. *If G is a group such that $K_n(G) = \langle 1 \rangle$, and $K_{n-1}(G) \neq \langle 1 \rangle$, then $K_{n-2}(G)$ is torsion free.*

Proof. We have $K_{n-1}(G) \leq L(G)$, $K_{n-1}(G)$ is torsion free. Let $x \in K_{n-2}(G)$, then $x^{-1}\alpha(x) \in K_{n-1}(G) \leq L(G)$, hence $|x|$ is infinite. Thus $K_{n-2}(G)$ is torsion free. □

Theorem 2.12. *Let G be a finitely generated group non-abelian such that $G^* \leq L(G)$. Then $Hom(G/Z(G), L(G)) \cong G/Z(G)$ if and only if $L(G)$ is cyclic or $L(G) \cong C_k \times Z^r$, where $exp(G^*)/k$ and r is torsion free rank of $Z(G)$.*

Proof. Suppose G is torsion. Since G is a finitely generated nilpotent group, therefore $|G|$ is finite. Hence $Hom(G/Z(G), L(G)) \cong G/Z(G)$ if $L(G)$ is cyclic as $exp(G/Z(G))$ divides $exp(L(G))$.

Suppose G is torsion free. Since G is a finitely generated nilpotent group of class 2, $G/Z(G)$ and $L(G)$ are free abelian groups of finite ranks m and n , respectively. Therefore $G/Z(G)$ and $L(G)$ are direct sum of infinite cyclic groups. Hence $Hom(G/Z(G), L(G)) \cong Hom(Z^m, Z^n) \cong Z^{mn}$, hence $Hom(G/Z(G), L(G)) \cong G/Z(G)$ if and only if

$n=1$. Thus $L(G)$ is cyclic.

Suppose G is a mixed group that is neither torsion nor torsion free. Now G is not torsion, therefore G is a finitely generated infinite group, and hence $L(G)$ contains an element of infinite order. Thus torsion free rank r of $L(G)$ is non-zero. Since G is not torsion free, therefore $L(G)$ is not torsion free. Thus $L(G) = A \times B$, where A and B are non-trivial, A is torsion and B is a torsion free abelian group of rank r . suppose $G/Z(G) = C \times D$, where C is finite and D is torsion free part of $G/Z(G)$ with rank s . Therefore $Hom(G/Z(G), L(G)) \cong Hom(C \times D, A \times B) \cong Hom(C, A) \times Hom(C, B) \times Hom(D, A) \times Hom(D, B)$. Since B is torsion free and C is torsion, $Hom(C, B) = 1$, and $Hom(D, B) \cong Z^{rs}$, $Hom(D, A) \cong A^s$. Hence $Hom(G/Z(G), L(G)) \cong Hom(C, A) \times Z^{rs} \times A^s$. Therefore $Hom(G/Z(G), L(G)) \cong G/Z(G) = C \times D$, thus $C \cong Hom(C, A) \times A^s, D \cong Z^{rs}$. We claim $D=1$. Suppose $D \neq 1$, then $A^s \neq 1$. Therefore $exp(C) = exp(A)$, as $exp(Hom(C, A))$ divides $exp(A) = k$. Let $A = C_k \times M$, where M is abelian and finite. Thus $Hom(C, A) \cong Hom(C, C_k) \times Hom(C, M) \cong C \times Hom(C, M)$. Thus $C \cong C \times Hom(C, M) \times A^s$. Thus $Hom(C, M) = A^s = 1$. Since A is nontrivial, therefore $s=0$. Hence $D=1$, a contradiction. Therefore $G/Z(G)$ is torsion. Also $G/Z(G) \cong C$, and $exp(G/Z(G))$ divides k . Thus we have $C \cong Hom(C, A)$, and therefore A must be cyclic. Hence $A \cong C_k$, and $L(G) = A \times B \cong C_k \times Z^r$.

Conversely, suppose that $L(G) \cong C_k \times Z^r$, where $exp(G/Z(G))$ divides k , and r is torsion free rank of $L(G)$. We have $Hom(G/Z(G), L(G)) \cong Hom(G/Z(G), C_k \times Z^r) \cong Hom(G/Z(G), C_k) \times Hom(G/Z(G), Z^r) \cong G/Z(G)$, for $Hom(G/Z(G), Z^r) = 1$, $exp(G/Z(G))$ divides k . □

Corollary 2.13. *If G is a finitely generated group such that $G^* \leq L(G)$. Then $Inn(G) \cong Var_{Z(G)}(G)$ if and only if $L(G)$ is cyclic or $L(G) \cong C_k \times Z^r$, where $exp(G')/k$ and r is torsion free rank of $Z(G)$.*

Now we write the following finite abelian p -groups as the direct product of cyclic p -groups as:

$$G/G' = C_{p^{a_1}} \times C_{p^{a_2}} \times \dots \times C_{p^{a_n}}, \quad a_1 \geq a_2 \geq \dots \geq a_n \geq 1.$$

$$L(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \dots \times C_{p^{b_m}}, \quad b_1 \geq b_2 \geq \dots \geq b_m \geq 1.$$

$$Z(G) = C_{p^{c_1}} \times C_{p^{c_2}} \times \dots \times C_{p^{c_r}}, \quad c_1 \geq c_2 \geq \dots \geq c_r \geq 1.$$

We now find the conditions under which $Var(G)$ coincides with $Aut_c(G)$.

Theorem 2.14. *Let G be a purely non-abelian finite p -group. Then $Aut_c(G) = Var(G)$ if and only if either $Z(G) = L(G)$ or $m = r$ and $b_i \leq c_i$, for all $i, 1 \leq i \leq m$ and $a_1 \leq b_t$ where t is the largest in $\{1, 2, 3, \dots, m\}$ such that $b_t < c_t$.*

Proof. Suppose $Aut_c(G) = Var(G)$. Since $Var(G)$ is abelian, therefore $Aut_c(G)$ is abelian. Suppose $Z(G) \neq L(G)$. Therefore $L(G) \subset Z(G)$. Let t be the largest in $\{1, 2, 3, \dots, m\}$ such that $b_t < c_t$. Therefore $b_j = c_j$, for all $j > t$. Since G is purely non-abelian, therefore by [1]

$$|Aut_c(G)| = |Hom(G/G', Z(G))| \text{ and}$$

$$|Var(G)| = |Hom(G/G', L(G))|.$$

Therefore, $|Hom(G/G', Z(G))| = |Hom(G/G', L(G))|$.

$$\prod_{i,j} p^{\min(a_i, b_j)} = \prod_{i,k} p^{\min(a_i, c_k)}$$

Since $\min(a_i, b_j) \leq \min(a_i, c_j)$, therefore $m = r$. Also $b_j = c_j, \forall j > t$, thus we have $\min(a_i, b_j) = \min(a_i, c_j), \forall j > t$. Hence for $j = t, \min(a_i, b_j) = \min(a_i, c_j) = a_i$ as $b_t < c_t$.

In particular, for $i = t$,

$$b_t \geq a_t \geq a_{t-1} \geq a_{t-2} \dots \geq a_1 \geq 1.$$

Therefore $b_t \geq a_1$.

Conversely, suppose the conditions hold. If $Z(G) = L(G)$ then clearly from the definitions of the $Var(G)$ and $Aut_c(G)$, $Aut_c(G) = Var(G)$. Now since G is purely non-abelian

$$|Aut_c(G)| = |Hom(G/G', Z(G))| = \prod_{i,j} p^{\min(a_i, b_j)}$$

and

$$|Var(G)| = |Hom(G/G', L(G))| = \prod_{i,k} p^{\min(a_i, c_k)}$$

Further $b_j = c_j, \forall j > t$, therefore we have $\min(a_i, b_j) = \min(a_i, c_j), \forall j > t$.

For $j \leq t$,

$$b_1 \geq b_2 \geq b_3 \geq \dots \geq b_{t-1} \geq b_t \geq a_1 \geq a_2 \dots \geq a_n \geq 1,$$

$$a_i \leq b_t < c_t \leq c_{t-1} \leq \dots \leq c_1.$$

$$a_i \leq b_j \leq c_j, \text{ for all } i, 1 \leq i \leq n \text{ and for all } j, 1 \leq j \leq t.$$

Therefore $\min(a_i, b_j) = a_i = \min(a_i, c_j)$, for all $i, 1 \leq i \leq n$ and for all $j, 1 \leq j \leq t$.

$$|Aut_c(G)| = |Var(G)|.$$

Since $Var(G) \subseteq Aut_c(G)$, therefore $Aut_c(G) = Var(G)$.

□

Proposition 2.15. *Let G be a purely non-abelian p -group. If $\exp(G/G') = p$ or $\exp(L(G)) = p$, then $Var(G)$ is elementary abelian. Converse is also true.*

Bidwell et al. in [4] proved that if G is a group such that $G = H \times K$, where H and K have no common direct factor, then $Aut_c(G) \cong A = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \begin{matrix} \alpha \in Aut_c(H), \beta \in Hom(K, Z(H)) \\ \gamma \in Hom(H, Z(K)), \delta \in Aut_c(K) \end{matrix} \right\}$. We prove the following result :

Theorem 2.16. Let $G = H \times K$, where H and K have no common direct factor, then $Var(G) \subseteq \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \begin{array}{l} \alpha \in Var(H), \beta \in \\ \gamma \in Hom(H, L(K)) \end{array} \right.$

Proof. We have [4], if $\alpha \in Aut(H)$, then $\beta \in Hom(K, Z(H))$ Similarly $\delta \in Aut(K)$, implies $\gamma \in Hom(H, Z(K))$. We prove if $\alpha \in Aut(H)$, then $\beta \in Hom(K, Z(H))$ Similarly $\delta \in Aut(K)$, implies $\gamma \in Hom(H, Z(K))$

$$\text{Let } \phi \in Var(G). \text{ Then } \phi \in Aut_c(G), \text{ therefore } \phi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \alpha \in Aut_c(H), \beta \in Hom(K, Z(H)) \gamma \in Hom(H, Z(K)), \delta \in \\ Aut_c(K). [h, k]^{-1} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \in L(G) \leq L(H) \times L(K)$$

We have $h^{-1}\alpha(h)\beta(k) \in L(H), h^{-1}\gamma(h)\delta(k) \in L(K)$. Thus if $\alpha \in Var(H)$, then $h^{-1}\alpha(h) \in L(H)$, therefore $\beta(k) \in L(H)$. Therefore, $\beta \in Hom(K, L(H))$. Now to prove main theorem, it is sufficient to prove $\alpha \in Var(G)$. Suppose there exist $h \in H$ such that $h^{-1}\alpha(h) \notin L(H)$. consider $(h, 1) \in G$. Then $(h, 1)^{-1}\phi(h, 1) \in L(G) \leq L(H) \times L(K)$. We have $h^{-1}\alpha(h)\beta(1) \in L(H), h^{-1}\gamma(h)\delta(1) \in L(K)$, thus $h^{-1}\alpha(h) \in L(H)$, a contradiction. Therefore $\alpha \in Var(H)$. Similarly we can prove $\delta \in Var(K)$. Hence the result follows. □

3 Examples

Example 1. Let $G = \langle x_1, x_2, x | [x_1, x_2] = x, [x_i, x] = 1, 1 \leq i \leq 2 \rangle$. Then $G' = Z(G) = \langle x \rangle \cong Z$ and $G/Z(G) = \langle x_1Z(G), x_2Z(G) \rangle \cong Z \times Z$. since $Z(G)$ is characteristic, therefore elements of $Z(G)$ maps to $Z(G)$ under any automorphism of G . Since $Z(G)$ is cyclic of infinite order, two automorphisms are possible, one is trivial and other is inverse map. Therefore $L(G) = \langle x^2 \rangle$. We have $Var(G) \cong Aut_c(G) \cong Inn(G)$.

Example 2. Let $G = \langle x_1, x_2, x | [x_1, x_2] = x^s (s \neq 1), [x_i, x] = 1, 1 \leq i \leq 2 \rangle$. Then $G' = \langle x^s \rangle, Z(G) = \langle x \rangle \cong Z$ and $G/Z(G) = \langle x_1Z(G), x_2Z(G) \rangle \cong Z \times Z$. By the same argument as above $Var(G) \cong Aut_c(G) \cong Inn(G)$.

Example 3. Let $G = \langle x_1, x_2, x | x^{p^2} = x_1^p = x_2^p = [x_1, x_2] = x^p (p \text{ is prime}) \rangle$. Then $G' = \langle x^p \rangle \cong C_p, Z(G) = \langle x \rangle \cong C_{p^2}$ and $G/Z(G) \cong C_p \times C_p$. G is purely non-abelian of class 2. Since $Z(G)$ is characteristic, therefore each automorphism of G maps $Z(G)$ to $Z(G)$. Therefore $L(G) \neq Z(G)$. Therefore either $L(G) = 1$ (for odd prime) or $L(G) = C_p$ (for even prime). Thus either $Var(G)$ is trivial or $Var_{Z(G)}(G) \cong Inn(G)$.

Example 4. We shall consider all non-abelian groups of order 12. First is alternating group of degree 4, second is semi direct product of C_4 and C_3 and third is direct product of S_3 and C_2 . In the first case $Var(G)$ is trivial as $L(G)$ is trivial. In the second case and in the third case $Var(G)$ is same as $Aut_c(G)$ as $L(G) = Z(G)$.

Example 5. Let $G = \langle a, b | a^4 = b^4 = 1, a^b = a^{-1} \rangle$. Then G is purely non-abelian of class 2. Here $G' \cong C_2, Z(G) \cong C_2 \times C_2$ and $G/G' \cong C_2 \times C_4$. Here $G' \leq Z(G)$, therefore $C_{Aut_c(G)}(Z(G)) \cong Hom(G/Z(G), Z(G)) \cong Hom(C_2 \times C_2, C_2 \times C_2)$. $|Aut_c(G)| = |Hom(G/G', Z(G))| = 2^4 = |C_{Aut_c(G)}(Z(G))|$. Thus, $Aut_c(G) = C_{Aut_c(G)}(Z(G))$. Therefore $Aut_c(G) \cong C_2^4$, is an elementary abelian. Since each central automorphism fixes $Z(G)$, therefore each autocentral automorphism also fixes $Z(G)$. We have $Var_{Z(G)}(G) = Var(G)$. Hence $Var(G) \cong Hom(G/Z(G), L(G))$. Since $L(G) \leq Z(G)$, therefore either $L(G)$ is cyclic of order 2 and in this case $Var(G) \cong Inn(G)$ and if $L(G) = Z(G)$, then $Var(G) = Aut_c(G)$.

Example 6. Let $G = \langle x, y \mid x^4 = y^2 = 1, yxy = x^{-1} \rangle$ be the dihedral group of order 8. Then we have the following structure: $G^* = \langle x \rangle, L(G) = Z(G) = G' = 1, x^2, \text{Aut}_c(G) = \text{Var}(G) = \text{Inn}(G) \cong G/Z(G) \cong C_2 \times C_2$. G is nilpotent group of class 2, but $K_2(G) \neq 1$.

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