On Roots of Apolar Polynomials

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Abstract

This article continues the exploration of methods based on that of Gian-Carlo Rota that involve apolar invariants used for solving cubic and quintic polynomial equations.These polynomial invariants were disclosed previously as an alternative to and to clarify the umbral method of Rota.Theorems are proved regarding quintic, cubic, and quadratic polynomials that are pairwise apolar in that they satisfy particular polynomial apolar invariants.

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1 Introduction

Let $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ and $q(x) = b_0x^2 + 2b_1x + b_2$ represent typical cubic and quadratic polynomials, respectively, each with complex coefficients a_i, b_j, written in binomial form. In [3], it is shown that if the coefficients of p and q satisfy the polynomial equation

$$A_{2,3}(q(x), p(x)) = (a_0b_2 - 2a_1b_1 + b_0a_2)x - (a_1b_2 - 2a_2b_1 + b_0a_3) = 0$$

for all x in which case $p(x)$ and $q(x)$ are said to be "apolar", or "$A_{2,3}-$ apolar", then this provides a method for finding the roots of $p(x)$. This expression for $A_{2,3}(q(x), p(x))$ appears in [5] for the case in which q and p are monic; that is, $a_0 = 1 = b_0$. These strategies for finding the roots of an arbitrary cubic open an avenue to be explored whereby these "apolar methods" are extended to solve for the roots of an arbitrary quintic polynomial. Given a general cubic $p(x)$, one may solve for a quadratic $q(x)$ which is $A_{2,3}-$ apolar to $p(x)$. In the case where this $q(x)$ has 2 distinct roots r_1 and r_2, it is shown in [3] that the cubic $p(x)$ may be written

$$p(x) = \left(\frac{a_0r_2 + a_1}{r_2 - r_1}\right)(x - r_1)^3 - \left(\frac{a_0r_1 + a_1}{r_2 - r_1}\right)(x - r_2)^3 \tag{1}$$

which allows for a solution of $p(x) = 0$ using algebraic methods [3, Cor. 1]. As Gian-Carlo Rota remarks in [5]: "This method of solving a cubic equation is the only one I can remember". This method is alluded to also in [4]. In this paper, it will be proved that if such a cubic $p(x)$ has more than 1 root, then such a $p(x)$ will have 3 distinct roots. Alternately, if the quadratic $q(x)$ has one repeated root r, then this r will also be a repeated root of $p(x)$. This fact along with division allows for the complete solution of $p(x) = 0$. So, in either case, whether the roots of the quadratic $q(x)$ are distinct or repeated, the roots of the cubic $p(x)$ may be found algebraically [3, Th. 2]. An example of this method of apolar polynomials used to solve a cubic equation appears in Sec. 3 of this paper.

The preceding methods are then extended to the case of a general quintic polynomial

$$s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$$

also written in binomial form. A system of 3 linear equations in 4 unknowns is used to solve for a cubic $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ whose coefficients satisfy the polynomial equation

$$A_{3,5}\left(p\left(x\right), s\left(x\right)\right) = 0 \text{ for all } x, \text{ where}$$

$$\begin{aligned} A_{3,5}\left(p\left(x\right), s\left(x\right)\right) &= x^2\left(d_0a_3 - 3d_1a_2 + 3d_2a_1 - d_3a_0\right) \\ &\quad -2x\left(d_1a_3 - 3d_2a_2 + 3d_3a_1 - d_4a_0\right) \\ &\quad +d_2a_3 - 3d_3a_2 + 3d_4a_1 - d_5a_0 \ . \end{aligned}$$

Such polynomials $p\left(x\right)$ and $s\left(x\right)$ are said to be "apolar" or "$A_{3,5}-$ apolar" in that they are zeroes of the apolar invariant $A_{3,5}\left(p\left(x\right), s\left(x\right)\right)$. It was shown in [3], that if the cubic $p\left(x\right)$ has 3 distinct roots r_1, r_2, and r_3, then

$$s(x) = k_1\left(x - r_1\right)^5 + k_2\left(x - r_2\right)^5 + k_3\left(x - r_3\right)^5$$

for constants k_1, k_2, k_3. This fact is mentioned also in [5]. Similar to Equation (1), explicit formulas for each k_i that depend on the roots of $p(x)$ and the coefficients of $s(x)$ will be found in Sec. 3.

The general apolar polynomial $A_{k,n}\left(q\left(x\right), p\left(x\right)\right)$ is defined in [3] and is proved to be an invariant under translation in the cases for which $(k,\, n) = (2,\, 3)$, $(3,\, 5)$, $(2,\, 5)$. What "invariant under translation" means in the case of $A_{k,n}\left(q\left(x\right), p\left(x\right)\right)$ is that

$$A_{k,n}\left(q\left(x + c\right), p\left(x + c\right)\right) = A_{k,n}\left(q\left(x\right), p\left(x\right)\right)$$

for all x and for any complex constant c . In other words, substituting in $x+c$ for x (which forces the coefficients with respect to x of $q(x+c)$ and $p(x+c)$ to equal corresponding polynomials in c) results, after simplification, in c disappearing completely via cancellation. So in the 3 cases $(k,\, n) = (2,\, 3)$, $(3,\, 5)$, $(2,\, 5)$, the polynomial $A_{k,n}\left(q\left(x\right), p\left(x\right)\right)$ may be referred as an "apolar invariant".

As shown in [3, Sec. 5], for integers k and n with $1 \leq k \leq n$, the general apolar expression is a polynomial of degree $n-k$ in x for which the coefficients with respect to x are divisible by polynomials in a_i and b_j , which define the coefficients of p and q, respectively. Explicitly $A_{k,\, n}(q(x),\, p(x)) = \sum\limits_{i=0}^{n-k} T_i$ where

$$T_i = (-1)^i \begin{pmatrix} n-k \\ i \end{pmatrix} x^{n-k-i} \left(a_ib_k - \begin{pmatrix} k \\ 1 \end{pmatrix} a_{i+1}b_{k-1} + \cdots + (-1)^{k-1} \begin{pmatrix} k \\ k-1 \end{pmatrix} a_{k+i-1}b_1 + (-1)^k a_{k+i}b_0\right)$$

The degree $n-k$ term of $A_{k,\, n}(q(x),\, p(x))$, that is, T_0, is divisible by a polynomial in a_i and b_j, that appears in [1, Equation 1.1].

Also it is proved in [3, Th. 3] if a is a root of a degree k polynomial $q(x)$, then $q(x)$ is $A_{k,n}-$ apolar to the degree n polynomial $(x-a)^n$, where $n \geq k$, as long as the polynomial $A_{k,n}(q(x), p(x))$ is invariant under translation. This result appears in [5, Th. 1].

We will now look to extend these results.

2. Apolar Cubics and Quadratics

We concentrate now on connections between repeated roots of apolar polynomials of degrees 2, 3, and 5. The following is an update of [3, Th. 2] that contributes to the goal of providing a more complete picture of the relationship between apolar 2nd and 3rd degree polynomials with regard to repeated roots.

Theorem 1 *Let* $q(x) = b_0x^2 + 2b_1x + b_2$ *and* $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ *represent a general quadratic and cubic polynomial, respectively.*

(a) *If* $q(x)$ *has a root* $x = r$ *of multiplicity 2, then any* $p(x)$ *which is* $A_{2,\,3}-$ *apolar to* $q(x)$ *has a root* $x = r$ *of multiplicity at least 2.*

(b) *If* $p(x)$ *has a root* $x = r$ *of multiplicity 2, then any* $q(x)$ *which is* $A_{2,\,3}-$ *apolar to* $p(x)$ *has a root* $x = r$ *of multiplicity 2.*

(c) *If* $p(x)$ *has a root* $x = r$ *of multiplicity 3, then any* $q(x)$ *that is* $A_{2,\,3}-$ *apolar to* $p(x)$ *has a root* $x = r$*. And there exists a particular* $q(x)$ *that is* $A_{2,\,3}-$ *apolar to* $p(x)$ *with a root of multiplicity* 2 *at* $x = r$*.*

Proof: (a) Assume first that $q(x) = b_0x^2 + 2b_1x + b_2$ has a root $x = r$ of multiplicity 2. It follows that $q(x) = b_0\,(x-r)^2 = b_0x^2 - 2b_0rx + b_0r^2$. Then $b_1 = -b_0r$ and $b_2 = b_0r^2$.

Solve the equation $A_{2,\,3}(q(x), p(x)) = (a_0b_2 - 2a_1b_1 + b_0a_2)\,x - (a_1b_2 - 2a_2b_1 + b_0a_3) = 0$ for all x for a cubic $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ through use of the matrix $\begin{bmatrix} b_2 & -2b_1 & b_0 & 0 & 0 \\ 0 & b_2 & -2b_1 & b_0 & 0 \end{bmatrix}$ in which the columns, left-to-right, display the coefficients for $a_0,\ a_1,\ a_2,\ a_3,$ respectively.

This yields for $q(x) = b_0x^2 - 2b_0rx + b_0r^2$ the augmented matrix $\begin{bmatrix} b_0r^2 & 2b_0r & b_0 & 0 & 0 \\ 0 & b_0r^2 & 2b_0r & b_0 & 0 \end{bmatrix}$.

Since we assume $b_0 \neq 0$, it follows that this system is equivalent to one with augmented matrix $\begin{bmatrix} r^2 & 2r & 1 & 0 & 0 \\ 0 & r^2 & 2r & 1 & 0 \end{bmatrix}$

<u>Case 1</u>: $r = 0$

Then $\begin{bmatrix} r^2 & 2r & 1 & 0 & 0 \\ 0 & r^2 & 2r & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ which implies that $a_2 = 0 = a_3$.

Therefore $p(x) = a_0x^3 + 3a_1x^2 = x^2\,(a_0x + 3a_1)$, which means that p has root $r = 0$ of multiplicity at least 2.

<u>Case 2</u>: $r \neq 0$

Then $\begin{bmatrix} r^2 & 2r & 1 & 0 & 0 \\ 0 & r^2 & 2r & 1 & 0 \end{bmatrix}$, row reduces to $\begin{bmatrix} 1 & 0 & -\frac{3}{r^2} & -\frac{2}{r^3} & 0 \\ 0 & 1 & \frac{2}{r} & \frac{1}{r^2} & 0 \end{bmatrix}$.

This implies that $a_0 = a_2\frac{3}{r^2} + a_3\frac{2}{r^3} = \frac{3ra_2+2a_3}{r^3}$ and $a_1 = -a_2\frac{2}{r} - a_3\frac{1}{r^2} = \frac{-2ra_2-a_3}{r^2}$.

Therefore $p(x)$ must have the form

$$p(x) = \left(\tfrac{3ra_2+2a_3}{r^3}\right)x^3 + 3\left(\tfrac{-2ra_2-a_3}{r^2}\right)x^2 + 3a_2x + a_3 \ .$$

Since $\left(\frac{(3a_2r+2a_3)x+ra_3}{r^3}\right)(x-r)^2 = \left(\frac{3ra_2+2a_3}{r^3}\right)x^3 + 3\left(\frac{-2ra_2-a_3}{r^2}\right)x^2 + 3a_2x + a_3$

then $p(x) = \left(\frac{(3a_2r+2a_3)x+ra_3}{r^3}\right)(x-r)^2$, so $p(x)$ has a root r of multiplicity at least 2.

Therefore, in either case, p has a root at $x = r$ of multiplicity at least 2.

(b) If $p(x)$ has a root $x = r$ of multiplicity 2 or more, then

$p(x) = a_0\,(x-r)^2\,(x-a) = a_0x^3 + (-2a_0r - a_0a)\,x^2 + \left(a_0r^2 + 2a_0ra\right)x - a_0r^2a,$ where a is a complex root of $p(x)$.

So then $a_3 = -a_0 r^2 a$, $a_2 = \frac{a_0\left(r^2+2ra\right)}{3}$, and $a_1 = -\frac{a_0(2r+a)}{3}$.

To solve the equation $A_{2,\,3}(q(x), p(x)) = (a_0 b_2 - 2a_1 b_1 + b_0 a_2)\, x - (a_1 b_2 - 2a_2 b_1 + b_0 a_3) = 0$ for all x for the quadratic $q(x) = b_0 x^2 + 2b_1 x + b_2$, it is sufficient to solve the system with augmented matrix $\left[\begin{array}{cccc} a_0 & -2a_1 & a_2 & 0 \\ a_1 & -2a_2 & a_3 & 0 \end{array}\right]$ in which the first 3 columns, left-to-right, display the coefficients for b_2, b_1, and b_0, respectively.

This yields for $p(x) = a_0 x^3 + (-2a_0 r - a_0 a)\, x^2 + \left(a_0 r^2 + 2a_0 ra\right) x - a_0 r^2 a$, the system with augmented matrix

$$\left[\begin{array}{cccc} a_0 & -2a_1 & a_2 & 0 \\ a_1 & -2a_2 & a_3 & 0 \end{array}\right] = \left[\begin{array}{cccc} a_0 & \frac{2a_0(2r+a)}{3} & \frac{a_0\left(r^2+2ra\right)}{3} & 0 \\ -\frac{a_0(2r+a)}{3} & \frac{-2a_0\left(r^2+2ra\right)}{3} & -a_0 r^2 a & 0 \end{array}\right].$$

Since the leading coefficient of p is $a_0 \neq 0$, this last matrix is equivalent to

$$\left[\begin{array}{cccc} 1 & \frac{2(2r+a)}{3} & \frac{\left(r^2+2ra\right)}{3} & 0 \\ -\frac{(2r+a)}{3} & \frac{-2\left(r^2+2ra\right)}{3} & -r^2 a & 0 \end{array}\right],$$

which can be row-reduced to $\left[\begin{array}{cccc} 3 & 2\,(2r+a) & r^2+2ra & 0 \\ 0 & (a-r)^2 & r(a-r)^2 & 0 \end{array}\right]$ ($\sharp$).

If $x = r$ is a multiplicity 2 root of $p(x)$, then $a \neq r$, so this last matrix ($\sharp$) reduces to $\left[\begin{array}{cccc} 1 & 0 & -r^2 & 0 \\ 0 & 1 & r & 0 \end{array}\right]$, which means that $b_2 = b_0 r^2$ and $b_1 = -b_0 r$. Therefore $q(x) = b_0 x^2 - 2b_0 rx + b_0 r^2$.

Since $b_0\,(x-r)^2 = b_0 x^2 - 2b_0 xr + b_0 r^2$, then $q(x) = b_0\,(x-r)^2$, and $q(x)$ has a multiplicity 2 root at $x = r$ also.

(c) If $p(x)$ has a multiplicity 3 root at r, then using the same notation as in the proof of Part (b), $a = r$, so the matrix ($\sharp$) $\left[\begin{array}{cccc} 3 & 2\,(2r+a) & r^2+2ra & 0 \\ 0 & (a-r)^2 & r(a-r)^2 & 0 \end{array}\right]$ becomes $\left[\begin{array}{cccc} 3 & 6r & 3r^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is equivalent to $\left[\begin{array}{cccc} 1 & 2r & r^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Therefore $b_2 = -2rb_1 - r^2 b_0$, so that $q(x) = b_0 x^2 + 2b_1 x - 2rb_1 - r^2 b_0$.

Using division, $q(x) = b_0(x-r)(x + 2\frac{b_1}{b_0} + r)$, so $q(x)$ has a root at $x = r$. In this last expression for $q(x)$, choosing $b_1 = -b_0 r$ provides that $q(x) = b_0(x-r)(x + 2\frac{b_1}{b_0} + r) = b_0(x-r)(x + 2\frac{(-b_0 r)}{b_0} + r) = b_0\,(x-r)^2$. Therefore there exists a $q(x)$ $A_{2,\,3}-$ apolar to the given $p(x)$ with a root at $x = r$ of multiplicity 2. ■

Corollary 1 *Let* $p(x) = a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3$ *be a given cubic polynomial. If* $q(x) = b_0 x^2 + 2b_1 x + b_2$ *is* $A_{2,\,3}-$ *apolar to* $p(x)$ *and* $q(x)$ *has 2 distinct roots, then either* $p(x)$ *has* 3 *distinct roots all different from those of* $q(x)$, *or* $p(x)$ *has* 1 *triple root that* q *and* p *share.*

Proof: In [3], in the paragraphs immediately preceding and following

[3, Cor. 1], it is argued that given a cubic $p(x)$ and a quadratic $q(x)$ derived as $A_{2,\,3}-$ apolar to the given p, that if q has 2 distinct roots r_1 and r_2, then the set of cubics that are apolar to $q(x)$ (and which contains $p(x)$) has the form

$$\left\{ c_1\,(x-r_1)^3 + c_2\,(x-r_2)^3 \mid c_1,\, c_2 \in \mathbf{C} \right\},$$

where $\mathbf{C}$ is understood to be the set of complex numbers.

There are 2 cases: Either p shares 1 or more roots with q, or not.

<u>Case 1</u>: p and q share a root. Without loss of generality, assume that $p(r_1) = 0$. Since $p(x) = c_1\,(x-r_1)^3 + c_2\,(x-r_2)^3$ for some $c_1,\, c_2 \in \mathbf{C}$ not both zero (since p is of degree 3), it follows that $p(r_1) = c_2\,(r_1 - r_2)^3 = 0$. Since $r_1 \neq r_2$, this can only be true if $c_2 = 0$ in which case it follows that $p(x) = c_1\,(x-r_1)^3$. This means that p has one root of multiplicity 3 that p and q share. Therefore, if p and q share a root, they share only 1 root, which is a root of multiplicity 3 of $p(x)$.

Case 2: On the other hand, if p and q have no roots in common, then Th. 1 would preclude $p(x)$ from having a root of multiplicity 2 or 3. If p has a root of multiplicity 2, then Th. 1(b) implies that $q(x)$ would have the same root with multiplicity 2, which contradicts the assumption that $q(x)$ has distinct roots and that p and q have no common roots. Likewise, if p has a root of multiplicity 3, Th. 1(c) implies that q would share this root, which contradicts our assumption. Therefore p must have 3 distinct roots all different from those of $q(x)$. ■

3. Apolar Quintics and Cubics

Before applying apolar methods to quintics, it is worth mentioning certain classes of quintics that will be left out of our discussion either because their roots may be found easily, or they are not $A_{3,\,5}-$ polar to any cubic polynomial.

First, any quintic with only 1 or 2 terms (including the 5th-degree term) such as x^5-3, $2x^5+5x$, $-3x^5+20x^2$, x^5-140x^3, $61x^5+2x^4$, or $-15x^5$ may be solved using algebraic methods, and thus do not warrant apolar methods.

Consider now quintics with exactly 3 nonzero terms and for which coefficients of 3 consecutive powers of x are 0. These are of the form $s(x)=d_0x^5+5d_1x^4+10d_2x^3$, $s(x)=d_0x^5+5d_1x^4+d_5$ or $s(x)=d_0x^5+5d_4x+d_5$, which have 0 coefficients in the consecutive powers of x: 2, 1, 0; 3, 2, 1; and 4, 3, 2, respectively. The first, of course, may be solved easily by factoring and solving the remaining quadratic. The remaining 2 are discussed below.

Definition: A quintic polynomial of the form $s(x)=d_0x^5+5d_1x^4+d_5$ or $s(x)=d_0x^5+5d_4x+d_5$ whereby the coefficients d_0, d_1, d_4, and d_5 are all nonzero, is each termed a **trivial** $A_{3,\,5}-$ **apolar quintic**.

The previous definition and the next theorem concern quintics that are $A_{3,\,5}-$ polar to polynomials of degree less than 3 for which 0 is the only root.

Theorem 2 *Trivial $A_{3,\,5}-$ apolar quintics are $A_{3,\,5}-$ apolar only to polynomials of the form $p(x)=3a_2x$ or $p(x)=3a_1x^2$.*

Proof: To solve for a cubic $p(x)=a_0x^3+3a_1x^2+3a_2x+a_3$ that is $A_{3,\,5}-$ apolar to a given quintic, having the form $s(x)=d_0x^5+5d_1x^4+10d_2x^3+10d_3x^2+5d_4x+d_5$, involves solving a system of equations represented by the matrix

$$\begin{bmatrix} d_0 & -3d_1 & 3d_2 & -d_3 & 0 \\ d_1 & -3d_2 & 3d_3 & -d_4 & 0 \\ d_2 & -3d_3 & 3d_4 & -d_5 & 0 \end{bmatrix}$$

in which the first 4 columns represent the coefficients of the variables a_3, a_2, a_1, and a_0 respectively.

Case 1: $s(x)=d_0x^5+5d_1x^4+d_5$. In this case $d_2=d_3=d_4=0$, and by definition, d_0, d_1, and d_5 are all nonzero. In this case, the above matrix becomes

$$\begin{bmatrix} d_0 & -3d_1 & 0 & 0 & 0 \\ d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_5 & 0 \end{bmatrix}$$

from which it can be seen that $a_3d_0-3a_2d_1=0$, $a_3d_1=0$, and $-a_0d_5=0$. With the assumption that $d_1\neq 0$, the 2nd equation implies that $a_3=0$. This means the 1st equation along with $d_1\neq 0$ implies that $a_2=0$. The 3rd equation along with the assumption that $d_5\neq 0$ yields that $a_0=0$. Therefore $p(x)=3a_1x^2$.

Case 2: $s(x)=d_0x^5+5d_4x+d_5$. In this case $d_1=d_2=d_3=0$, and by definition, d_0, d_4, and d_5 are all nonzero. The above matrix

$$\begin{bmatrix} d_0 & -3d_1 & 3d_2 & -d_3 & 0 \\ d_1 & -3d_2 & 3d_3 & -d_4 & 0 \\ d_2 & -3d_3 & 3d_4 & -d_5 & 0 \end{bmatrix} \text{ becomes } \begin{bmatrix} d_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_4 & 0 \\ 0 & 0 & 3d_4 & -d_5 & 0 \end{bmatrix}.$$

The 1st row of this matrix implies that $a_3d_0=0$, and with $d_0\neq 0$ implies that $a_3=0$. Next the 2nd row implies that $-a_0d_4=0$, which means that $a_0=0$ since $d_4\neq 0$. These conclusions along with the equation $3a_1d_4-a_0d_5=0$ implies that $a_1=0$. Therefore $p(x)\ =3a_2x$.

In either case, these trivial $A_{3,\,5}-$ apolar quintic polynomials are apolar only to polynomials of the form $p(x) = 3a_2x$ or $p(x) = 3a_1x^2$, which only have 0 as a root. ■

Corollary 2 *If a quintic polynomial $s(x)$ is $A_{3,\,5}-$ apolar to a cubic polynomial $p(x)$, then $s(x)$ cannot be of either form $d_0x^5 + 5d_1x^4 + d_5$ or $d_0x^5 + 5d_4x + d_5$.*

Proof: This is a direct result of Theorem 2. ■

Lemma 3 *Let $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$ be a quintic polynomial and $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$, a cubic polynomial that is apolar to $s(x)$. Then $s(x)$ has a root $x = t$ of multiplicity at least 3 if and only if t is a root of $p(x)$ of multiplicity 3 .*

Proof: We know that a quintic polynomial $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$ and cubic $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ are apolar if

$$A_{3,5}\left(p\left(x\right), s\left(x\right)\right) = 0, \text{ where}$$

$$\begin{aligned} A_{3,5}\left(p\left(x\right), s\left(x\right)\right) = {} & x^2\left(d_0a_3 - 3d_1a_2 + 3d_2a_1 - d_3a_0\right) \\ & -2x\left(d_1a_3 - 3d_2a_2 + 3d_3a_1 - d_4a_0\right) \\ & +d_2a_3 - 3d_3a_2 + 3d_4a_1 - d_5a_0 \end{aligned}$$

So given $s(x)$ we may solve for an $A_{3,5}-$ apolar $p(x)$ by solving the linear system

$$\begin{aligned} d_0a_3 - 3d_1a_2 + 3d_2a_1 - d_3a_0 &= 0 \\ d_1a_3 - 3d_2a_2 + 3d_3a_1 - d_4a_0 &= 0 \\ d_2a_3 - 3d_3a_2 + 3d_4a_1 - d_5a_0 &= 0 \end{aligned}$$

for the coefficients a_3, a_2, a_1, and a_0, of $p(x)$, which can be found by row-reducing the matrix

$$\begin{bmatrix} d_0 & -3d_1 & 3d_2 & -d_3 & 0 \\ d_1 & -3d_2 & 3d_3 & -d_4 & 0 \\ d_2 & -3d_3 & 3d_4 & -d_5 & 0 \end{bmatrix}$$

If $s(x)$ has a repeated root t of multiplicity at least 3, then

$$\begin{aligned} s(x) &= d_0\left(x-t\right)^3\left(x-u\right)\left(x-v\right) \\ &= d_0x^5 + \left(-3td_0 - ud_0 - vd_0\right)x^4 + \left(3t^2d_0 + 3tud_0 + 3tvd_0 + uvd_0\right)x^3 \\ &\quad + \left(-t^3d_0 - 3t^2ud_0 - 3t^2vd_0 - 3tuvd_0\right)x^2 \\ &\quad + \left(t^3ud_0 + t^3vd_0 + 3t^2uvd_0\right)x - t^3uvd_0 \end{aligned}$$

which means that $d_1 = \left(-3td_0 - ud_0 - vd_0\right)/5$,

$d_2 = \left(3t^2d_0 + 3tud_0 + 3tvd_0 + uvd_0\right)/10$,

$d_3 = \left(-t^3d_0 - 3t^2ud_0 - 3t^2vd_0 - 3tuvd_0\right)/10$,

$d_4 = \left(t^3ud_0 + t^3vd_0 + 3t^2uvd_0\right)/5$, and $d_5 = -t^3uvd_0$.

To solve for the a_3, a_2, a_1, a_0 that define the cubic $p(x)$ apolar to this $s(x)$ we row-reduce the previous augmented matrix that now becomes with these substitutions

$$\left[\begin{matrix} d_0 & \frac{3}{5}d_0\left(3t+u+v\right) & \cdots \\ -\frac{1}{5}d_0\left(3t+u+v\right) & -\frac{9}{10}t^2d_0 - \frac{9}{10}tud_0 - \frac{9}{10}tvd_0 - \frac{3}{10}uvd_0 & \\ \frac{3}{10}t^2d_0 + \frac{3}{10}tud_0 + \frac{3}{10}tvd_0 + \frac{1}{10}uvd_0 & \frac{3}{10}td_0\left(3tu + 3tv + 3uv + t^2\right) & \end{matrix}\right.$$

$$\cdots \left. \begin{matrix} \frac{9}{10}t^2d_0+\frac{9}{10}tud_0+\frac{9}{10}tvd_0+\frac{3}{10}uvd_0 & \frac{1}{10}td_0\left(3tu+3tv+3uv+t^2\right) & 0 \\ -\frac{3}{10}td_0\left(3tu+3tv+3uv+t^2\right) & -\frac{1}{5}t^2d_0\left(tu+tv+3uv\right) & 0 \\ \frac{3}{5}t^2d_0\left(tu+tv+3uv\right) & t^3uvd_0 & 0 \end{matrix} \right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & t^3 & 0 \\ 0 & 1 & 0 & -t^2 & 0 \\ 0 & 0 & 1 & t & 0 \end{bmatrix}.$$

Then $a_3=-t^3a_0$, $a_2=t^2a_0$, and $a_1=-ta_0$, so that $p(x)=a_0x^3-3ta_0x^2+3t^2a_0x-t^3a_0=a_0\left(x-t\right)^3$. Therefore $p(x)$ has a root of multiplicity 3 at $x=t$ also.

On the other hand, if $p(x)$ has a root t of multiplicity 3, then $p(x)=a_0\left(x-t\right)^3=a_0x^3-3ta_0x^2+3t^2a_0x-t^3a_0$, so that $a_1=-ta_0$, $a_2=t^2a_0$, and $a_3=-t^3a_0$. To find the vector space of quintics $s(x)=d_0x^5+5d_1x^4+10d_2x^3+10d_3x^2+5d_4x+d_5$ apolar to $p(x)$ requires the solving of the linear system used in the first part of this proof for the d_i rather than the a_i, which was the essence of the proof of the "forward argument" of the biconditional statement in question. The system of equations

$$\begin{aligned} d_0a_3-3d_1a_2+3d_2a_1-d_3a_0 &= 0 \\ d_1a_3-3d_2a_2+3d_3a_1-d_4a_0 &= 0 \\ d_2a_3-3d_3a_2+3d_4a_1-d_5a_0 &= 0 \end{aligned}$$

which can be rewritten

$$\clubsuit \quad \begin{array}{cccccccl} d_0a_3 & -3d_1a_2 & +3d_2a_1 & -d_3a_0 & & & & =0 \\ & d_1a_3 & -3d_2a_2 & +3d_3a_1 & -d_4a_0 & & & =0 \\ & & d_2a_3 & -3d_3a_2 & +3d_4a_1 & -d_5a_0 & & =0 \end{array}$$

can be represented by the 3×7 matrix $\begin{bmatrix} a_3 & -3a_2 & 3a_1 & -a_0 & 0 & 0 & 0 \\ 0 & a_3 & -3a_2 & 3a_1 & -a_0 & 0 & 0 \\ 0 & 0 & a_3 & -3a_2 & 3a_1 & -a_0 & 0 \end{bmatrix}$.

With $a_1=-ta_0$, $a_2=t^2a_0$, and $a_3=-t^3a_0$, this becomes

$$\begin{bmatrix} -t^3a_0 & -3t^2a_0 & -3ta_0 & -a_0 & 0 & 0 & 0 \\ 0 & -t^3a_0 & -3t^2a_0 & -3ta_0 & -a_0 & 0 & 0 \\ 0 & 0 & -t^3a_0 & -3t^2a_0 & -3ta_0 & -a_0 & 0 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} t^3 & 3t^2 & 3t & 1 & 0 & 0 & 0 \\ 0 & t^3 & 3t^2 & 3t & 1 & 0 & 0 \\ 0 & 0 & t^3 & 3t^2 & 3t & 1 & 0 \end{bmatrix}$$

since it's assumed that $a_0\neq 0$ as p is degree 3.

We consider two cases.

<u>Case 1</u>: $t=0$ Then we have

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This implies that $d_3=d_4=d_5=0$.

Therefore, $s(x)=d_0x^5+5d_1x^4+10d_2x^3=\left(d_0x^2+5d_1x+10d_2\right)x^3$, which implies that $s(x)$ has the root $t=0$ with multiplicity at least 3.

<u>Case 2</u>: $t\neq 0$ Then

$$\begin{bmatrix} t^3 & 3t^2 & 3t & 1 & 0 & 0 & 0 \\ 0 & t^3 & 3t^2 & 3t & 1 & 0 & 0 \\ 0 & 0 & t^3 & 3t^2 & 3t & 1 & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{10}{t^3} & \frac{15}{t^4} & \frac{6}{t^5} & 0 \\ 0 & 1 & 0 & -\frac{6}{t^2} & -\frac{8}{t^3} & -\frac{3}{t^4} & 0 \\ 0 & 0 & 1 & \frac{3}{t} & \frac{3}{t^2} & \frac{1}{t^3} & 0 \end{bmatrix}$$

Therefore $d_0 = -\frac{10}{t^3}d_3 - \frac{15}{t^4}d_4 - \frac{6}{t^5}d_5$, $d_1 = \frac{6}{t^2}d_3 + \frac{8}{t^3}d_4 + \frac{3}{t^4}d_5$, and $d_2 = -\frac{3}{t}d_3 - \frac{3}{t^2}d_4 - \frac{1}{t^3}d_5$.

This gives for $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$, which is apolar to $p(x) = a_0(x-t)^3$, that

$$\begin{aligned} s(x) &= \left(-\frac{10}{t^3}d_3 - \frac{15}{t^4}d_4 - \frac{6}{t^5}d_5\right)x^5 + 5\left(\frac{6}{t^2}d_3 + \frac{8}{t^3}d_4 + \frac{3}{t^4}d_5\right)x^4 \\ &\quad +10\left(-\frac{3}{t}d_3 - \frac{3}{t^2}d_4 - \frac{1}{t^3}d_5\right)x^3 + 10d_3x^2 + 5d_4x + d_5 \end{aligned}$$

By repeated division by $(x-t)$, it can be shown that

$$s(x) = (x-t)^3\left[\left(-\frac{10}{t^3}d_3 - \frac{15}{t^4}d_4 - \frac{6}{t^5}d_5\right)x^2 + -\frac{1}{t^4}(3d_5 + 5td_4)x - \frac{1}{t^3}d_5\right]$$

and that t is a root of $s(x)$ of multiplicity at least 3 . ■

The following example shows how previous results may be combined to find the roots of a given quintic.

Example: Given the quintic polynomial

$$s(x) = x^5 + (-14-9i)x^4 + (44+108i)x^3 + (100-396i)x^2 + (-533+468i)x + 506 - 99i$$

using the binomial notation $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$ we identify $d_0 = 1$, $d_1 = (-14-9i)/5 = -\frac{14}{5} - \frac{9}{5}i$, $d_2 = (44+108i)/10 = \frac{22}{5} + \frac{54}{5}i$, $d_3 = (100-396i)/10 = 10 - \frac{198}{5}i$, $d_4 = (-533+468i)/5 = -\frac{533}{5} + \frac{468}{5}i$, and $d_5 = 506 - 99i$.

An apolar cubic $p(x)$ may be found by row-reducing the matrix

$$\begin{bmatrix} d_0 & -3d_1 & 3d_2 & -d_3 & 0 \\ d_1 & -3d_2 & 3d_3 & -d_4 & 0 \\ d_2 & -3d_3 & 3d_4 & -d_5 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3\left(\frac{14}{5} + \frac{9}{5}i\right) & 3\left(\frac{22}{5} + \frac{54}{5}i\right) & -\left(10 - \frac{198}{5}i\right) & 0 \\ -\frac{14}{5} - \frac{9}{5}i & -3\left(\frac{22}{5} + \frac{54}{5}i\right) & 3\left(10 - \frac{198}{5}i\right) & \left(\frac{533}{5} - \frac{468}{5}i\right) & 0 \\ \frac{22}{5} + \frac{54}{5}i & -3\left(10 - \frac{198}{5}i\right) & 3\left(-\frac{533}{5} + \frac{468}{5}i\right) & -(506 - 99i) & 0 \end{bmatrix}.$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -46+9i & 0 \\ 0 & 1 & 0 & 5-12i & 0 \\ 0 & 0 & 1 & 2+3i & 0 \end{bmatrix}$$

So $a_3 = (46-9i)a_0$, $a_2 = (-5+12i)a_0$, and $a_1 = -(2+3i)a_0$, and

$$\begin{aligned} p(x) &= a_0x^3 + 3a_1x^2 + 3a_2x + a_3 \\ &= a_0x^3 - 3((2+3i)a_0)x^2 + 3(-5+12i)a_0x + (46-9i)a_0 \end{aligned}$$

This $p(x)$ represents a family of cubics which is the null space of the original 3×4 matrix $\begin{bmatrix} d_0 & -3d_1 & 3d_2 & -d_3 \\ d_1 & -3d_2 & 3d_3 & -d_4 \\ d_2 & -3d_3 & 3d_4 & -d_5 \end{bmatrix}$.

Next we solve for the family of quadratics $q(x) = b_0x^2 + 2b_1x + b_2$ that are $A_{2,3}-$ apolar to this $p(x)$ by solving the system represented by the matrix

$$\begin{bmatrix} a_0 & -2a_1 & a_2 & 0 \\ a_1 & -2a_2 & a_3 & 0 \end{bmatrix}$$

in which the columns, left-to-right, display the coefficients for b_2, b_1, and b_0, respectively.

With $a_1 = -(2+3i)\,a_0$, $a_2 = (-5+12i)\,a_0$, and $a_3 = (46-9i)\,a_0$,

$$\begin{bmatrix} a_0 & -2a_1 & a_2 & 0 \\ a_1 & -2a_2 & a_3 & 0 \end{bmatrix} = \begin{bmatrix} a_0 & 2\,(2+3i)\,a_0 & (-5+12i)\,a_0 & 0 \\ -(2+3i)\,a_0 & 2\,(5-12i)\,a_0 & (46-9i)\,a_0 & 0 \end{bmatrix}.$$

This is equivalent to

$$\begin{bmatrix} 1 & 2\,(2+3i) & -5+12i & 0 \\ -(2+3i) & 2\,(5-12i) & 46-9i & 0 \end{bmatrix}$$

which row reduces to $\begin{bmatrix} 1 & 4+6i & -5+12i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which implies that $b_2 = -(4+6i)\,b_1 + (5-12i)\,b_0$.

Therefore $q(x) = b_0x^2 + 2b_1x + b_2 = b_0x^2 + 2b_1x - (4+6i)\,b_1 + (5-12i)\,b_0$.

Solving the quadratic equation $q(x) = b_0x^2 + 2b_1x - (4+6i)\,b_1 + (5-12i)\,b_0 = 0$ leads to $x = 2+3i$ or $x = -\frac{2b_1}{b_0} - (2+3i)$

Therefore $q(x)$ has the root $x = 2+3i$. And for certain choices of b_0 and b_1 this root is of multiplicity 2. Choosing $b_1 = -b_0\,(2+3i)$ results in $x = -\frac{2b_1}{b_0} - (2+3i) = -\frac{2(-b_0(2+3i))}{b_0} - (2+3i) = 2+3i$.

Therefore $2+3i$ is a root of $q(x)$ of multiplicity 2. Theorem 1 implies that $p(x)$ also has a root at $x = 2+3i$ of multiplicity at least 2. Since $p(x) = a_0 f(x)$ with $f(x) = x^3 - 3\,(2+3i)\,x^2 + 3\,(-5+12i)\,x + (46-9i)$ (synthetic) division results in an $f(x) = (x-(2+3i))^3$ in this case.

Therefore $p(x) = a_0(x-(2+3i))^3$. Lemma 3 implies that $s(x)$ must have a root of multiplicity at least 3 at $x = 2+3i$ as well. In evidence, synthetic division leads to

$$\begin{aligned} s(x) &= x^5 + (-14-9i)\,x^4 + (44+108i)\,x^3 + (100-396i)\,x^2 \\ &\quad + (-533+468i)\,x + 506 - 99i \\ &= (x-(2+3i))^3\left(x^2 - 8x + 11\right) \end{aligned}$$

From this all roots of $s(x)$ may be found as $x^2 - 8x + 11 = 0 \Rightarrow x = 4 \pm \sqrt{5}$.

The Method of $A_{3,\,5}-$Apolar Polynomials with Distinct Roots of the Cubic

The following theorem builds upon concepts discussed and shown true at least partially in [3], but here a more complete result is obtained. The first part of the conclusion, that $s(x) = k_1\,(x-a)^5 + k_2\,(x-b)^5 + k_3\,(x-c)^5$ where a, b, and c are roots of an apolar cubic polynomial was mentioned by Rota in [5].

Theorem 3 *If* $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ *and* $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$ *are polynomials that are* $A_{3,\,5}-$ *apolar, and* $p(x)$ *has 3 distinct roots denoted by* a, b, *and* c, *then*

$$s(x) = k_1\,(x-a)^5 + k_2\,(x-b)^5 + k_3\,(x-c)^5$$

in which k_1, k_2, k_3 *are constants defined as follows:*

$$\begin{aligned} k_1 &= \frac{bd_0c + bd_1 + d_2 + cd_1}{(c-a)(b-a)} \\ k_2 &= \frac{cd_1 + cad_0 + d_2 + ad_1}{(b-c)(b-a)} \\ k_3 &= \frac{bd_1 + bad_0 + d_2 + ad_1}{(b-c)(a-c)} \end{aligned}$$

Proof: In [3, Th. 3], it is proved that if $q(x)$ is a degree k polynomial and r is a root of $q(x)$, then for any integer $n \geq k$, if $A_{k,\,n}$ is an apolar invariant, then $q(x)$ is $A_{k,\,n}-$ apolar to $(x-r)^n$. This fact is also mentioned in [5]. Since the $A_{3,\,5}-$ apolar polynomial is proved to be an invariant in [3, Sec. 5] we have that the cubic $p(x)$ is $A_{3,\,5}-$ apolar to each of the polynomials $(x-a)^5$, $(x-b)^5$, and $(x-c)^5$. The set of degree 5 polynomials, $S_{p(x)}$, defined as the set of quintics that are $A_{3,\,5}-$ apolar to $p(x)$ is found as the null space of a system of 3 homogeneous linear equations in 6 unknowns. This system is given in Section 3 (in the proof of Lemma 3) and is denoted by (♣). One can confirm that the rank of this 3×6 coefficient matrix will always be 3. This implies that the null space of this linear system, $S_{p(x)}$, is a vector space of dimension at most 3. Following Rota's argument in [5], under the assumption that $p(x)$ has 3 distinct roots a, b, and c, then $(x-a)^5$, $(x-b)^5$, and $(x-c)^5$ are linearly independent elements of $S_{p(x)}$ so these three degree 5 polynomials span $S_{p(x)}$ hence form a basis for $S_{p(x)}$. Therefore,

$$S_{p(x)} = \left\{ k_1 (x-a)^5 + k_2 (x-b)^5 + k_3 (x-c)^5 \mid k_1,\ k_2,\ k_3 \in \mathbf{C} \right\} .$$

Since it is assumed that $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$ is $A_{3,\,5}-$ apolar to $p(x)$, then $s(x) \in S_{p(x)}$ and so

$$s(x) = k_1 (x-a)^5 + k_2 (x-b)^5 + k_3 (x-c)^5$$

for some constants k_1, k_2, k_3.

Since

$$\begin{aligned} s(x) &= k_1 (x-a)^5 + k_2 (x-b)^5 + k_3 (x-c)^5 \\ &= (k_1 + k_2 + k_3) x^5 \\ &\quad + (-5k_1a - 5k_2b - 5k_3c) x^4 \\ &\quad + \left(10k_1a^2 + 10k_2b^2 + 10k_3c^2\right) x^3 \\ &\quad + \left(-10k_1a^3 - 10k_2b^3 - 10k_3c^3\right) x^2 \\ &\quad + \left(5k_1a^4 + 5k_2b^4 + 5k_3c^4\right) x \\ &\quad - k_1a^5 - k_2b^5 - k_3c^5 \end{aligned}$$

k_1, k_2, k_3 can be found by solving the system of equations

$$\begin{array}{lcl} k_1 + k_2 + k_3 & = & d_0 \\ -5k_1a - 5k_2b - 5k_3c & = & 5d_1 \\ 10k_1a^2 + 10k_2b^2 + 10k_3c^2 & = & 10d_2 \end{array} \Rightarrow \begin{array}{lcl} k_1 + k_2 + k_3 & = & d_0 \\ k_1a + k_2b + k_3c & = & -d_1 \\ k_1a^2 + k_2b^2 + k_3c^2 & = & d_2 \end{array}$$

which is represented by the augmented matrix

$$\left[\begin{array}{cccc} 1 & 1 & 1 & d_0 \\ a & b & c & -d_1 \\ a^2 & b^2 & c^2 & d_2 \end{array}\right] \text{ that reduces to } \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{bd_0c+bd_1+d_2+cd_1}{(c-a)(b-a)} \\ 0 & 1 & 0 & \frac{cd_1+cad_0+d_2+ad_1}{(b-c)(b-a)} \\ 0 & 0 & 1 & \frac{bd_1+bad_0+d_2+ad_1}{(b-c)(a-c)} \end{array}\right]$$

Therefore $k_1 = \frac{bd_0c+bd_1+d_2+cd_1}{(c-a)(b-a)}$, $k_2 = \frac{cd_1+cad_0+d_2+ad_1}{(b-c)(b-a)}$, and $k_3 = \frac{bd_1+bad_0+d_2+ad_1}{(b-c)(a-c)}$. ■

The following corollary summarizes and synthesizes results of this paper.

Corollary 3 *Let $s(x)$ be a given quintic polynomial and assume that $p(x)$ is a cubic polynomial found to be $A_{3,\,5}-$ apolar to $s(x)$. In turn, assume that $q(x)$ is a quadratic polynomial found to be $A_{2,\,3}-$ apolar to this $p(x)$. Then one of the following 4 possibilities is true.*

(a) $q(x)$ has 2 distinct roots. One of the following is also true:

(i) $p(x)$ has 3 distinct roots with none in common with $q(x)$, and $s(x)$ can be written in the form given in the conclusion of Th. 3. Also $s(x)$ has 3-5 roots each of multiplicity 2 or less.

(ii) $p(x)$ has 1 triple root and shares this root with $q(x)$. $s(x)$ will have this same root with multiplicity 3 or more, and the roots of $s(x)$ can be solved for completely.

(b) $q(x)$ has 1 double root. One of the following is also true:

(i) $p(x)$ has 1 triple root and shares this root with $q(x)$. $s(x)$ will have this same root with multiplicity 3 or more, and the roots of $s(x)$ can be solved for completely.

(ii) $p(x)$ shares a root with $q(x)$ of multiplicity 2, and $s(x)$ will have 3-5 roots each of multiplicity 2 or less.

Proof: Clearly $q(x)$ either has 2 distinct roots or 1 repeated root, so we have either cases (a) or (b).

(a) Assume that $q(x)$ has 2 distinct roots. Cor. 1 implies that either (i) $p(x)$ has 3 distinct roots all different from $q(x)$, or (ii) $p(x)$ has 1 triple root that is shared by q. Case a(i): If $p(x)$ has 3 distinct roots, then Th. 3 implies that $s(x)$ may be written in the form of the conclusion of Th. 3. Lemma 3 implies that $s(x)$ will not have a root of multiplicity 3 or more. Therefore $s(x)$ must have 3-5 roots each of multiplicity 2 or less. Case a(ii): If $p(x)$ has 1 triple root shared with $q(x)$, then Lemma 3 implies that $s(x)$ also has this root with multiplicity at least 3, and the roots of $s(x)$ may be solved for completely by division.

(b) If we assume that $q(x)$ has only 1 double root, then Th. 1 implies that $p(x)$ shares this root with multiplicity 2 or 3. Case b(i): If $p(x)$ has 1 root of multiplicity 3, then the conclusion is the same as Case a(ii): $s(x)$ also has this root with multiplicity at least 3, and the roots of $s(x)$ may be solved for completely by division. Case b(ii): If $p(x)$ shares a root of $q(x)$ but with multiplicity 2, then by Lemma 3, $s(x)$ will not have a root of multiplicity 3 or more. Therefore $s(x)$ will have 3-5 roots each of multiplicity 2 or less. ■

Example: Consider $s(x) = x^5 - 3x^2 + 1$. This quintic has no rational roots by the Rational Roots Test, which allows that the only possible rational roots are $x = 1$ or -1. However since $s(1) = -1$ and $s(-1) = -3$, then $s(x)$ has no rational roots. If $s(x) = d_0x^5 + 5d_1x^4 + 10d_2x^3 + 10d_3x^2 + 5d_4x + d_5$, then $d_0 = 1$, $d_3 = -\frac{3}{10}$, $d_5 = 1$, and $d_1 = d_2 = d_4 = 0$. As before, to solve for an $A_{3,5}-$ apolar cubic $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ it is sufficient to solve the linear system with augmented matrix

$$\left[\begin{array}{ccccc} d_0 & -3d_1 & 3d_2 & -d_3 & 0 \\ d_1 & -3d_2 & 3d_3 & -d_4 & 0 \\ d_2 & -3d_3 & 3d_4 & -d_5 & 0 \end{array}\right]$$

for the coefficients of $p(x)$.

Using the particular values d_i that define $s(x) = x^5 - 3x^2 + 1$, we arrive at the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{10} & 0 \\ 0 & 0 & -\frac{9}{10} & 0 & 0 \\ 0 & \frac{9}{10} & 0 & -1 & 0 \end{bmatrix} \text{ which reduces to } \begin{bmatrix} 1 & 0 & 0 & \frac{3}{10} & 0 \\ 0 & 1 & 0 & -\frac{10}{9} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore $a_3 = -\frac{3}{10}a_0$, $a_2 = \frac{10}{9}a_0$, and $a_1 = 0$, so that $p(x) = a_0x^3 + 3\left(\frac{10}{9}a_0\right)x - \frac{3}{10}a_0 = a_0x^3 + \frac{10}{3}a_0x - \frac{3}{10}a_0$ and the space of cubics $S_{s(x)}$ $A_{3,5}-$ apolar to $s(x) = x^5 - 3x^2 + 1$ using $a_0 = 30t$ equals

$$S_{s(x)} = \left\{\left(30x^3 + 100x - 9\right)\ t \mid t \in \mathbf{C}\right\}$$

Consider $p(x) = 30x^3 + 100x - 9 = 30x^3 + 3\left(\frac{100}{3}\right)x - 9$ where for $p(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$, $a_0 = 30$, $a_1 = 0$, $a_2 = \frac{100}{3}$, and $a_3 = -9$.

To solve for an $A_{2,3}-$ apolar $q(x) = b_0x^2 + 2b_1x + b_2$, it is sufficient to solve the system of equations with augmented matrix

$$\begin{bmatrix} a_0 & -2a_1 & a_2 & 0 \\ a_1 & -2a_2 & a_3 & 0 \end{bmatrix}$$

for the coefficients of $q(x)$. Using the values $a_0 = 30$, $a_1 = 0$, $a_2 = \frac{100}{3}$, and $a_3 = -9$ produces the matrix

$$\begin{bmatrix} 30 & 0 & \frac{100}{3} & 0 \\ 0 & -\frac{200}{3} & -9 & 0 \end{bmatrix} \text{ which row-reduces to } \begin{bmatrix} 1 & 0 & \frac{10}{9} & 0 \\ 0 & 1 & \frac{27}{200} & 0 \end{bmatrix}$$

meaning that $b_2 = -\frac{10}{9}b_0$, and $b_1 = -\frac{27}{200}b_0$, so that generally $q\left(x\right) = b_0x^2 - \frac{27}{100}b_0x - \frac{10}{9}b_0$. Therefore the space of quadratic polynomials $S_{p(x)}$ that are $A_{2,3}-$ apolar to $p(x)$ is defined, with $b_0 = 900t$, to be

$$S_{p(x)} = \left\{\left(900x^2 - 243x - 1000\right)\ t \mid t \in \mathbf{C}\right\}$$

If we solve $900x^2 - 243x - 1000 = 0$, we find the solutions $r_1 = \frac{27}{200} + \frac{1}{600}\sqrt{406\,561}$, and $r_2 = \frac{27}{200} - \frac{1}{600}\sqrt{406\,561}$.

Then

$$30x^3 + 100x - 9 = c_1\left(x - r_1\right)^3 + c_2\left(x - r_2\right)^3$$

for some constants c_1 and c_2.

Using the formula found in [3, Cor. 1]:

$$p(x) = \left(\frac{a_0r_2 + a_1}{r_2 - r_1}\right)\left(x - r_1\right)^3 - \left(\frac{a_0r_1 + a_1}{r_2 - r_1}\right)\left(x - r_2\right)^3$$

with $a_0 = 30$ and $a_1 = 0$, we have that

$$p(x) = \left(\frac{30r_2}{r_2 - r_1}\right)(x - r_1)^3 - \left(\frac{30r_1}{r_2 - r_1}\right)(x - r_2)^3$$

Therefore

$$\begin{aligned} 30x^3 + 100x - 9 &= \frac{30r_2}{r_2 - r_1}(x - r_1)^3 - \frac{30r_1}{r_2 - r_1}(x - r_2)^3 \\ &= \frac{30}{r_2 - r_1}\left[r_2 (x - r_1)^3 - r_1 (x - r_2)^3\right] \end{aligned}$$

To now solve $30x^3 + 100x - 9 = 0$, one need only solve

$$\begin{aligned} r_2 (x - r_1)^3 - r_1 (x - r_2)^3 &= 0 \\ &\Rightarrow r_2 (x - r_1)^3 = r_1 (x - r_2)^3 \\ &\Rightarrow \frac{(x - r_2)^3}{(x - r_1)^3} = \frac{r_2}{r_1} \Rightarrow \left(\frac{x - r_2}{x - r_1}\right)^3 = \frac{r_2}{r_1} \end{aligned}$$

Define $y = \frac{x-r_2}{x-r_1}$ and $d = \frac{r_2}{r_1} = \frac{-413\,122+162\sqrt{406\,561}}{400\,000}$.

Next solve $y^3 = d$ using methods relating to DeMoivre's Theorem.

Since $d = \frac{-413\,122+162\sqrt{406\,561}}{400\,000}$ is a negative real number, the polar form for d is $d = |d|(\cos\pi + i\sin\pi) = |d|(\cos(\pi + 2\pi k) + i\sin(\pi + 2\pi k))$

and the solutions for $y = y_k$ are $y_k = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\cos\left(\frac{\pi+2\pi k}{3}\right) + i\sin\left(\frac{\pi+2\pi k}{3}\right)\right)$ for $k = 0, 1, 2$.

Then $y_0 = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$= \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$,

$y_1 = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}(\cos(\pi) + i\sin(\pi)) = -\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}$, and

$y_2 = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

$= \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

That is, $y_0 = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$, $y_1 = -\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}$,

and $y_2 = \left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

With $y_k = \frac{x_k-r_2}{x_k-r_1}$ for $k = 0, 1, 2$, one can solve for x_k, which yields $x_k = \frac{y_k r_1 - r_2}{y_k - 1}$.

Then

$$x_0 = \frac{\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)r_1 - r_2}{\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - 1}$$

$$= \frac{1}{600}\frac{81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+81i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}-200}$$

$$+\frac{1}{600}\frac{i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}\sqrt{406\,561}-16\,200+200\sqrt{406\,561}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}-200}$$

$$x_1=\frac{-\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}r_1-r_2}{-\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}-1}$$

$$=\frac{1}{600}\frac{81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+8100-100\sqrt{406\,561}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+100}\quad\text{and}$$

$$x_2=\frac{\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)r_1-r_2}{\left(\frac{413\,122-162\sqrt{406\,561}}{400\,000}\right)^{1/3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)-1}$$

$$=\frac{1}{600}\frac{-81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+81i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}}{-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}+200}$$

$$+\frac{1}{600}\frac{i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}\sqrt{406\,561}+16\,200-200\sqrt{406\,561}}{-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}+200}$$

That is, the roots of $30x^3+100x-9=0$ are

$$a=\frac{1}{600}\frac{81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+81i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}-200}$$

$$+\frac{1}{600}\frac{i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}\sqrt{406\,561}-16\,200+200\sqrt{406\,561}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}-200}$$

$$\approx -0.044\,891\,439\,5-1.827\,396\,798i,$$

$$b=\frac{1}{600}\frac{81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+8100-100\sqrt{406\,561}}{\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+100}$$

$$\approx 8.978\,287\,903\times 10^{-2},\quad\text{and}$$

$$c=\frac{1}{600}\frac{-81\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{406\,561}+81i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}}{-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}+200}$$

$$+\frac{1}{600}\frac{i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}\sqrt{406\,561}+16\,200-200\sqrt{406\,561}}{-\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}+i\sqrt[3]{\left(1032\,805-405\sqrt{406\,561}\right)}\sqrt{3}+200}$$

$$\approx -0.044\,891\,439\,5+1.827\,396\,798i\quad .$$

Therefore $s(x)=x^5-3x^2+1=k_1(x-a)^5+k_2(x-b)^5+k_3(x-c)^5$ for some constants k_1, k_2, k_3 in the complex numbers.

Using the results of Th. 3, with $d_0=1$ and $d_1=0=d_2$

$$\begin{aligned}
k_1 &= \frac{bd_0c+bd_1+d_2+cd_1}{(c-a)(b-a)}=\frac{bc}{(c-a)(b-a)}\\
k_2 &= \frac{cd_1+cad_0+d_2+ad_1}{(b-c)(b-a)}=\frac{ca}{(b-c)(b-a)}\\
k_3 &= \frac{bd_1+bad_0+d_2+ad_1}{(b-c)(a-c)}=\frac{ba}{(b-c)(a-c)}
\end{aligned}$$

So that $x^5 - 3x^2 + 1 = \frac{bc}{(c-a)(b-a)}(x-a)^5 + \frac{ac}{(b-c)(b-a)}(x-b)^5 + \frac{ab}{(b-c)(a-c)}(x-c)^5$.

By Cor. 3, since we know that $p(x) = 30x^3 + 100x - 9$ has 3 distinct roots, then $s(x) = x^5 - 3x^2 + 1$ has 3 or more roots each of multiplicity 2 or less.

Given the form for $s(x) = x^5 - 3x^2 + 1$ found above in terms of the 3 roots of $p(x)$, one must question whether these 3-5 different roots may be solved for algebraically, or by some method of exact mathematics.

4. Conclusions and Future Work

Why does anyone care about solving quintic equations or polynomial equations of any degree? Extending the ancient results of solving quadratic equations by radicals has been a quest of mathematicians for many centuries. The algebraists Cardano and Ferrari extended solutions by radicals to the cubic and quartic, respectively in the 1500's, and it wasn't until the early 1800's when someone (Niels Abel) proved there was no universal formula by radicals for 5th degree equations, and Galois in the same time period invented group theory to explain the criteria for discerning which quintic equations were solvable by radicals and which were not. See [2, Ch. 6], [6].

The method of apolar invariants used to solve polynomial equations is flawed in that it definitely does not work in every case. There are definitely cases in which the process breaks down, such as for those quintic polynomials that are not apolar to any cubic polynomials. But the method of apolar invariants does provide a different tool that sheds light on classes of quintic polynomial equations. This inspires more questions.

Question 1: Can the roots of a quintic polynomial $s(x)$ of the form given in the conclusion of Th. 3 (as in the last example, $s(x) = x^5 - 3x^2 + 1$) in which all 3 k_i's are nonzero, be solved for algebraically? Is there some method of exact mathematics that might be harnessed to find these roots?

Question 2: How do these methods of apolar polynomials ally with Galois Theory? Can the specific forms given for the k_i's found in Th. 3 (or other formulas using other of the quintic coefficients) be manipulated to shed light on which quintic polynomial equations in one variable may be solved for by radicals? Can a new criteria for solvability by radicals be offered in the language of apolar polynomials?

Question 3: Another result of [3] provides a very special class of quintics that are $A_{2,\,5}-$ apolar to a quadratic. What light might this shed on questions relating to the solvability of quintics?

Question 4: What more can be learned regarding the roots of a quintic $s(x)$ in the situation in which a cubic $p(x)$ found as apolar to the given $s(x)$ has a root of multiplicity 2 which is also a multiplicity 2 root of the corresponding quadratic $q(x)$?

Question 5: What methods might be brought into action to solve those quintics not apolar to cubic polynomials such as those of the forms $s(x) = d_0x^5 + 5d_1x^4 + d_5$ or $s(x) = d_0x^5 + 5d_4x + d_5$?

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