A Riccati-Bernoulli Sub-ODE Method for the Resonant Nonlinear Schrödinger Equation with Both Spatio-Temporal Dispersions and Inter-Modal

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Abstract

This work uses the Riccati-Bernoulli sub-ODE method in constructing various new optical soliton solutions to the resonant nonlinear Schrödinger equation with both spatio-temporal dispersion and inter-modal dispersion. Actually, the proposed method is effective tool to solve many other nonlinear partial differential equations in mathematical physics. Moreover this method can give a new infinite sequence of solutions. These solutions are expressed by hyperbolic functions, trigonometric functions and rational functions. Finally, with the aid of Matlab release 15, some graphical simulations were designed to see the behavior of these solutions.

Keywords: Riccati-Bernoulli Sub-ODE Method, Resonant Nonlinear SchrÖdinger Equation, Solitons, Spatio-Temporal Dispersion, Inter-Modal Dispersion, Matlab Release 15.

AMS Subject Classification. 35A20, 35A99, 83C15, 65Z05

Introduction

Over the years, many complex nonlinear aspects arising in various fields of nonlinear sciences, such as; plasma physics, biology, optical fibers, fluid dynamics, physics, chemical kinematics, quantum mechanics, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Thus, many new methods have been proposed to investigate these equations, such as the tanh-sech method [13, 14, 15], Jacobi elliptic function method [16, 17, 18], exp-function method [19, 20], sine-cosine method [21, 22, 23], homogeneous balance method [24, 25], F-expansion method [26, 27], extended tanh-method [28, 29], $(\frac{G'}{G})-$ expansion method [30, 31] and so on.

This paper concerned with the resonant nonlinear Schrödinger equation [32, 33, 34] given by

$$i(\psi_t - \delta\psi_x) + \alpha\psi_{xx} + \beta\psi_{xt} + \lambda F\left(\mid \psi \mid^2\right)\psi + \gamma\left(\frac{\mid \psi \mid_{xx}}{\mid \psi \mid}\right)\psi = 0, \tag{0.1}$$

where $\psi(x,t)$ is the complex wave profile, x and t are the spatial and temporal variables, respectively. Here, α and β represent the coefficients of group-velocity dispersion and spatio-temporal (STD), respectively, while λ and γ are the coefficients of non-Kerr law nonlinearity and resonant nonlinearity, respectively. Moreover δ represents the coefficient of inter-modal dispersions (IMD). Obviously, this system is a strongly nonlinear and it is quiet difficult to obtain its solitary wave solutions. In this paper, we use the Riccati-Bernoulli sub-ODE method [10, 11, 35, 36, 37], to construct exact solutions, solitary wave solutions of nonlinear partial differential equations (NPDEs). We choose the resonant nonlinear Schrödinger equation in order to illustrate the validity of the proposed method. Actually the solutions of this equation turn out to be very useful in order to prescribe physical interpretation in a completely unified way. Indeed the proposed method can be used to solve so many other NPDEs.

The rest of the paper is arranged as follows. The Riccati-Bernoulli sub-ODE method is described in Section 1. In Section 2, some exact solutions for the resonant nonlinear Schrödinger equation are given. In Section 3 we compare our results with other results in order to show that the proposed methods in this paper are efficacious, robust and adequate. Namely, we clarify that the Riccati-Bernoulli sub-ODE method superior to other methods. Conclusion will appear in Section 4.

1 Description of the method

Consider the following nonlinear evolution equation

$$P(\phi, \phi_t, \phi_x, \phi_{tt}, \phi_{xx}, ...) = 0, \tag{1.1}$$

where P is a polynomial in $\phi(x,t)$ and its partial derivatives with respect to x or t. We review the steps of this method [37] as follows:

Step 1. Substituting the traveling wave transformation

$$\phi(x,t) = \phi(\xi), \qquad \xi = k(x + vt), \tag{1.2}$$

into Eq. (1.1), gives the following nonlinear ordinary differential equations (NODE):

$$H(\phi, \phi', \phi'', \phi''', ...) = 0, \tag{1.3}$$

where H is a polynomial in $\phi(\xi)$ and its total derivatives with respect to ξ.
Step 2. We assume that Eq. (1.3) has the formal solution in the following form:

$$\phi' = a\phi^{2-n} + b\phi + c\phi^n, \tag{1.4}$$

where a, b, c and n are constants to be calculated in sequel . From equation (1.4), we have

$$\phi'' = ab(3-n)\phi^{2-n} + a^2(2-n)\phi^{3-2n} + nc^2\phi^{2n-1} + bc(n+1)\phi^n + (2ac+b^2)\phi, \tag{1.5}$$

$$\begin{aligned}\phi''' = (&ab(3-n)(2-n)\phi^{1-n} + a^2(2-n)(3-2n)\phi^{2-2n} \\ &+n(2n-1)c^2\phi^{2n-2} + bcn(n+1)\phi^{n-1} + (2ac+b^2))\phi'.\end{aligned} \tag{1.6}$$

Remark 1.1. *Eq. (1.4) is called the Riccati-Bernoulli equation. At $ac \neq 0$ and $n = 0$, Eq. (1.4) is called a Riccati equation. At $a \neq 0$, $c = 0$, and $n \neq 0$, Eq. (1.4) is called a Bernoulli equation.*

The solutions for the Riccati-Bernoulli Eq. (1.4) are:

Case 1: When $n = 1$, the solution is

$$\phi(\xi) = \mu e^{(a+b+c)\xi} . \tag{1.7}$$

Case 2: When $n \neq 1$, $b = 0$ and $c = 0$, the solution is

$$\phi(\xi) = (a(n-1)(\xi + \mu))^{\frac{1}{n-1}} . \tag{1.8}$$

Case 3: When $n \neq 1$, $b \neq 0$ and $c = 0$, the solution is

$$\phi(\xi) = \left(\frac{-a}{b} + \mu e^{b(n-1)\xi}\right)^{\frac{r}{n-1}} . \tag{1.9}$$

Case 4: When $n \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, the solution is

$$\phi(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a} tan\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \tag{1.10}$$

and

$$\phi(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{4ac-b^2}}{2a} cot\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \tag{1.11}$$

Case 5: When $n \neq 1$,$a \neq 0$ and $b^2 - 4ac > 0$, the solution of Eq. (1.4) is

$$\phi(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} coth\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \tag{1.12}$$

and

$$\phi(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} tanh\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \tag{1.13}$$

Case 6: When $n \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, the solution of Eq. (1.4) is

$$\phi(\xi) = \left(\frac{1}{a(n-1)(\xi+\mu)} - \frac{b}{2a}\right)^{\frac{1}{1-n}} . \tag{1.14}$$

Here μ is an arbitrary constant.

Step 3. Superseding the derivatives of ϕ into Eq. (1.3) gives an algebraic equations of ϕ. Using the symmetry of the right-hand item of equation (1.4) and setting the highest power exponents of ϕ to equivalence in Eq. (1.3), n can be determined. Comparing the coefficients of ϕ^i gives algebraic equations of $a, b, c, and\ v$. Solving these equations and substituting n, a, b, c, v and $\xi = k(x + vt)$ into Eqs. (1.7)-(1.14)), give the traveling wave solutions for Eq. (1.1).

1.1 Bäcklund transformation

When $\phi_m(\xi)$ and $\phi_{m+1}(\xi)(\phi_{m+1}(\xi) = \phi_{m+1}(\phi_m(\xi)))$ are the solutions of Eq. (1.4), we get

$$\frac{d\phi_{m+1}(\xi)}{d\xi} = \frac{d\phi_{m+1}(\xi)}{d\phi_m(\xi)}\frac{d\phi_m(\xi)}{d\xi} = \frac{d\phi_{m+1}(\xi)}{d\phi_m(\xi)}(a\phi_m^{2-n} + b\phi_m + c\phi_m^n),$$

namely

$$\frac{d\phi_{m+1}(\xi)}{a\phi_{m+1}^{2-n}+b\phi_{m+1}+c\phi_{m+1}^{n}}=\frac{d\phi_{m+1}(\xi)}{a\phi_{m}^{2-n}+b\phi_{m}+c\phi_{m}^{n}}. \tag{1.15}$$

Integrating equation (1.15) once with respect to ξ, we get

$$\phi_{m+1}(\xi)=\left(\frac{-cA_1+aA_2\left(\phi_m(\xi)\right)^{1-n}}{bA_1+aA_2+aA_1\left(\phi_m(\xi)\right)^{1-n}}\right)^{\frac{1}{1-n}}, \tag{1.16}$$

where A_1 and A_2 are arbitrary constants.

Eq. (1.16) is a Bäcklund transformation of Eq. (1.4). If we get a solution of this equation, we use Eq. (1.16) to obtain infinite sequence of solutions of Eq. (1.4), and like wise of Eq. (1.1).

2 Application

Here, the application of the Riccati-Bernoulli sub-ODE method to Eq. (0.1) is introduced. We first use the following complex wave transformation:

$$\psi(x,t)=e^{i\omega(x,t)}\phi(\xi),\quad \omega=-\vartheta x+wt+h,\quad \xi=x-\kappa t, \tag{2.1}$$

where, $\phi(\xi)$ represents the shape of the traveling wave, $\omega(x,t)$ is the phase component of the wave, while ϑ, w, h and κ are real constants.

Substituting Eq. (2.1) into Eq. (0.1) and separating the real and the imaginary parts, respectively, we have:

$$c=\frac{\delta+2\alpha\vartheta-\beta w}{\beta\vartheta-1}. \tag{2.2}$$

$$E\phi''+H\phi+\lambda F(\phi^2)\phi=0, \tag{2.3}$$

where $E=\alpha-\beta\kappa+\gamma$, $H=w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2$.

Eq. (2.2) gives the velocity of the traveling wave, while Eq. (2.3) will be solved in the following subsections by using the Riccati-Bernoulli sub-ODE method.

For Kerr law nonlinearity, $F(\phi^2)=\phi^2$, then Eq. (2.3) reduces to

$$E\phi''+H\phi+\lambda\phi^3=0. \tag{2.4}$$

Substituting Eq. (1.5) into Eq. (2.4), we obtain

$$E\left(ab(3-m)\phi^{2-m}+a^2(2-m)\phi^{3-2m}+mc^2\phi^{2m-1}+bc(m+1)\phi^m+(2ac+b^2)\phi\right)+H\phi+\lambda\phi^3=0 \tag{2.5}$$

Setting $m=0$, equation (2.5) is reduced to

$$E(3ab\phi^2+2a^2\phi^3+bc+(2ac+b^2)\phi)+H\phi+\lambda\phi^3=0. \tag{2.6}$$

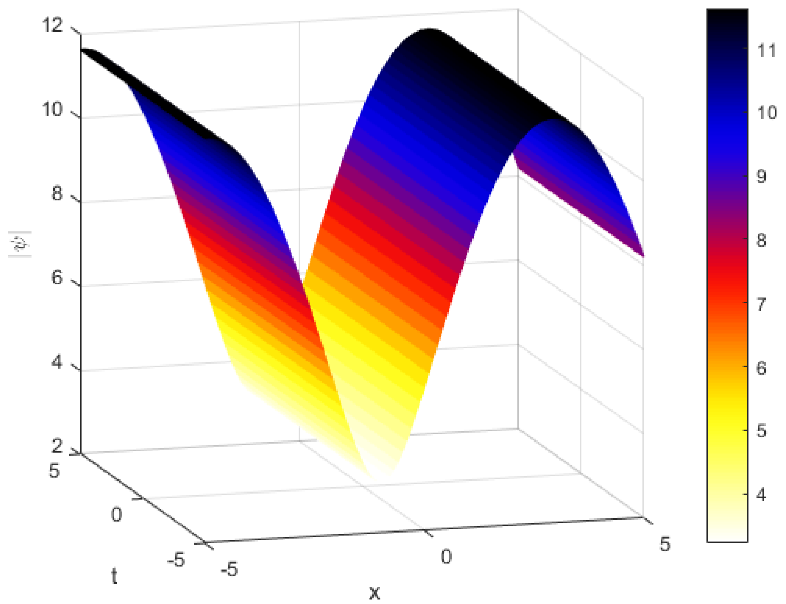

Figure 1: The solution $\psi_2(x,t)$ in (2.18) for $\vartheta = 2.4$, $h = 1.4$, $w = 2$, $\alpha = 1.2$, $\beta = 2$, $\gamma = 0.6$, $\delta = 1.5$, $\kappa = 3$, $\lambda = 3$, $\mu = 1$, and $-5 \leq t, x \leq 5$.

Setting each coefficient of $\phi^i (i = 0, 1, 2, 3)$ to zero, we get

$$E\ bc = 0\,, \tag{2.7}$$

$$E\ (2ac + b^2) + H = 0\,, \tag{2.8}$$

$$3E\ ab = 0\,, \tag{2.9}$$

$$2E\ a^2 + \lambda = 0\,. \tag{2.10}$$

Solving equations (2.7)-(2.10), we get

$$b = 0\,, \tag{2.11}$$

$$ac = \frac{-H}{2E}\,, \tag{2.12}$$

$$a = \pm\sqrt{\frac{-\lambda}{2E}}. \tag{2.13}$$

Hence, we give the cases of solutions for the equations (2.4) and (0.1), respectively

1. When $b = 0$ and $c = 0$, the solution of equation (2.4) is

$$\phi_1(x,t) = (-a(x - \kappa t + \mu))^{-1}\,. \tag{2.14}$$

Using equations (2.14), and (2.1) the solutions of equation (0.1) take the forms:

$$\psi_1(x,t) = e^{i(-\vartheta x + wt + h)}\,(-a(x - \kappa t + \mu))^{-1}\,. \tag{2.15}$$

where ϑ, w, h, κ and μ are arbitrary constants.

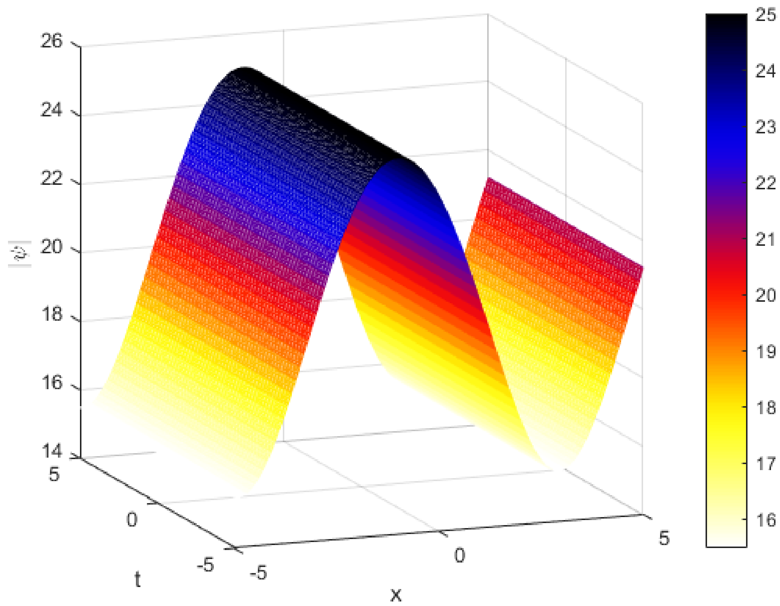

Figure 2: The solution $\psi_6(x,t)$ in (2.22) for $\vartheta = 1.6$, $h = 2.1$, $w = 2.3$, $\alpha = 1.5$, $\beta = 1.1$, $\gamma = 0.3$, $\delta = 1.4$, $\kappa = 2$, $\lambda = 1.5$, $\mu = 1$, and $-5 \le t, x \le 5$.

2. When $\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\alpha-\beta\kappa+\gamma} < 0$, substituting equation (2.11)-(2.13) and (2.1) into equations (1.10) and (1.11), we obtain traveling wave solutions of equation (0.1),

$$\phi_{2,3}(x,t) = \pm\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}}\tan\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x-\kappa t+\mu)\right) \tag{2.16}$$

and

$$\phi_{4,5}(x,t) = \pm\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}}\cot\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x-\kappa t+\mu)\right). \tag{2.17}$$

Using equations (2.16), (2.17) and (2.1) the solutions of equation (0.1) take the forms:

$$\psi_{2,3}(x,t) = \pm e^{i(-\vartheta x+wt+h)}\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}}\tan\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x-\kappa t+\mu)\right)\right) \tag{2.18}$$

and

$$\psi_{4,5}(x,t) = \pm e^{i(-\vartheta x+wt+h)}\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}}\cot\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x-\kappa t+\mu)\right)\right), \tag{2.19}$$

where $\vartheta, w, h, \beta, \alpha, \gamma, \kappa$ and μ are arbitrary constants. Figure 1 illustrated the solution ψ_2 with some certain values of the parameters. This figure give the behaviour of this solution.

3. When $\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\alpha-\beta\kappa+\gamma} > 0$, substituting equations (2.11)-(2.13) and (2.1) into equations (1.12) and (1.13), we obtain traveling wave solutions of equation (0.1),

$$\phi_{6,7}(x,t) = \pm\sqrt{\frac{\delta\vartheta + \alpha\vartheta^2 - w(\beta\vartheta - 1)}{\lambda}} \tanh\left(\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{2(\alpha - \beta\kappa + \gamma)}}(x - \kappa t + \mu)\right) \tag{2.20}$$

and

$$\phi_{8,9}(x,t) = \pm\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{\lambda}} \coth\left(\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{2(\beta\kappa - \alpha - \gamma)}}(x - \kappa t + \mu)\right). \tag{2.21}$$

Using equations (2.20), (2.21) and (2.1) the solutions of equation (0.1) take the forms:

$$\psi_{6,7}(x,t) = \pm e^{i(-\vartheta x + wt + h)}\sqrt{\frac{\delta\vartheta + \alpha\vartheta^2 - w(\beta\vartheta - 1)}{\lambda}} \tanh\left(\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{2(\alpha - \beta\kappa + \gamma)}}(x - \kappa t + \mu)\right) \tag{2.22}$$

and

$$\psi_{8,9}(x,t) = \pm e^{i(-\vartheta x + wt + h)}\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{\lambda}} \coth\left(\sqrt{\frac{w(\beta\vartheta - 1) - \delta\vartheta - \alpha\vartheta^2}{2(\beta\kappa - \alpha - \gamma)}}(x - \kappa t + \mu)\right), \tag{2.23}$$

where $\vartheta, w, h, \beta, \alpha, \gamma, \kappa$ and μ are arbitrary constants. Figure 2 illustrated the solution ψ_6 with some certain values of the parameters. This figure give the behaviour of this solution.

Remark 2.1. *Applying equation (1.16) to $E_i(x,t)$, i =1,2,...,8,9, we obtain an infinite sequence of solutions of equation (0.1). For illustration, by applying equation (1.16) to $E_i(x,t)$, i =1,2,...,9, once, we have new solutions of equation (0.1)*

$$\phi_1^\star(x,t) = \frac{B_3}{-aB_3\,(x - \kappa t + \mu) \pm 1}, \tag{2.24}$$

$$\phi_{2,3}^\star(x,t) = \frac{\pm\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\sqrt{2\lambda(\beta\kappa-\alpha-\gamma)}} \pm B_3\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}} \tan\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta c-\alpha-\gamma)}}(x - \kappa t + \mu)\right)}{B_3 \pm \sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}} \tan\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta c-\alpha-\gamma)}}(x - \kappa t + \mu)\right)}, \tag{2.25}$$

$$\phi_{4,5}^\star(x,t) = \frac{\pm\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\sqrt{2\lambda(\beta\kappa-\alpha-\gamma)}} \pm B_3\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}} \cot\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x - \kappa t + \mu)\right)}{B_3 \pm \sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\lambda}} \cot\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\beta\kappa-\alpha-\gamma)}}(x - \kappa t + \mu)\right)}, \tag{2.26}$$

$$\phi_{6,7}^\star(x,t) = \frac{\pm\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\sqrt{2\lambda(\beta\kappa-\alpha-\gamma)}} \pm B_3\sqrt{\frac{\delta\vartheta+\alpha\vartheta^2-w(\beta\vartheta-1)}{\lambda}} \tanh\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\alpha-\beta\kappa+\gamma)}}(x - \kappa t + \mu)\right)}{B_3 \pm \sqrt{\frac{\delta\vartheta+\alpha\vartheta^2-w(\beta\vartheta-1)}{\lambda}} \tanh\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\alpha-\beta\kappa+\gamma)}}(x - \kappa t + \mu)\right)}, \tag{2.27}$$

$$\phi_{8,9}^\star(x,t) = \frac{\pm\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{\sqrt{2\lambda(\beta\kappa-\alpha-\gamma)}} \pm B_3\sqrt{\frac{\delta\vartheta+\alpha\vartheta^2-w(\beta\vartheta-1)}{\lambda}} \coth\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\alpha-\beta\kappa+\gamma)}}(x - \kappa t + \mu)\right)}{B_3 \pm \sqrt{\frac{\delta\vartheta+\alpha\vartheta^2-w(\beta\vartheta-1)}{\lambda}} \coth\left(\sqrt{\frac{w(\beta\vartheta-1)-\delta\vartheta-\alpha\vartheta^2}{2(\alpha-\beta\kappa+\gamma)}}(x - \kappa t + \mu)\right)}, \tag{2.28}$$

where $B_3, \vartheta, w, h, \beta, \alpha, \gamma, \kappa$ and μ are arbitrary constants.

3 Comparisons

Here we compare our results with other results in order to show that the our methods are robust, adequate and efficient. Namely, we compare between our solutions and the solutions given in [32, 34]. Indeed, we clarify that the Riccati-Bernoulli sub-ODE method superior to other methods. Zhou et al. [32] have introduced only three solutions for the resonant nonlinear Schrödinger equation, using the $(\frac{G'}{G})$- expansion method. Whereas Bulut et al. [34] given twelve solutions of the resonant nonlinear Schrödinger equation, using the extended sinh-Gordon equation expansion method. Indeed his proposed method is simple, flexible and easy to use and produces very accurate results. His result is better than the Zhou's result [32]. In this article, we given new and so many solutions, using the Riccati-Bernoulli sub-ODE method. It can be seen that by choosing suitable values for the parameters of the solutions given in [32, 34], similar solutions can be verified. The main interesting feature of the Riccati-Bernoulli sub-ODE method over the other methods is that it produce many new exact traveling wave solutions with additional free parameters. Another positive side, that the Riccati-Bernoulli sub-ODE technique is more effective in providing many new solutions than these methods. Above the all, the Riccati-Bernoulli sub-ODE method has a very important characteristic, that provides infinite sequence of solutions of equation. In fact, this feature has never given for any another method. Consequently, this method is efficient, robust and proper to solve other nonlinear problems in mathematical physics and nonlinear science.

Remark 3.1. *The Riccati-Bernoulli sub-ODE technique can easily applied to solve nonlinear fractional differential equations, see [35, 36, 38, 39].*

4 Conclusions

In this work we consider the solution of the resonant nonlinear Schrödinger equation with both spatio-temporal dispersion and inter-modal dispersion, using the Riccati-Bernoulli sub-ODE technique. As a result, some new exact solutions for this equation have successfully been gained. Indeed, this method can give an infinite sequence of solutions, which is consider an interesting feature of this method. Currently, work is in progress on the applications of the Riccati-Bernoulli sub-ODE technique to other NPDEs in nonlinear science.

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