The Classification of Permutation Groups with Maximum Orbits

*Mehdi Alaeiyan,**Behname Razzaghmaneshi

Iran University of Science and Technology, Narmak, Tehran 16844, Iran

*alaeiyan@iust.ac.ir, **b.razagh@yahoo.com

Abstract

Let $G$ be a permutation group on a set $\Omega$ with no fixed points in $\Omega$ and let $m$ be a positive integer. If no element of $G$ moves any subset of $\Omega$ by more than $m$ points (that is, if $|\Gamma^g \setminus \Gamma| \leq m$ for every $\Gamma \subseteq \Omega$ and $g \in G$), and the lengths two of orbits is $p$, and the rest of orbits have lengths equal to 3. Then the number $t$ of $G$-orbits in $\Omega$ is at most $\lfloor \frac{1}{2}(3m - 2) + \frac{5}{2p} \rfloor$. Moreover, we classify all groups for $t = \lfloor \frac{1}{2}(3m - 2) + \frac{5}{2p} \rfloor$ is hold. (For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.)

Keywords: permutation group, bounded movement, orbits.

2000 AMS classification subjects: 20B25

Introduction

1 Introduction

Let $G$ be a permutation group on a set $\Omega$ with no fixed points in $\Omega$ and let $m$ be a positive integer. If no element of $G$ moves any subset of $\Omega$ by more than $m$ points (that is, if $|\Gamma^g \setminus \Gamma| \leq m$ for every $\Gamma \subseteq \Omega$ and $g \in G$), and the lengths two of orbits is $p$, and the rest of orbits have lengths equal to 3. Then the number $t$ of $G$-orbits in $\Omega$ is at most $\lfloor \frac{1}{2}(3m - 2) + \frac{5}{2p} \rfloor$. Moreover, we classify all groups for $t = \lfloor \frac{1}{2}(3m - 2) + \frac{5}{2p} \rfloor$ is hold. (For $x \in R$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.)

This notion was introduced in [3]. By [3,Theorem 1], if $G$ has bounded movement $m$, then $\Omega$ is finite. Moreover both the number of $G$-orbits in $\Omega$ and the length of each $G$-orbit are bounded above by linear functions of $m$. In particular it was shown that the number of $G$-orbits is at most $2m-1$. In this paper we will improve this to $\frac{1}{2}(3m - 2) + \frac{5}{2p}$, if the lengths two of orbits is $p$, and the rest of orbits have lengths equal to 3. If $m=1$, then $t = \lfloor \frac{1}{2} \rfloor$, $|\Omega| = 2$ and $G$ is $Z_2$ or $S_2$. So in this paper we suppose that $m$ greater than 1. We present here a classification of all groups for which the bound $\frac{1}{2}(3m - 2) + \frac{5}{2p}$ is attained. We shall say that an orbit of permutation group is nontrivial if its length is greater than 1. The main result is the following theorem.

**Theorem 1.1.** Let $m$ be a positive integer and suppose that $G$ is a permutation group on a set $\Omega$ such that $G$ has no fixed points in $\Omega$, and $G$ has bounded movement equal to $m$. If the lengths two of orbits is $p$, and the rest of orbits have lengths...
equal to 3. Then the number \( t \) of \( G \)-orbits in \( \Omega \) is at most \( \frac{1}{2}(3m - 2) + \frac{5}{2p} \). And also if \( t = \frac{1}{2}(3m - 1) + \frac{1}{2} \), then \( m \) is product of \( p \) in power of 3, and \( G \) is order \( pm \), all \( G \)-orbits have length 3, and the pointwise stabilizers of the \( G \)-orbits are precisely the \( \frac{1}{2}(3m - 2) + \frac{5}{2p} \) distinct subgroups of \( G \) of index 3.

Note that an orbit of a permutation group is non trivial if its length is greater than 1. The groups described below are examples of permutation groups with bounded movement equal to \( m \) which have exactly \( \frac{1}{2}(3m - 2) + \frac{5}{2p} \) nontrivial orbits.

### 2 Examples and Preliminaries

Let \( 1 \neq g \in G \) and suppose that \( g \) in its disjoint cycle representations has \( t \) nontrivial cycles of lengths \( l_1, \ldots, l_t \), say. We might represent \( g \) as \( g = (a_1a_2\ldots a_{k_1})(b_1b_2\ldots b_{k_2})\ldots(z_1z_2\ldots z_{k_t}) \). Let \( \Gamma(g) \) denote a subset of \( \Omega \) consisting \( \lfloor l_i/2 \rfloor \) points from the \( i \)th cycle, for each \( i \), chosen in such a way that \( \Gamma(g) \cap \Gamma(g') = \emptyset \). For example, we could choose \( \Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, \ldots, z_2, z_4, \ldots, z_{k_t} \} \), where \( k_i = l_i - 1 \) if \( l_i \) is odd and \( k_i = l_i \) if \( l_i \) is even. Note that \( \Gamma(g) \) is not uniqueness determined as it depends on the way each cycle is written. For any set \( \Gamma(g) \) consists of every point of every cycle of \( g \). From the definition of \( \Gamma(g) \) we see that

\[
|\Gamma(g)^g \setminus \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.
\]

The next lemma shows that this quantity is an upper bound for \( |\Gamma^g \setminus \Gamma| \) for an arbitrary subset \( \Gamma \) of \( \Omega \).

**Lemma 2.1.** [5, Lemma 2.1.]. Let \( G \) be a permutation group on a set \( \Omega \) and suppose that \( \Gamma \subseteq \Omega \). Then for each \( g \in G \),

\[
|\Gamma^g \setminus \Gamma| \leq \sum_{i=1}^{t} \lfloor l_i/2 \rfloor,
\]

where \( l_i \) is the length of the \( i \)th cycle of \( g \) and \( t \) is the number of nontrivial cycles of \( g \) in its disjoint cycle representation. This upper bound is attained for \( \Gamma = \Gamma(g) \) defined above.

Now we will show that there certainly is an infinite family of 3-groups for which the maximum bound obtained in Theorem 1.1 holds.

**Example 2.2.** Let \( r \) be a positive integer, let \( G:=\mathbb{Z}_p^2 \mathbb{Z}_3^{r-2} \), let \( t := \frac{1}{2}(3m - 2) + \frac{5}{2p} \), and let the lengths two of orbits is \( p \), and the rest of orbits have lengths equal to 3, and \( H_1, \ldots, H_t \) be an enumeration of the subgroups of index 3 in \( G \).

Define \( \Omega_i \) to be the coset space of \( H_i \) in \( G \) and \( \Omega := \Omega_1 \cup \ldots \cup \Omega_t \). If \( g \in G \setminus \{1\} \) then \( g \) lies in \( \frac{1}{2}(p^2.3^{r-1}-1) + \frac{5}{2p} \) of the groups \( H_i \) and therefore acts on \( \Omega \) as a permutation with \( \frac{1}{2}(p^2.3^{r-1}-1) + \frac{5}{2p} = m - 1 \) fixed points and \( 3^{r-3} \) disjoint 3-cycles. Taking one point from each of these 3-cycles to form a set \( \Gamma \) we see that \( m(G) \geq 3^{r-3} \), and it is not hard to prove that in fact \( m(G) = 3^{r-3} \). Thus \( n = 2t = (p^2.3^{r-1}-1) + \frac{5}{p} \). This proves bound of \( G - \text{orbits} \) of Theorem 1.1. It follows that \( G \) has bounded movement equal to \( m \), and \( G \) has \( \frac{1}{2}(3m - 2) + \frac{5}{2p} \) nontrivial orbits in \( \Omega \).

When \( m > 1 \) the classification in Theorem 1.1 follows immediately from the following theorem about subsets with movement \( m \).

**Definition** Let \( G \) be a permutation group on a set \( \Omega \) with orbits \( \Omega_i \), for \( i \in I \). We shall say that a subset \( \Gamma \subseteq \Omega \) cuts across each \( G \)-orbit if \( \Gamma_i := \Gamma \cap \Omega_i \notin \{\emptyset, \Omega_i\} \), for every \( i \in I \).

**Theorem 2.3.** Let \( GL\text{Sym}(\Omega) \) be a permutation group with \( t \) orbits for positive integer \( t \), such that the lengths two of orbits is \( p \), and the rest of orbits have lengths equal to 3. Moreover suppose that \( \Gamma \subseteq \Omega \) such that the move (\( \Gamma \)) = \( m > 1 \), and \( \Gamma \) cuts across each \( G \)-orbit. Then \( t \leq \frac{1}{2}(3m - 2) + \frac{5}{2p} \) and moreover, if \( t = \frac{1}{2}(3m - 2) + \frac{5}{2p} \), then:

1. \( G \) is an 3-group and all \( G \)-orbits of \( G \) has size 3 \\
2. If the rank of the group \( G \) is \( r \) then \( r \geq 2, t = \left( \frac{1}{2}(p^2.3^{r-1}-1) + \frac{5}{p} \right) \) and \( m = p(3^{r-3}) \);
3. If one of the \( G \)-orbits is 3, then The \( t \) different \( G \)-orbits are (isomorphic to) the coset spaces of the \( \left( \frac{1}{2}(p^2.3^{r-1}-1) + \frac{5}{2p} \right) \) different subgroups of index 3 in \( G \).
3 Proof of Theorem 2.3.

Proof: Let \( \Omega_1, \ldots, \Omega_t \) be \( t \) orbits of \( G \) of lengths \( n_1, \ldots, n_t \). Choose \( \alpha_i \in \Omega \) and let \( H_i := G_{\alpha_i} \), so that \( |G : H_i| = n_i \). For \( g \in G \), let \( \Gamma(g) = \{ \alpha_i | \alpha_i^g \neq \alpha_i \} \), be every second point of every cycle of \( g \) and let \( \gamma(g) := |\Gamma(g)| \). Since \( \Gamma(g) \cap \Gamma(g)^g = \emptyset \) it follows that \( \gamma(g) \leq m \) for all \( g \in G \). Let \( \Omega := \Omega_1 \cup \ldots \cup \Omega_t \), and let \( G \) and \( H_1, \ldots, H_t \) denote the finite permutation groups on \( \Omega \) induced by \( G \) and \( H_1, \ldots, H_t \) respectively. Then \( n_i = |G : H_i| \).

For \( g \in G \), let \( g \in G \) denote the permutation of \( \Omega \) induced by \( g \). Then as \( \gamma(1_G) = 0 \), we have \( \sum_{g \in G} \gamma(g) < m|G| \).

Now, counting the pairs \((g, i)\) such that \( g \in G \) and \( \alpha_i^g \neq \alpha_i \) gives

\[
\sum_{g \in G} \gamma(g) = \sum_{i} |\{ g \in G | \alpha_i^g \neq \alpha_i \}| = \sum_{i} |\{ g \in G | g \notin H_i \}| = \sum_{i} (|G| - |H_i|) = |G| \sum_{i} (1 - \frac{1}{n_i}).
\]

It follows that \( \sum_{i} (1 - \frac{1}{n_i}) < m \). Since \( n_i \geq 3, p^2 \) for each \( i \), it follows that \( \sum_{i} (1 - \frac{1}{n_i}) \geq \frac{2(p-1)}{p} + \frac{2}{3} (t-2) \) and hence \( \frac{2(p-1)}{p} + \frac{2}{3} (t-2) < m \), that is, \( t \frac{1}{2} (3m-2) + \frac{5}{2p} \).

Consequently \( G \) has at most \( \frac{1}{2} (3m-2) + \frac{5}{2p} \) orbits in \( \Omega \). Now let \( m \) be a positive integer greater than \( 1 \). Suppose that \( G \leq Sym(\Omega) \) with orbits \( \Omega_1, \Omega_2, \ldots, \Omega_t \), where \( t = \frac{1}{2} (3m-1) + \frac{1}{2} \). Suppose further that \( \Gamma \subseteq \Omega \) has move \( (\Gamma) = m \) and that \( G \) cuts across each of the \( G \)-orbits \( \Omega_i \). For each \( i \) set \( n_i = |\Omega_i| \) and \( \Gamma_i = \Gamma \cap \Omega_i \). Note that \( 0 < |\Gamma_i| < n_i \).

Claim 3.1 If Theorem 2.3 holds for the special case in which \( |\Gamma_i| = 1 \) for \( i = 1, \ldots, (\frac{1}{2} (3m-2) + \frac{5}{2p}) \), then it holds in general.

Proof: Suppose that Theorem 2.3 holds for the case where each \( |\Gamma_i| = 1 \). For \( i = 1, \ldots, t \), define \( \sum_{i} := \{ \Gamma_i^g | g \in G \} \), and note that \( |\sum_{i}| \geq 3 \) since \( \Gamma \) cuts across \( \Omega \). Set \( \Sigma = \cup_{i \geq 1} \sum_{i} \). Then \( G \) induces a natural action on \( \Sigma \) for which the \( G \)-orbits are \( \Sigma_1, \ldots, \Sigma_t \). Let \( G^\Sigma \) denote the permutation group induced by \( G \) on \( \Sigma \), and let \( K \) denote the kernel of this action.

We claim that the t-element subset \( \Gamma_\Sigma = \{ \Gamma_1, \ldots, \Gamma_t \} \subseteq \Sigma \) has movement equal to \( m \) relative to \( G^\Sigma \), and that \( \Gamma_\Sigma \) cuts across each \( \Gamma^\Sigma \)-orbit \( \Sigma_i \). For each \( g \in G \), \( \Gamma(g) = \Gamma^g - \Gamma^g |\Sigma| \) and hence \( \Gamma^g = \Gamma^g \Sigma \Sigma \Sigma \Sigma \). Thus move \( (\Gamma^g) |\Sigma| \). Also, since \( |\Sigma_i| \geq 3 \) and \( \Gamma \cap \Sigma_i \) consists of the single element \( \Gamma_1 \) of \( \Sigma_i \), the set \( \Gamma \Sigma \) cuts across each of the \( \frac{1}{2} (3m-2) + \frac{5}{2p} \) orbits \( \Sigma_i \). However, it follows that the number of \( G^\Sigma \)-orbits is at most \( \frac{1}{2} (3 \text{move}(\Gamma_\Sigma) - 2) + \frac{5}{2p} \), and hence move \( (\Gamma_\Sigma) = m \).

Thus the hypotheses of theorem 2.3 hold for the subset \( \Gamma_\Sigma \subseteq \Sigma \) relative to \( G^\Sigma \), and \( \Gamma_\Sigma \) cuts each \( G^\Sigma \)-orbit in exactly one point. By our assumption it follows that \( t = \frac{1}{2} (p^23^{r-1} - 2) \frac{5}{2p} = \frac{1}{2} (3m-2) + \frac{5}{2p} \) for some \( r > 1 \), and that \( G^\Sigma = Z_3^r \) and each \( |\Sigma_i| = 3 \). Further, the subgroups \( H_i \) of \( G \) fixing \( \Gamma_1 \) setwise range over the \( \frac{1}{2} (p^23^{r-1} - 2) + \frac{5}{2p} \) distinct subgroups which have index 3 in \( G \) and which contain \( K \). In particular, for each \( i, H_i \) is normal in \( G \) and hence the \( H_i \)-orbits of \( \Omega_i \) are blocks of imprimitivity for \( G \), and their number is at most \( |G : H| = 3 \). Since \( H_i \) fixes \( \Gamma_1 \) setwise it follows that \( \Gamma_1 \) is an \( H_i \)-orbit and \( n_i = 3 |\Gamma_1| \).

Let \( g \in G \setminus K \). Then in its action on \( \Sigma \), \( g \) moves exactly \( m \) of the \( \Gamma_i \). Since the \( \Gamma_i \) are blocks of imprimitivity for \( G \), each \( \Gamma_i^g \) is equal to either \( \Gamma_i \) or \( \Omega_i - \Gamma_i \). It follows that \( |\Gamma^g \setminus \Gamma| \) is equal to the sum of the sizes of the \( m \) subsets \( \Gamma_i \) moved by \( g \). However, since \( \text{move}(\Gamma) = m \), each of these \( m \) subsets \( \Gamma_i \) must have size 1. Since for each \( i \) we may
choose an element \( g \) which moves \( \Gamma_i \), we deduce that each of the \( \Gamma_i \) has size 1, and that \( K \) is the identity subgroup. It follows that theorem 2.3 hold for \( G \). Thus the claim is proved.

From now on we may and shall assume that each \( |\Gamma_i| = 1 \). Let \( \Gamma_i = \{\Omega_i\} \). Further we may assume that \( n_1 n_2 \ldots n_t \). For \( g \in G \) let \( c(g) \) denote the number of integers \( I \) such that \( \omega_i^g = \omega_i \). Note that since move \( (\Gamma) = m \), we have \( c(g) = t - m = \frac{1}{2}(3m - 2) + \frac{5}{2p} - m = \frac{m-2}{2} + \frac{5}{2p} \) and also \( c(1_G) = t = \frac{m-2}{2} + \frac{5}{2p} \).

**Lemma 3.2.** If two of the orbits of \( G \) has length equal to \( p \), then the rest orbits of \( G \) has size 3.

**Proof:** Let \( X \) denote the number of pairs \( (g,i) \) such that \( g \in G, 1 \leq i \leq \frac{t}{2} \), and \( \omega_i^g = \omega_i \). Then \( X = \sum_{g \in G \setminus H} c(g) \), and by our observations, \( X > |G| \frac{m-2}{2} + \frac{5}{2p} \). On the other hand, for each \( i \), the number of elements of \( G \) which fix \( \omega_i \) is \( |G_{\omega_i}| = \frac{|G|}{n_i} \), and hence \( X = |G| \sum_{i=1}^t n_i^{-1} \) if all the \( n_i \geq 3 \), and one of \( n_i \) is equal to \( m \), then \( X \leq |G| \left( \frac{2}{p} + \frac{t-1}{2} \right) = |G|(\frac{2}{p} + \frac{3n-2}{6p} + \frac{5}{2p}) \) (since \( n \geq 3 \) ) which is a contradiction. Hence \( n=3 \).

A similar argument to this enables us to show that except one of \( n_i \) the rest of \( n_i \) is \( n_i = 3 \), and hence that \( G \) is an 3 - group.

**Lemma 3.3.** The group \( G = Z_p^2 Z_3^t \) for some \( r \geq 2 \). Moreover for each \( n_i = 3 \), except one , the stabilizers \( G_{\omega_i}(2i\text{ll}) \) are pair wise distinct subgroups of index 3 in \( G \), and for each \( g \neq 1, c(g) = \left( \frac{m-2}{2} + \frac{5}{6p} \right) \).

**Proof:** By Lemma 3.2, except one of \( n_i \) the rest of \( n_i \) is \( n_i = 3 \). Thus \( H = G_{\omega_i} \) is a subgroup of index 3. This time we compute the number \( Y \) of pairs \( (g,i) \) such that \( g \in G \setminus H, 2i\text{ll} \), and \( \omega_i^g = \omega_i \). For each such \( g, \omega_i^g \neq \omega_1 \) and hence there are \( c(g) \) of these pairs with first entry \( g \). Thus \( Y = \sum_{g \in G \setminus H} c(g) \geq |G \setminus H| \left( \frac{m-1}{2} + \frac{5}{2p} \right) = |G| \left( \frac{m-1}{6} + \frac{5}{6p} \right) \).

On the other hand, for each \( i \geq 2 \), the number of elements of \( G \), which fix \( \omega_i \) is \( |G_{\omega_i}| \setminus \frac{3}{4} \). If \( H = G_{\omega_i} \) then \( |G_{\omega_i}| \setminus \frac{2}{3} = 0 \), while if \( G_{\omega_i} \neq H \), then \( |G_{\omega_i}| \setminus \frac{2}{3} = |G_{\omega_i}| \setminus \frac{|G|}{3} \). Hence

\[
Y = \left( \sum_{i=2}^t |G_{\omega_i}| \setminus \frac{2}{3} \right) \left( \frac{3}{4} \sum_{i=2}^t \frac{1}{n_i} \left( \frac{2}{p} + \frac{3n-2}{6p} \right) \right).
\]

It follows that equality holds in both of the displayed approximations for \( Y \). This means in particular that each \( n_i = 2 \), whence \( G = Z_p^2 Z_3^r \) for some \( r \). Further, for each \( i \geq 3, G_{\omega_i} \neq H \) and so \( r \geq 2 \). Arguing in the same way with \( H \) replaced by \( G_{\omega_i} \), for some \( i \geq 2 \), we see that \( G_{\omega_i} \neq G_{\omega_j} \) if \( j \neq i \), and also if \( g \in G_{\omega_j} \) then \( c(g) = \left( \frac{m-2}{2} + \frac{5}{6p} \right) \). Thus the stabilizers \( G_{\omega_i}(1i\text{ll}) \) are pair wise distinct , and if \( gl1 \) then \( c(g) = \left( \frac{m-2}{2} + \frac{5}{6p} \right) \). Finally we determine \( m \).

**Lemma 3.4.** \( m = p(3^{r-2}) \)

**Proof:** We use the information in lemma3.3 to determine precise the quantity \( X = \sum_{g \in G \setminus H} c(g) : X = t + (|G| - 1)(\frac{1}{2}(m-2) + \frac{5}{2p}) = \frac{1}{2}(3m - 2) + \frac{5}{2p} + (p^2.3^{r-2} - 1)(\frac{1}{2}(m-2) + \frac{5}{2p}). \)

\[
X = |G| \sum_{i=1}^t n_i^{-1} = |G| \left( \frac{2}{p} + \frac{t-2}{3} \right) = p^2.3^{r-2} \left( \frac{2}{p} + \frac{3m-2}{6} + \frac{5}{6p} - \frac{2}{3} \right).
\]

Thus implies that \( m = p(3^{r-3}) \).

The proof of theorem 2.3 now follows from lemmas 3.2-3.4.
References


