Some Results of Anti Fuzzy Subrings Over t-Conorms

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Abstract

In this paper, we define anti fuzzy subrings by using t-conorm C and study some of their algebraic properties. We consider properties of intersection, direct product and homomorphisms for anti fuzzy subbrings with respect to t-conorm C. Thereafter, we define anti fuzzy quotient subrings over t -conorm C.

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1 Introduction

In 1965, L. Zadeh first introduced the theory of fuzzy sets in his pioneer paper [36]. After the birth of fuzzy set theory it has achieved manifold applications than the ordinary set theory. From the beginning of fuzzy set theory it was clear that this theory was an extra ordinary tool for representing human knowledge. However, L. Zadeh himself established that sometimes, in decision-making processes, knowledge is better represented by means of some generalizations of fuzzy sets. Many researchers have centered on giving an algebraic structure to the universe space, defining the classic algebraic topics on a fuzzy environment and studying their properties. For instance, the reader may consult the papers [6] or [7] about fuzzy semigroups; [5], [4], [9], [34] or [37] about fuzzy ideals and fuzzy rings; [8] or [13] about fuzzy modules; [10] about fuzyy vector spaces; [3] about fuzzy coalgebras over a field; [35] about Lie algebras, and so on. The author by using norms, investigated some properties of fuzzy algebraic structures [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The aim of this paper is to introduce the concept of anti fuzzy subrings by using t-conorm C. By using t-conorm C, we consider the relationship between subrings and anti fuzzy subrings and we prove that the intersection and direct product of two anti fuzzy subrings are also anti fuzzy subring. Also we obtain some results for anti fuzzy subrings under the ring homomorphisms. Finally, we show that anti fuzzy of quotient subring is also an anti fuzzy subring with respect to t-conorm C.

2 Preliminaries

In this section, some basic definitions and resuls that are essential for this paper are assembled.

Definition 2.1. (See [11]) A fuzzy subset of a set X, is a function from X into $[0,1]$. The set of all fuzzy subsets of X is called the $[0,1]$-power set of X and is denoted $[0,1]^X$.

Definition 2.2. (See [2]) A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(C1) $C(x,0) = x$
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$
(C3) $C(x,y) = C(y,x)$
(C4) $C(x,C(y,z)) = C(C(x,y),z)$,
for all $x,y,z \in [0,1]$.

Example 2.3. The basic t-conorms are $C_m(x,y) = \max\{x,y\}$,$C_b(x,y) = \min\{1,x+y\}$ and $C_p(x,y) = x+y-xy$ for all $x,y \in [0,1]$.
C_m is standard union, C_b is bounded sum, C_p is algebraic sum.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, $C(x,x) = x$.

Lemma 2.4. *(See [1]) Let C be a t-conorm. Then*

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 2.5. (See [11]) Let f be a mapping from ring R into S, $\mu \in [0,1]^R$ and $\nu \in [0,1]^S$. Define $f(\mu) \in [0,1]^S$ and $f^{-1}(\nu) \in [0,1]^R$, such that $\forall y \in S$,

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in R, f(x) = y\} \text{ if } \mathrm{f}^{-1}(y) \neq \emptyset \\ 0 \quad \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Also $\forall x \in R$, $f^{-1}(\nu)(x) = \nu(f(x))$.

Theorem 2.6. *(See [12]) Let R be a ring. A nonempty subset S of R is a subring of R if and only if $x - y \in S$ and $xy \in S$ for all $x,y \in S$.*

Definition 2.7. (See [12]) Let R be a ring and I be a nonempty subset of R. We say that I is a left(right) ideal of R if for all $x,y \in I$ and for all $r \in R$, $x - y \in I$, $rx \in I$ ($x - y \in I$, $xr \in I$).

3 Main Results

Definition 3.1. Let μ be a fuzzy subset of a ring R. Then μ is called an anti fuzzy subring of R under a t-conorm C iff for all $x,y \in R$
(1) $\mu(x-y) \leq C(\mu(x),\mu(y))$
(2) $\mu(xy) \leq C(\mu(x),\mu(y))$.
Denote by $AF(R)$, the set of all anti fuzzy subrings of R under a t-conorm C.

Definition 3.2. Let $\mu_1,\mu_2 \in AF(R)$ and $x \in R$. We define
(1) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x) \leq \mu_2(x)$,
(2) $\mu_1 = \mu_2$ iff $\mu_1(x) = \mu_2(x)$,
(3)$(\mu_1 \cap \mu_2)(x) = C(\mu_1(x),\mu_2(x))$. Also $\mu_1 \cap \mu_2 = \mu_2 \cap \mu_1$ and $\mu_1 \cap \mu_2 \cap \mu_3 = (\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$ (property (C3 and C4)).

Proposition 3.3. *Let $\mu_1, \mu_2 \in AF(R)$. Then $\mu_1 \cap \mu_2 \in AF(R)$.*

Proof. Let $x, y \in R$.

$$\begin{aligned}(\mu_1 \cap \mu_2)(x-y) &= C(\mu_1(x-y), \mu_2(x-y)) \\ &\leq C(C(\mu_1(x), \mu_1(y)), C(\mu_2(x), \mu_2(y))) \\ &= C(C(\mu_1(x), \mu_2(x)), C(\mu_1(y), \mu_2(y))) \\ &= C((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)).\end{aligned}$$

Also

$$\begin{aligned}(\mu_1 \cap \mu_2)(xy) &= C(\mu_1(xy), \mu_2(xy)) \\ &\leq C(C(\mu_1(x), \mu_1(y)), C(\mu_2(x), \mu_2(y))) \\ &= C(C(\mu_1(x), \mu_2(x)), C(\mu_1(y), \mu_2(y))) \\ &= C((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)).\end{aligned}$$

Thus $\mu_1 \cap \mu_2 \in AF(R)$. □

Corollary 3.4. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq AF(R)$, Then $\mu = \cap_{i \in I_n} \mu_i \in AF(R)$.*

Example 3.5. Let $R = (\mathbb{Z}, +, .)$ be a ring of integers. Define fuzzy subset $\mu \in [0,1]^R$ as

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \{\pm 1, \pm 3, ...\} \\ 0.5 & \text{if } x \in \{0, \pm 2, \pm 4, ...\}. \end{cases}$$

Let $C_p(x, y) = x + y - xy$ for all $x, y \in R$, then $\mu \in AF(R)$.

Proposition 3.6. *Let $\mu \in AF(R)$ and C be idempotent. Then for all $t \in [0,1]$, $R_t = \{x \in R \mid \mu(x) \leq t\}$ is a subring of R.*

Proof. Let $x, y \in R_t$. Then

$$\mu(x-y) \leq C(\mu(x), \mu(y)) \leq C(t,t) = t.$$

Hence $x - y \in R_t$. Also

$$\mu(xy) \leq C(\mu(x), \mu(y)) \leq C(t,t) = t.$$

Hence $xy \in R_t$ and then R_t is a subring of R. □

Proposition 3.7. *Let μ be a fuzzy subset of ring R and C be idempotent, such that $\mu(x-y) \leq C(\mu(x), \mu(y))$ and $\mu(rx) \leq \mu(x)$ for all $x, y, r \in R$. Then*
(1) $\mu(0) \leq \mu(x)$ for all $x \in R$;
(2) $\mu(x) = \mu(-x)$ for all $x \in R$;
(3) for all $t \in [0,1]$, $R_t = \{x \in R \mid \mu(x) \leq t\}$ is a left ideal of R;
(4) $R_0 = \{x \in R \mid \mu(x) = \mu(0)\}$ is a left ideal of R.

Proof. Let $x \in R$. Then
(1) $\mu(0) = \mu(x-x) \leq C(\mu(x), \mu(x)) = \mu(x)$.
(2)

$$\mu(x) = \mu(0-(-x))$$
$$\leq C(\mu(0), \mu(-x)) \leq C(\mu(-x), \mu(-x))$$
$$= \mu(-x) = \mu(0-x)$$
$$\leq C(\mu(0), \mu(x)) \leq C(\mu(x), \mu(x)) = \mu(x)$$

and then $\mu(x) = \mu(-x)$.
(3) If $x, y \in R_t, r \in R$, then

$$\mu(x-y) \leq C(\mu(x), \mu(y)) \leq C(t,t) = t$$

and $\mu(rx) \leq \mu(x) \leq t$. Hence R_t is a left ideal of R.
(4) Let $x, y \in R_0$ and $r \in R$. We have

$$\mu(x-y) \leq C(\mu(x), \mu(y)) = C(\mu(0), \mu(0)) = \mu(0)$$

and $\mu(rx) \leq \mu(x) = \mu(0)$.Thus R_0 is a left ideal of R. □

Proposition 3.8. *Let $\mu \in AF(R)$ and C be idempotent. Then $\mu(x-y) = \mu(-y)$ if and only if $\mu(x) = \mu(0)$ for all $x, y \in R$.*

Proof. Let $x, y \in R$ and $\mu(x-y) = \mu(-y)$. Then by letting $y = 0$, we get that $\mu(x) = \mu(0)$. Conversely, suppose that $\mu(x) = \mu(0)$ then from Proposition 3.7 (1) we get $\mu(x) \leq \mu(x-y), \mu(-y)$. Now

$$\mu(x-y) \leq C(\mu(x), \mu(y)) \leq C(\mu(-y), \mu(y)) =$$
$$C(\mu(-y), \mu(-y)) = \mu(-y) = \mu(x-y-x))$$
$$\leq C(\mu(x-y), \mu(x)) \leq C(\mu(x-y), \mu(x-y)) = \mu(x-y)$$

and so $\mu(x-y) = \mu(-y)$. □

Proposition 3.9. *Let $\mu \in AF(R)$ and S be a ring. Suppose that f is onto homomorphism of R into S. Then $f(\mu) \in AF(S)$.*

Proof. Let $s_1, s_2 \in S$ then there exist $x, y \in R$ such that $s_1 = f(x)$ and $s_2 = f(y)$. Now

$$f(\mu)(s_1 - s_2) = \sup\{\mu(x-y) \mid s_1 = f(x), s_2 = f(y)\}$$
$$\leq \sup\{C(\mu(x), \mu(y)) \mid s_1 = f(x), s_2 = f(y)\}$$
$$= C(\sup\{\mu(x) \mid s_1 = f(x)\}, \sup\{\mu(y) \mid s_2 = f(y)\})$$
$$= C(f(\mu)(s_1), f(\mu)(s_2)).$$

Also

$$f(\mu)(s_1 s_2) = \sup\{\mu(xy) \mid s_1 = f(x), s_2 = f(y)\}$$
$$\leq \sup\{C(\mu(x), \mu(y)) \mid s_1 = f(x), s_2 = f(y)\}$$

$$= C(\sup\{\mu(x) \mid s_1 = f(x)\}, \sup\{\mu(y) \mid s_2 = f(y)\})$$
$$= C(f(\mu)(s_1), f(\mu)(s_2)).$$

Therefore $f(\mu) \in AF(S)$. □

Proposition 3.10. *Let S be a ring and $\nu \in AF(S)$. If f be a homomorphism of R into S, then $f^{-1}(\nu) \in AF(R)$.*

Proof. Let $x, y \in R$. Then

$$f^{-1}(\nu)(x-y) = \nu(f(x-y)) = \nu(f(x) - f(y))$$
$$\leq C(\nu(f(x)), \nu(f(y))) = C(f^{-1}(\nu)(x), f^{-1}(\nu)(y)).$$

Also

$$f^{-1}(\nu)(xy) = \nu(f(xy)) = \nu(f(x)f(y))$$
$$\leq C(\nu(f(x)), \nu(f(y))) = C(f^{-1}(\nu)(x), f^{-1}(\nu)(y)).$$

Hence $f^{-1}(\nu) \in AF(R)$. □

Definition 3.11. Let R and S be two rings such that $\mu \in AF(R)$ and $\nu \in AF(S)$. The direct product of μ and ν, denoted by $\mu \times \nu$, is the fuzzy subset of ring $R \times S$ such that for all $x \in R$ and $y \in S$ defined by $(\mu \times \nu)(x, y) = C(\mu(x), \nu(y))$.

Now we prove that the direct product of two anti fuzzy subrings is also an anti fuzzy subring.

Proposition 3.12. *If $\mu_i \in AF(R_i)$ for $i = 1, 2$, then $\mu_1 \times \mu_2 \in AF(R_1 \times R_2)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in R_1 \times R_2$. Then

$$(\mu_1 \times \mu_2)((x_1, y_1) - (x_2, y_2))$$
$$= (\mu_1 \times \mu_2)(x_1 - x_2, y_1 - y_2)$$
$$= C(\mu_1(x_1 - x_2), \mu_2(y_1 - y_2))$$
$$\leq C(C(\mu_1(x_1), \mu_1(x_2)), C(\mu_2(y_1), \mu_2(y_2)))$$
$$= C(C(\mu_1(x_1), \mu_2(y_1), C(\mu_1(x_2), \mu_2(y_2))$$
$$= C((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

Also

$$(\mu_1 \times \mu_2)((x_1, y_1)(x_2, y_2))$$
$$= (\mu_1 \times \mu_2)(x_1 x_2, y_1 b_2)$$
$$= C(\mu_1(x_1 x_2), \mu_2(y_1 y_2))$$
$$\leq C(C(\mu_1(x_1), \mu_1(x_2)), C(\mu_2(y_1), \mu_2(y_2)))$$
$$= C(C(\mu_1(x_1), \mu_2(y_1), C(\mu_1(x_2), \mu_2(y_2))$$
$$= C((\mu_1 \times \mu_2)(x_1, y_1), (\mu_1 \times \mu_2)(x_2, y_2)).$$

□

Corollary 3.13. *Let $\mu_i \in AF(R_i)$ for $i = 1, 2, ..., n$. Then*

$$\mu_1 \times \mu_2 \times ... \times \mu_n \in AF(R_1 \times R_2 \times ... \times R_n).$$

The following definition introduces the concept of anti fuzzy quotient subrings.

Definition 3.14. Let R be a ring, I be an ideal of R and $\mu_I \in AF(R)$. Define $\mu : R/I \to [0, 1]$ by

$$\mu(x + I) = \begin{cases} C(\mu_I(x), \mu_I(i)) & \text{if } x \neq i \\ 1 & \text{if } x = i. \end{cases}$$

for all $x \in R$ and $i \in I$.

Proposition 3.15. *If C be idempotent, then $\mu \in AF(R/I)$.*

Proof. Let $x + I, y + I \in R/I, i \in I$ and $\mu_I \in AF(I)$ such that $x \neq i \neq y$.

$$\begin{aligned} &\mu((x + I) - (y + I)) \\ &= \mu((x - y) + I) \\ &= C(\mu_I(x - y), \mu_I(i)) \\ &\leq C(C(\mu_I(x), \mu_I(y)), \mu_I(i)) \\ &= C(C(\mu_I(x), \mu_I(y)), C(\mu_I(i), \mu_I(i))) \\ &= C(C(\mu_I(x), \mu_I(i)), C(\mu_I(y), \mu_I(i))) \\ &= C(\mu(x + I), \mu(y + I)). \end{aligned}$$

Also

$$\begin{aligned} &\mu((x + I)(y + I)) \\ &= \mu(xy + I) \\ &= C(\mu_I(xy), \mu_I(i)) \leq C(C(\mu_I(x), \mu_I(y)), \mu_I(i)) \\ &= C(C(\mu_I(x), \mu_I(y)), C(\mu_I(i), \mu_I(i))) \\ &= C(C(\mu_I(x), \mu_I(i)), C(\mu_I(y), \mu_I(i))) \\ &= C(\mu(x + I), \mu(y + I)). \end{aligned}$$

□

4 Conclusions

The concept of anti fuzzy in rings setting has been extended by t-conorms. We lucidly exemplified intersection, direct product and homomorphisms for anti fuzzy subbrings with respect to t-conorm C and deduced several results. Finally, homomorphism and some of its properties were proposed in the context of anti fuzzy subbrings with respect to t-conorms.

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