Invariant solutions of generalized Fisher-KPP equation

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Abstract

In this paper, we consider a hyperbolic generalized Fisher-KPP equation: $\varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u)$ where f, g and k are arbitrary smooth functions of variable u and ε is a speed parameter. We find invariant solutions by Lie method. Also, we study standard and weak conditional and approximate symmetries.

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Introduction

The symmetry group method plays a key role in the study of differential equation. The basic method for computing symmetry groups, using the prolongation formula for their generators, goes back to Sophus Lie. He was the first one who actually did introduce the general method for finding symmetry groups and invariant solutions [17].

The symmetry group of a PDE constructs some new solutions from known ones. G.W. Bluman and J.D. Cole generalized the Lie's method and presented a non-classical method of group invariant solutions for the linear heat equation [2]. The concept conditional symmetry was introduced and developed by W.I. Fushchych [8, 9]. Non-trivial conditional symmetries of a PDE allows us to obtain a clear form of solutions of the equation which can not be found by Lie method. P.J. Olver and P. Rosenau developed this method and proved that any vector field X is a conditional symmetry and any solution of the equation is an invariant solution under some X [18].

Recently, many mathematicians, mechanicians, and physicists, such as Euler, D'Alembert, Poincare, Bateman used conditional symmetries for the construction of exact symmetries of the linear wave equation, which some solutions can not be obtained by Lie's method. Classical Lie group theory was provided an efficient tool for computing symmetry groups of a PDE, but any small perturbation in an equation changed the symmetry group, so this method isn't always applicable. An approximate theorem is provided us to construct approximate symmetries that are stable under small perturbations of a PDE. V.A. Baikov, R.K. Gazizov, and N.H. Ibragimov were the first people who worked on this subject [4]. In this method, the Lie operator is expanded in a perturbation series so that an approximate operator can be found [13–16, 20].

One of the most important and most useful equations is the Fisher-KPP equation which plays an important role in the medicine and biology science $u_t = u_{xx} + F(u)$, where F is monostable and $F(0) = F(1) = 0$. S.A. Gourley showed that traveling front of a nonlocal Fisher-KPP equation exists if the nonlocality is sufficiently weak in a certain sense [10]. In the [11, 12], this equation is generalized to the following form $m(u)u_t = (k(u)u_x)_x + f(u)$. As in Mckean's approach, S. Dunbar and H. Othmer [6, 7] showed that the function $u(t, x)$ satisfies a nonlinear hyperbolic equation of the general form

$$\varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u), \tag{0.1}$$

that is derived from models of movement cells and celled organisms and from a mathematical treatment of a branching random walk. They studied the equation (0.1) while $k(u)$ is a constant function and $g(u)$, $f(u)$ are polynomials functions and obtained traveling wave solutions of certain speeds. K.P. Hadeler introduced a simplifying transformation for general parameter functions g, k, u and provided a complete description for the case of a constant or a monotone function k [11].

In this paper, we consider the equation (0.1) and find invariant solutions of it by the classical, non-classical and approximate method. In the first section, we suppose that the parameter ε is constant and obtain symmetry groups and invariant solutions by using Lie method. In the second section, we assume that the parameter ε is a constant and the conditional symmetry is found. In the third section, we obtain first and second order approximate symmetries by the Ibragimov approximate method. In the final section, we analyze the important and useful case of the equation (0.1) and invariant solutions are found.

1 Lie Symmetries

In this section, we illustrate a general Lie method. Also we find symmetry groups and invariant solutions of the partial differential equation (0.1) by Lie method. We consider a system of a PDE of order n

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{1.2}$$

which $x = (x^1, \cdots, x^p)$ and $u = (u^1, \cdots, u^q)$ are independent and dependent variables, respectively and $u^{(n)} := \left(u^\alpha_{i_1}; u^\alpha_{i_1 i_2}; \cdots u^\alpha_{i_1 \cdots i_n}\right)$ is derivatives of u with respect to x from 0 to n. We consider a one parameter translation group G that acts on independent and dependent variables of (1.2) by the following form

$$\begin{aligned} \tilde{x}^i &= x^i + s\xi^i(x, u) + O(s^2), \qquad & i = 1, \cdots, p \\ \tilde{u}^\alpha &= u^\alpha + s\varphi^\alpha(x, u) + O(s^2), \qquad & \alpha = 1, \cdots, q \end{aligned}$$

where ξ^i and φ^α are the infinitesimal of the translation for the independent and dependent variables, respectively.

The general form of the infinitesimal generator of the translation group G is

$$X = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u)\partial_{u^\alpha}. \tag{1.3}$$

By theorem 2.36 of [17], the n-th order prolongation of the infinitesimal generator X is the vector field

$$\mathbf{p}r^{(n)}X = X + \sum_{\alpha=1}^{q}\sum_{J} \varphi^J_\alpha(x, u^{(n)})\partial_{u^\alpha_J}, \tag{1.4}$$

with $1 \le k \le n$, $J = (j_1, \cdots, j_k)$, $1 \le j_k \le p$. The coeffcient functions φ^J_α are given by the following formula:

$$\varphi^J_\alpha(x, u^{(n)}) = D_J\Big(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i\Big) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}, \tag{1.5}$$

where $u^\alpha_i := \partial u^\alpha/\partial x^i$ and $u^\alpha_{J,i} := \partial u^\alpha_J/\partial x^i$ and D is the total derivative.

By theorem 2.71 of [17], G is a symmetry group of the nondegenerate system (1.2) if and only if

$$\mathbf{p}r^{(n)}X(\Delta_\nu(x, u^{(n)})) = 0, \qquad \nu = 1, \cdots, l, \tag{1.6}$$

whenever $\Delta(x, u^{(n)}) = 0$.

By solving the above system, we obtain coefficients of the infinitesimal generator X. By using Lie equations, we gain the symmetry group G.

To earn invariant solutions, we must acquire functional independent invariants of X by integrating a corresponding characteristic system.

Now, we consider the equation (0.1) which the parameter ε is a constant. Let $x^1 = t$, $x^2 = x$ and $u(t, x)$ be independent and dependent variables. So the one parameter translation group acts:

$$(\tilde{t}, \tilde{x}, \tilde{u}) = (t, x, u) + s(\tau, \xi, \varphi)(t, x, u) + O(s^2).$$

The infinitesmal vector field corresponding with G is $X = \tau\,\partial_t + \xi\,\partial_x + \varphi\,\partial_u$. The second prolongation of X is the form:

$$\mathbf{p}r^{(2)}X = X + \varphi^t\,\partial_{u_t} + \varphi^x\,\partial_{u_x} + \varphi^{tt}\,\partial_{u_{tt}} + \varphi^{tx}\,\partial_{u_{tx}} + \varphi^{xx}\,\partial_{u_{xx}}$$

Suppose that $Q = \varphi - \tau u_t - \xi u_x$. Applying (1.5), the coefficients φ^J leads as:

$$\begin{array}{ll} \varphi^t = D_t Q + \tau u_{tt} + \xi u_{xt}, & \varphi^x = D_x Q + \tau u_{xt} + \xi u_{xx}, \\ \varphi^{tt} = D_t^2 Q + \tau u_{ttt} + \xi u_{ttx}, & \varphi^{tx} = D_t D_x Q + \tau u_{ttx} + \xi u_{txx}, \\ \varphi^{xx} = D_x^2 Q + \tau u_{xxt} + \xi u_{xxx}. & \end{array}$$

Therefore, the invariant condition (1.6) is equivalent to solving the system:

$$\begin{aligned} &\mathbf{pr}^2 X(\varepsilon^2 u_{tt} + g(u)u_t - (k(u)u_x)_x - f(u)) = 0, \\ &\varepsilon^2 u_{tt} + g(u)u_t - (k(u)u_x)_x - f(u) = 0. \end{aligned}$$

Then we obtain a polynomial equation including derivative u that the coefficients u are derivatives ξ, τ, φ with respect to t, x, u. The coefficients of derivatives must be zero; therefore, we obtain a determining system of 16 equations:

$$k(u)\,\xi_{uu} = 0, \quad \tau_{uu} = 0, \quad k(u)\,(\varphi_{uu} - 2\xi_{ux}) = 0, \quad \cdots$$

Solving the above system, leads:

Theorem. *The one-parameter Lie group of point symmetry of the equation (0.1) has an infinitesimal generator X, whose coefficient functions $\tau = c_1$, $\xi = c_2$ and $\varphi = 0$ are constants.*

Therefore, this symmetry group has the infinitesimal generators $X_1 = \partial_t, X_2 = \partial_x$. By integrating Lie equation system, we obtain one parameter symmetry groups with generators X_1 and X_2:

$$G_1(s) : (t, x, u) \mapsto (t + s, x, u), \qquad G_2(s) : (t, x, u) \mapsto (t, x + s, u). \tag{1.7}$$

Thus we can say:

Corollary. *If $u = f(t, x)$ is a solution of (0.1), then*

$$u^{(1)} = f(t - s, x), \qquad u^{(2)} = f(t, x - s), \qquad s \in \mathbb{R},$$

are solutions of (0.1).

Proof: Now, we want to find nontrivial invariant solution of equation (0.1). Consider the symmetry generator $X_1 = \partial_t$, by solving characteristic equations, we obtain functional independent invariants $y = x$ and $w = u$. Thus the reduced equation is

$$k(u)w_{yy} + f(w) = 0,$$

which is an invariant solution of the equation (0.1) corresponding with X_1 by integrating the above reduced equation.

Similarly, we find functional independent invariants $y = t$, $w = u$ corresponding with $X_2 = \partial_x$, and reduced equation:

$$\varepsilon^2 w_{yy} - g(w) + w_y - f(w) = 0.$$

Solving the above reduced equation leads to an invariant solution of the equation (0.1) corresponding with X_2.

For example, suppose that $f(u) = c_1$, $g(u) = c_2$ and $k(u) = c_3$ are constants. Then the reduced equations corresponding with X_1 and X_2 are, respectively:

$$c_3 w_{yy} + c_1 = 0, \qquad \varepsilon^2 w_{yy} - c_2 + w_y - c_1 = 0.$$

The following invariant solutions are respectively:

$$\begin{aligned} w_1(y) &= -\frac{c_1}{2c_3} y^2 + c_1 y + c_2, \\ w_2(y) &= -\frac{c_1}{c_2} \varepsilon^2 \exp(-\frac{c_2}{\epsilon^2} y) + \frac{c_1}{c_2} y + c_2. \end{aligned}$$

Therefore we can prove that:

Theorem. *If $f(u) = c_1$, $g(u) = c_2$ and $k(u) = c_3$ are constant functions then invariant solutions of the equation (0.1) corresponding with exact symmetries $X_1 = \partial_t$ and $X_2 = \partial_x$ are respectively:*

$$u_1 = -\frac{c_1}{2c_3}x^2 + c_1 x + c_2,$$
$$u_2 = -\frac{c_1}{c_2}\varepsilon^2 \exp(-\frac{c_2}{\varepsilon^2}t) + \frac{c_1}{c_2}t + c_2.$$

2 Conditional Symmetries

In this section, we desrcibe Cicogna method for finding conditional symmetry. By definition of Fuschych, X is a conditional symmetry (CS) of the equation $\Delta = 0$ if there is a supplementary equation $E = 0$ such that X is an exact symmetry of the system $\Delta = E = 0$.

Cicogna considered the simplest and more common case of supplementary equation that is called 'side condition' or invariant surface condition: $X_Q u = \xi_i\, u_{x_i} - \varphi = 0$, where X_Q is the symmetry written in evolutionary form [5,17]. This condition indicates that we are finding precisely solutions which are invariant under X. By proposition 1 in [5], we have:

Proposition. *A vector field X is a standard conditional symmetry for the PDE $\Delta = 0$ if it is a symmetry for the system*

$$\Delta = 0, \qquad X_Q u = 0,$$

and this corresponds to the existence of a reduced equation in $p-1$- independent variables which gives X- invariant solutions of $\Delta = 0$. Also, a vector field X is a weak conditional symmetry of order σ if it is a symmetry of the system

$$\Delta = 0, \quad \Delta^{(1)} := \mathbf{pr}^{(1)}X(\Delta) = 0, \quad \cdots, \quad \Delta^{(\sigma-1)} = 0, \quad X_Q u = 0,$$

and this corresponds to the existence of a system of σ reduced equations which gives X -invariant solutions of $\Delta = 0$.

Proof. Now, we want to find conditional symmetry of the equation (0.1). We consider the evolutional form of X, so $X_Q = \varphi(t,x,u) - \tau(t,x,u)\partial_t - \xi(t,x,u)\partial_x$. Then by proposition 2, X is a standard conditional symmetry if X is a symmetry of the following system:

$$\varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u), \qquad \varphi - \tau u_t - \xi u_x = 0.$$

Integrating of the above system implies that $\tau = c_1$, $\xi = c_2 = 0$ and $\varphi = 0$. Thus we can state:

Theorem. *The infinitesmal standard conditional symmetry of the equation (0.1) are $X_1 = \partial_t$, $X_2 = \partial_x$ and $X_3 = \partial_t + \partial_x$.*

Proof. Infinitesmal generator symmetries X_1 and X_2 are exact symmetries that we obtained invariant solutions of them by Lie method. The corresponding characteristic equation with X_3 is $dt = dx$ and $du = 0$. Then functional independent invariants are $y = x - t$ and $w = u$ and the reduced equation is:

$$\varepsilon^2 w_{yy} = g(w)w_y + k(w)w_{yy} + f(w). \tag{2.8}$$

Now, we can find invariant solutions of the equation (0.1) by integrating the reduced equation (2.8). For example, suppose that $f(u) = c_1$, $g(u) = c_2$ and $k(u) = c_3$ are constant functions, so the reduced equation corresponding with X_3 is $\varepsilon^2 w_{yy} = c_2 w_y + c_3 w_{yy} + c_1$ and the solution is:

$$w(y) = \frac{c_1}{c_2}(c_3 - \varepsilon^2)\exp\left(\frac{c_2 y}{c_3 - \varepsilon^2}\right) + \frac{c_1 y}{c_2} + c_2.$$

Therefore, we can state the following theorem:

Theorem. *Let functions $f(u) = c_1$, $g(u) = c_2$ and $k(u) = c_3$ in the equation (0.1) are constants, then the invaraint solution corresponding with the standard conditional symmetry $X = \partial_t + \partial_x$ is:*

$$u = \frac{c_1}{c_2}(c_3 - \varepsilon^2) \exp\left(\frac{c_2(x-t)}{c_3 - \varepsilon^2}\right) + \frac{c_1}{c_2}(x-t) + c_2.$$

Proof. Now, let $X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \varphi(t, x, u)\partial_u$ be a vector field. For finding weak conditional symmetry of order 1 of the equation (0.1), we must solve the following system:

$$\begin{cases} \Delta & : \quad \varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u), \\ \Delta^{(1)} & : \quad \mathbf{pr}^{(1)} X(\Delta) = 0, \\ X_Q u & : \quad \varphi - \tau u_t - \xi u_x = 0. \end{cases}$$

Then we gain $\tau = c_1$, $\xi = c_2$, $\varphi = 0$. Thus weak conditional symmetry of order 1 are exactly the standard conditional symmetries $X = \partial_t$, $X_2 = \partial_x$.

The weak conditional symmetry of order 2 are obtained by integrating the following system:

$$\begin{cases} \Delta & : \quad \varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u), \\ \Delta^{(1)} & : \quad \mathbf{p}r^{(1)} X(\Delta) = 0, \\ \Delta^{(2)} & : \quad \mathbf{p}r^{(2)} X(\Delta) = 0, \\ X_Q u & : \quad \varphi = \tau u_t + \xi u_x, \end{cases}$$

that these are the same as the weak conditional symmetry of order 1.

3 Approximate Symmetries

In this section, we illustrate approximate symmetries of the equation (0.1) by Ibragimov approximate method. Now, consider an approximate equation:

$$F(z, \varepsilon) = F_0(z) + \varepsilon F_1(z) = 0. \tag{3.9}$$

By theorem 2.2.1 in [13], the equation (3.9) is an approximate invariant under approximate transformation group G with the generator

$$X = X^0 + \varepsilon X^1 = \xi_0^i \partial_{z^i} + \varepsilon \xi_1^i(z)\partial_{z^i},$$

if and only if $\left[XF(z, \varepsilon)\right]_{F=0} = O(\varepsilon)$.

The Theorem 2.2.2 of [13] statted that "If the equation (3.9) admits an approximate transformation group with the generator $X = X^0 + \varepsilon X^1$, then the operator $X^0 = \xi_0^i(z)\partial_{z^i}$ is an exact symmetry of the equation $F_0(z) = 0$". Therefore, we can give an infinitesmal method for calculating approximate symmetries X of the first order for differential equation (3.9):

1. Computation of the exact symmetry X^0 of the unperturbed equation $F_0(z) = 0$.
2. Determination of the auxiliary function H by the equation:
$$H = \frac{1}{\varepsilon}\left[X^0(F_0(z) + \varepsilon F_1(z))\Big|_{F_0(z)+\varepsilon F_1(z)=0}\right],$$
3. Calculation of the operator X^1 by solving the determining equation:
$$X^1 F_0(z)\big|_{F_0(z)=0} + H = 0.$$

By similar method, we find approximate symmetry of the second order for the equation $F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) = 0$.

By using Ibragimov method, we gain approximate symmetries of the first and second order of the equation (0.1), then we can state the following theorem:

Theorem. *Approximate symmetries of order 1 of the equation (0.1) are ∂_t, ∂_x, $\varepsilon^2\partial_t$, $\varepsilon^2\partial_x$. And approximate symmetries of order 2 of the equation (0.1) are ∂_t, ∂_x, $\varepsilon\partial_t$, $\varepsilon\partial_x$, $\varepsilon^2\partial_t$ and $\varepsilon^2\partial_x$.*

4 Illustration

We consider the problem of traveling fronts proceeded by growth together with cell dispersal. This is the appearance in populations of bacteria swimming inside a narrow channel [1, 19]. S. Dunbar and H. Othmer [6, 7] introduced a model of cell dispersal. They consider a position migration process with branching. A newborn particle moves with constant speed to the right. It remains in this state and reverses its direction if it leaves this state. Moreover, the particle may split into two daughters which each of them chooses its direction of movement with probability 1/2. Due to this phenomenon, they were modeling equation (0.1). E. Bouin, V. Calvezyz, and G. Nadin used this model and studied a special case of it [3]. In fact, they supposed that $g(u) = 1 - \varepsilon^2 F'(\rho_\varepsilon(t,x))$, $k(u) = -1, f(u) = F(\rho_\varepsilon(t,x))$ in equation (0.1) where (t,x) and $\rho_\varepsilon(t,x)$ are independent and dependent variables and the growth function F is a concave function. So the equation (0.1) changes to the following form:

$$\varepsilon^2\partial_{tt}\rho_\varepsilon + (1 - \varepsilon^2 F'(\rho_\varepsilon))\partial_t\rho_\varepsilon - \partial_{xx}\rho_\varepsilon = F(\rho_\varepsilon), \tag{4.10}$$

where $\rho_\varepsilon(t,x)$ is cell density and the parameter ε is a scaling factor. We consider the equation (4.10) and obtained invariant solutions of it.

The equation (4.10) is equivalent to the hyperbolic system

$$\partial_t\rho_\varepsilon + \varepsilon^{-1}\partial_x(j_\varepsilon) = F(\rho_\varepsilon), \qquad \varepsilon\partial_t j_\varepsilon + \partial_x\rho_\varepsilon = -\varepsilon^{-1} j_\varepsilon, \tag{4.11}$$

which $\rho_\varepsilon(t,x)$ and $j_\varepsilon(t,x)$ are dependent variables. Then solutions of (4.11) are corresponding one to one with solutions of (4.10), therefore we integrate the hyperbolic system (4.11) instead of the equation (4.10).

For easy to work, suppose that $F(\rho) = \rho(1-\rho)$. Then by using Lie method, we can prove that:

Theorem. *The hyperbolic system (4.11) admits one parameter symmetry group G which generator by $X_1 = \partial_t$, $X_2 = \partial_x$ and $X_3 = \exp(-t/\varepsilon^2)\partial_{j_\varepsilon}$.*

Functional independent invariantes of $X_1 = \partial_t$ are $f(x) = \rho_\varepsilon$, $h(x) = j_\varepsilon$. Then the reduced system is the form:

$$h_x/\varepsilon - f + f^2 = 0, \qquad f_x + h/\varepsilon = 0.$$

By solving the above system, we find invariant solutions of the system (4.11). The reduced system with respect to $X_2 = \partial_x$ is:

$$f_t - f + f^2 = 0, \qquad \varepsilon h_t + h = 0.$$

Thus invariant solution is $h(t) = c_2\exp(-t/\varepsilon^2)$, $f(t) = 1/(1+\exp(-t)c_1)$.

Finally, the reduced system for $X_3 = \exp(-t/\varepsilon^2)\partial_{j_\varepsilon}$ is:

$$f_t - f + f^2 = 0, \qquad f_x = 0.$$

and invariant solution is $f(t,x) = 1/(1+\exp(-t)c_1)$.

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