# Giving Birth to Vectorial Coordinate Geometry II

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**Abstract**

This paper is an extension of the earlier work titled "Giving birth to vectorial coordinate geometry", which appeared in volume 3 of MathLab journal in 2019. While the novel concept of vectorial coordinates has been applied in the said work for the derivation of some useful results of two-dimensional Cartesian coordinate geometry, the concept of novel vectorial coordinates has been employed in the present scheme for the derivation of some standard results of three-dimensional Cartesian coordinate geometry. The fact that Vector algebra is a more powerful mathematical tool gets justified from this study. The vectorial treatments offered are novel, simple and straight forward. Furthermore, they will enhance the deepening of thought and understanding about Vector algebra and its application. Thus this contribution will enrich the relevant literature. At the same time, it increases the range of applicability of Vector algebra as well.

**Keywords**: Cartesian Coordinate Geometry; Vector Algebra; Dot Product; Cross Product; Rectangular Unit Vectors.

## Introduction

'Vector algebra' is one of the important mathematical tools having a wide range of applications in Science and Engineering. It has been used as a standard tool by the physicists. It has been proved to be an important tool for describing the electro-magnetic-interaction [1]. Classical electrodynamics [2], Quantum mechanics [3], Theory of relativity [4] etc. have been dealt with satisfactorily on the basis of Vector algebra and Vector calculus. In [5], a Vector algebraic approach has been reported to obtain molecular fields from canonical intersections. Vector algebraic concepts have been applied in [6] for the study of genome-wide expression data. In [7], the discovery of the generalized equations of motion has been reported by making use of vector algebra. More realistic definitions of trigonometric ratios have been provided in [8] with the help of vector algebra. In a recent paper [9], a novel approach has been provided for the derivation of the standard formulae in relation to properties of the triangle on the basis of vector algebra as an extension of the earlier work [8].

With a view to dealing with the real-world problems in which the concept of 'negative distance' involved in the 'sign convention' of the traditional Cartesian coordinate geometry is not applicable, Vectorial coordinate geometry has been introduced in [12]. Unlike the long-running Cartesian coordinate geometry, each coordinate of a point in the new frame work is to be denoted by a vector rather than a scalar. With the introduction of vectorial coordinates, some useful results for the two-dimensional Cartesian coordinate geometry have been derived, making use of vector algebra.

This paper is an extension of the published paper [12]. With the incorporation of the novel concept of vectorial coordinates, some useful results for the three-dimensional Cartesian coordinate geometry have been developed. The fact that the techniques of the derivation of the long-running standard results of the three-dimensional Cartesian coordinate geometry [10]-[11] are amenable to Vector algebra clearly follows from the derivations offered. Vectorial treatments offered in the said context are simple, straight forward and novel. They will enrich the traditional literature as well as enhance the deepening of thought and understanding in regard to Vector algebra and its applications.

**Dealing with The Three Dimensional Cases on The Basis of Vector Algebra**

**(i) To find the distance between two given points** ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**) and ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**).

**Solution**: Let the two given points be A ≡ ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**) and B ≡ ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**). Now, considering Figure 1, we have, $\mathbf{OA} = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$, and $\mathbf{OB} = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$.
Then by the triangle law of addition of vectors we obtain,
$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$

Now, AB = $|\mathbf{AB}| = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

Thus the distance between the two given points ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**) and ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**) is

$\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$.

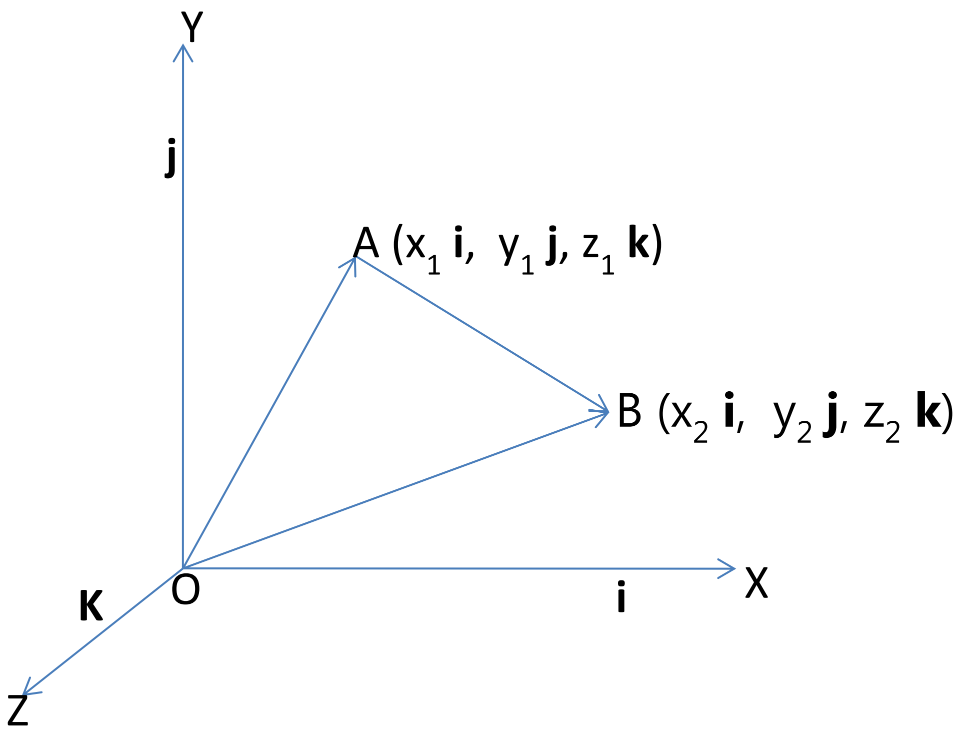


Figure 1. Diagram for finding the distance between two given points

**(ii) To find the area of the triangle formed by three given points** ($x_1$ **i**, $y_1$ **j**, $z_1$**k**), ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**) and ($x_3$ **i**, $y_3$ **j**, $z_3$ **k**).

**Solution**: Let the vertices of the triangle ABC be A ≡ ($x_1$ **i**, $y_1$ **j**, $z_1$**k**), B ≡ ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**), and C ≡ ($x_3$ **i**, $y_3$ **j**, $z_3$ **k**). Then from Figure 2, we have, $\mathbf{AB} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$, and $\mathbf{AC} = (x_3 - x_1)\,\mathbf{i} + (y_3 - y_1)\,\mathbf{j} + (z_3 - z_1)\,\mathbf{k}$.

Now, $\mathbf{AB} \times \mathbf{AC} = \{(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}\} \times \{(x_3 - x_1)\,\mathbf{i} + (y_3 - y_1)\,\mathbf{j} + (z_3 - z_1)\,\mathbf{k}\}$
$= \{y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)\}\mathbf{i} + \{z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)\}\mathbf{j} + \{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\mathbf{k}$

Then the magnitude of the area of the triangle ABC = $\frac{1}{2}$ $|\mathbf{AB} \times \mathbf{AC}|$ = $\frac{1}{2}$ $\sqrt{[\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}^2 + \{y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)\}^2 + \{z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)\}^2]}$

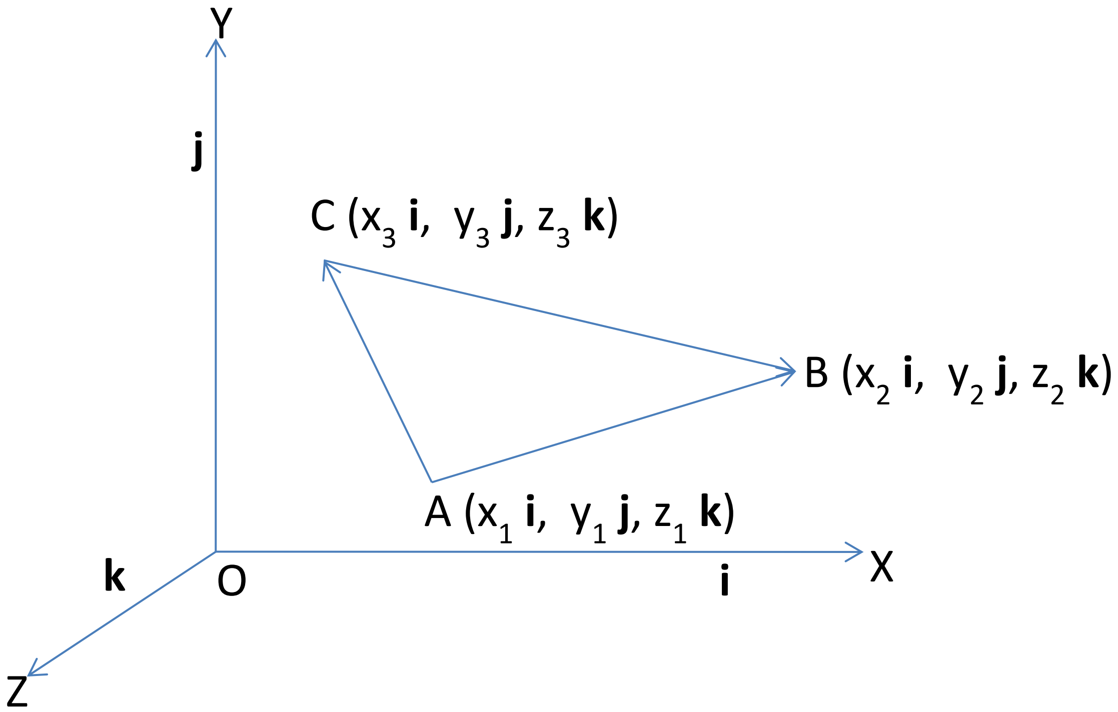


Figure 2. Diagram for finding the area of a triangle formed by joining three given points

**(iii) To find the equation of a plane in intercept form**

**Solution**: Let the intercepts made by the plane ABC from the X, Y, and Z axes of coordinates be 'a', 'b', and 'c' respectively. Then, as shown in Figure 3, coordinates of the points A, B, and C are A ≡ (a $\mathbf{i}$, 0 $\mathbf{j}$, 0 $\mathbf{k}$), B ≡ (0 $\mathbf{i}$, b $\mathbf{j}$, 0 $\mathbf{k}$), and C ≡ (0 $\mathbf{i}$, 0 $\mathbf{j}$, c $\mathbf{k}$). Let P be any point on the plane ABC having coordinates (x $\mathbf{i}$, y $\mathbf{j}$, z $\mathbf{k}$).

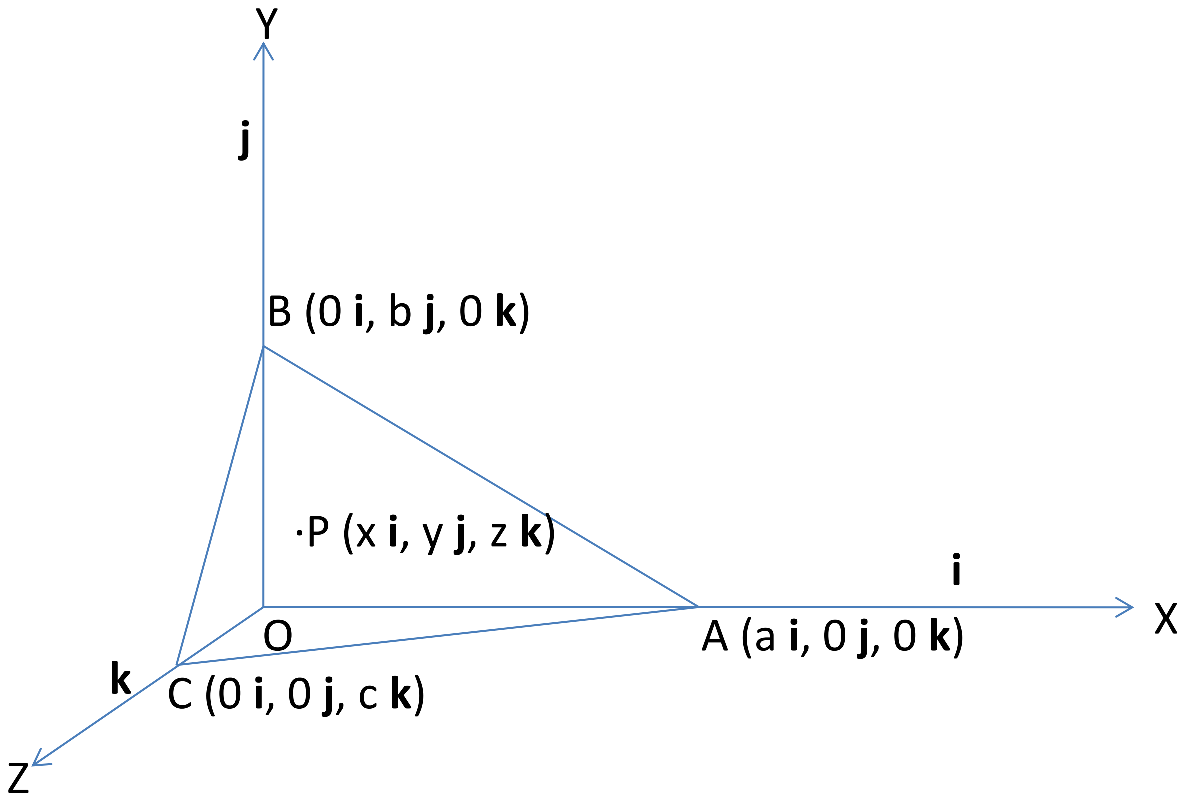


Figure 3. Diagram for finding equation of a plane in intercept form

Now, $\mathbf{AP} = (x - a)\,\mathbf{i} + (y - 0)\,\mathbf{j} + (z - 0)\,\mathbf{k} = (x - a)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$\mathbf{BP} = (x - 0)\,\mathbf{i} + (y - b)\,\mathbf{j} + (z - 0)\,\mathbf{k} = x\,\mathbf{i} + (y - b)\,\mathbf{j} + z\,\mathbf{k}$

$\mathbf{CP} = (x - 0)\,\mathbf{i} + (y - 0)\,\mathbf{j} + (z - c)\,\mathbf{k} = x\,\mathbf{i} + y\,\mathbf{j} + (z - c)\,\mathbf{k}$

Since **AP**, **BP**, and **CP** are coplanar, we must have,

$$\mathbf{AP} \cdot (\mathbf{BP} \times \mathbf{CP}) = 0$$

or, $\{(x - a)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\} \cdot [\{x\,\mathbf{i} + (y - b)\,\mathbf{j} + z\,\mathbf{k}\} \times \{x\,\mathbf{i} + y\,\mathbf{j} + (z - c)\,\mathbf{k}\}] = 0$

or, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, which is the required equation of the plane in intercept form.

**(iv) To find the equation of a plane passing through three given points** ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**), ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**) and ($x_3$ **i**, $y_3$ **j**, $z_3$ **k**).

**Solution**: Let the three given points lying on the plane be A ≡ $(x_1\,\mathbf{i}, y_1\,\mathbf{j}, z_1\mathbf{k})$, B ≡ $(x_2\,\mathbf{i}, y_2\,\mathbf{j}, z_2\,\mathbf{k})$, and C ≡ $(x_3\,\mathbf{i}, y_3\,\mathbf{j}, z_3\,\mathbf{k})$. We are to find the equation of the plane.

Let P be any point on the plane having coordinates (x **i**, y **j**, z **k**).

Then we have, $\mathbf{PA} = (x_1 - x)\,\mathbf{i} + (y_1 - y)\,\mathbf{j} + (z_1 - z)\,\mathbf{k}$

$$\mathbf{PB} = (x_2 - x)\,\mathbf{i} + (y_2 - y)\,\mathbf{j} + (z_2 - z)\,\mathbf{k}$$
$$\mathbf{PC} = (x_3 - x)\,\mathbf{i} + (y_3 - y)\,\mathbf{j} + (z_3 - z)\,\mathbf{k}$$

Since here **PA**, **PB**, and **PC** lie on the same plane, they are coplanar.

Hence, we must have,

$$\mathbf{PA} \cdot (\mathbf{PB} \times \mathbf{PC}) = 0$$

or, $\{(x_1 - x)\,\mathbf{i} + (y_1 - y)\,\mathbf{j} + (z_1 - z)\,\mathbf{k}\} \cdot [\{(x_2 - x)\,\mathbf{i} + (y_2 - y)\,\mathbf{j} + (z_2 - z)\,\mathbf{k}\} \times \{(x_3 - x)\,\mathbf{i} + (y_3 - y)\,\mathbf{j} + (z_3 - z)\,\mathbf{k}\}] = 0$

or,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

which is the required equation of the plane under consideration.

**(v) To find the perpendicular distance of the plane** ax + by + cz + d = 0 **from the given point** ($x_1$ **i**, $y_1$ **j**, $_1$**k**).

**Solution**: As shown in Figure 4, let ABC be the plane whose equation is, ax + by + cz + d = 0 and P ≡ ($x_1$ **i**, $y_1$ **j**, $z_1$**k**) be the given point. From the point P, let us drop a perpendicular PM on to the plane ABC. Also, let M ≡ ($x_2$ **i**, $y_2$**j**, $z_2$**k**), be the foot of the perpendicular.

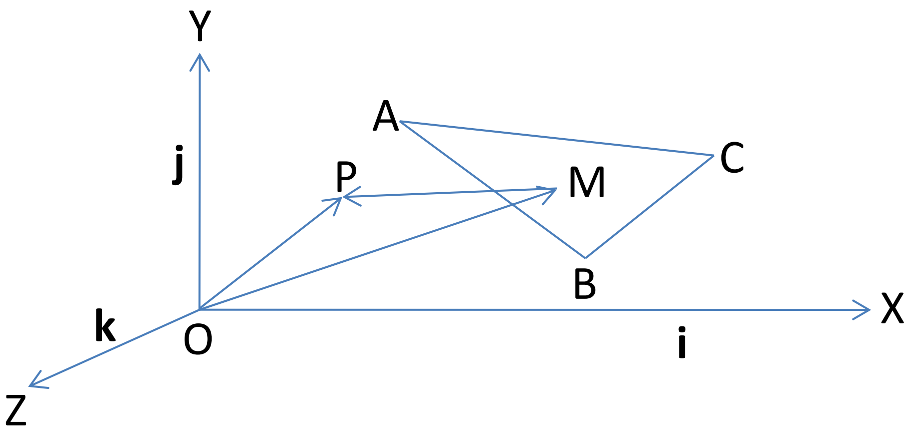


Figure 4. Diagram for finding the perpendicular distance of a given plane from a given point

Then considering Figure 4, we get,

$$\mathbf{MP} = \mathbf{OP} - \mathbf{OM} = (x_1 - x_2)\,\mathbf{i} + (y_1 - y_2)\,\mathbf{j} + (z_1 - z_2)\,\mathbf{k}$$

Then the unit vector ($\mathbf{n_1}$) along the direction of **MP** is given by,

$$\mathbf{n_1} = \{(x_1 - x_2)\,\mathbf{i} + (y_1 - y_2)\,\mathbf{j} + (z_1 - z_2)\,\mathbf{k}\}/[\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}]$$

Now, the equation of the plane is

$$\varphi(x, y, z) = ax + by + cz + d = 0$$

Then, grad $\varphi = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$

Hence, the unit vector ($\mathbf{n_2}$) along the direction of grad φ is given by,

$$\mathbf{n_2} = (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})/\left\{\sqrt{a^2 + b^2 + c^2}\right\}$$

Now, since ($\mathbf{n_1}$) and ($\mathbf{n_2}$) are parallel vectors, we must have,

$$\mathbf{n_1} \times \mathbf{n_2} = \mathbf{0}$$

or, $|\mathbf{n_1} \times \mathbf{n_2}| = 0$

Using the above expressions for ($\mathbf{n_1}$) and ($\mathbf{n_2}$) this relation will ultimately yield the following three relations.

$$b\,z_2 - c\,y_2 = b\,z_1 - c\,y_1 \quad \ldots \quad (1)$$

$$c\,x_2 - a\,z_2 = c\,x_1 - a\,z_1 \quad \ldots \quad (2)$$

$$a\,y_2 - b\,x_2 = a\,y_1 - b\,x_1 \quad \ldots \quad (3)$$

Again since the point ($x_2\,\mathbf{i}$, $y_2\,\mathbf{j}$, $z_2\,\mathbf{k}$) lies on the plane $ax + by + cz + d = 0$, we must have,

$$a\,x_2 + b\,y_2 + c\,z_2 + d = 0 \quad \ldots \quad (4)$$

Now, multiplying equation (2) by c, equation (4) by a, and then adding we get,

$$(c^2 + a^2)\,x_2 + aby_2 + ad = c^2\,x_1 - acz_1 \quad \ldots \quad (5)$$

Now solving equations (3) and (5) for $x_2$ and $y_2$, we get,

$$x_2 = (c^2\,x_1 - acz_1 + b^2\,x_1 - aby_1 - ad)/(a^2 + b^2 + c^2)$$

$$y_2 = (-bcz_1 - bd + a^2\,y_1 + c^2\,y_1 - abx_1)/(a^2 + b^2 + c^2)$$

Putting this value of $y_2$ in equation (1) we get,

$$z_2 = (a^2\,z_1 + b^2\,z_1 - bcy_1 - acx_1 - cd)/(a^2 + b^2 + c^2)$$

The required perpendicular distance (S, say) of the plane $ax + by + cz + d = 0$ from the given point ($x_1\,\mathbf{i}$, $y_1\,\mathbf{j}$, $z_1\,\mathbf{k}$) is then given by,

$$S = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}$$

$$= \sqrt{[\{x_1 - (c^2\,x_1 - acz_1 + b^2\,x_1 - aby_1 - ad)/(a^2 + b^2 + c^2)\}^2 +}$$

$\{y_1 - (-bcz_1 - bd + a^2 y_1 + c^2 y_1 - abx_1)/(a^2 + b^2 + c^2)\}^2 +$
$\{z_1 - (a^2 z_1 - bcy_1 + b^2 z_1 - acx_1 - cd)/(a^2 + b^2 + c^2)\}^2]$

$= \pm (a x_1 + b y_1 + c z_1 + d)/\sqrt{(a^2 + b^2 + c^2)}$,

where the '+ ve' or ' _ve' sign is to be taken according as whether the quantity '$ax_1 + by_1 + cz_1 + d = 0$' is positive or negative.

**(vi) If** l, m, **and** n **denote the direction cosines of a straight line, prove that,** $l^2 + m^2 + n^2 = 1$.

**Solution**: If α, β, and γ denote the angles which a straight line makes with the coordinate axes, then cos α, cos β, cos γ are called the direction cosines of the straight line. Here, l = cos α, m = cos β, and n = cos γ.

Let **A** ≡ ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**), and **B** ≡ ($x_2$ **i**, $y_2$ **j**, $z_2$ **k**) be two points lying on the straight line having direction cosines l, m, and n.

Then clearly, $l = \frac{\mathbf{AB} \cdot \mathbf{i}}{|\mathbf{AB}|} = [\{(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}\} \cdot \mathbf{i}]/\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}} = (x_2 - x_1)/\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

Similarly we have, $m = \frac{\mathbf{AB} \cdot \mathbf{j}}{|\mathbf{AB}|} = (y_2 - y_1)/\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

and $n = \frac{\mathbf{AB} \cdot \mathbf{k}}{|\mathbf{AB}|} = (z_2 - z_1)/\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

Thus, we have,

$$l^2 + m^2 + n^2 = \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}/\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\} = 1$$

In other words, we must have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

**(vii) To find the equation of a plane in perpendicular (or normal) form**

**Solution**: As shown in Figure 5, let ABC be the plane whose equation is to be found out. Let OM = p, where OM is the perpendicular dropped from the origin on to the plane ABC. Also let us assume that cos α, cos β, cos γ be the direction cosines of OM,

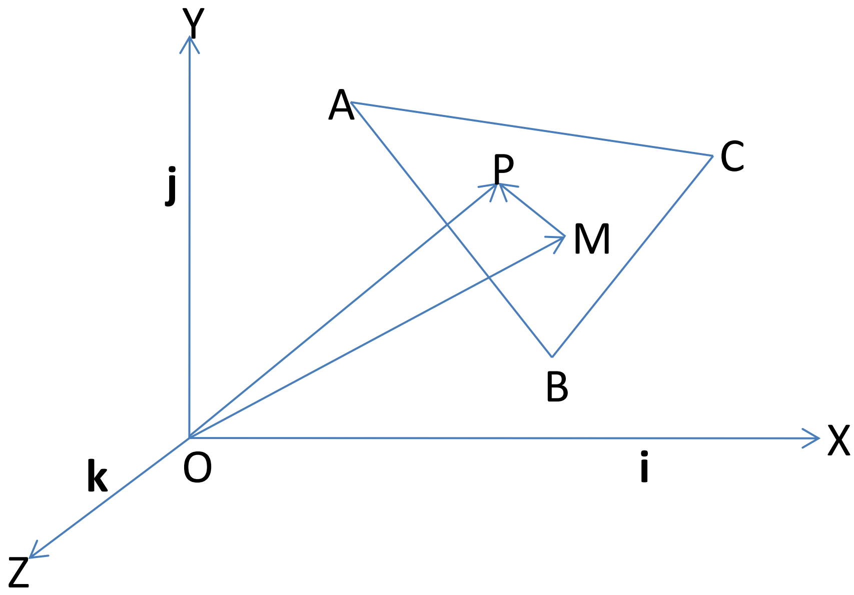


Figure 5. Diagram for finding the equation of a plane in perpendicular (or normal) form

Then, $\mathbf{OM} = (p\cos\alpha)\,\mathbf{i} + (p\cos\beta)\,\mathbf{j} + (p\cos\gamma)\,\mathbf{k}$.

Now, let P(x **i**, y **j**, z **k**) be any point on the plane ABC. Then we have, $\mathbf{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Hence, $\mathbf{MP} = \mathbf{OP} - \mathbf{OM} = (x - p\cos\alpha)\,\mathbf{i} + (y - p\cos\beta)\,\mathbf{j} + (z - p\cos\gamma)\,\mathbf{k}$.

Now, since here **MP** is perpendicular to **OM**, we must have,

$$\mathbf{MP} \cdot \mathbf{OM} = 0$$

or, $\{(x - p\cos\alpha)\,\mathbf{i} + (y - p\cos\beta)\,\mathbf{j} + (z - p\cos\gamma)\,\mathbf{k}\} \cdot \{(p\cos\alpha)\,\mathbf{i} + (p\cos\beta)\,\mathbf{j} + (p\cos\gamma)\,\mathbf{k}\} = 0$

or, $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$,

which is the required equation of the plane in perpendicular (or normal) form.

**(viii) To find the equation of the straight line having direction cosines** l, m, n **and which passes through the point** ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**).

**Solution**: Let the straight line having direction cosines l, m, and n pass through the point A ≡ ($x_1$ **i**, $y_1$ **j**, $z_1$ **k**). It is required to find the equation of the straight line.

Let us consider any point P on the straight line and let P ≡ (x **i**, y **j**, z **k**).

Now, $\mathbf{AP} = (x - x_1)\,\mathbf{i} + (y - y_1)\,\mathbf{j} + (z - z_1)\,\mathbf{k}$

Then, $l = \frac{\mathbf{AP}\cdot\mathbf{i}}{|\mathbf{AP}|} = (x - x_1)/\sqrt{\{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\}}$

$m = \frac{\mathbf{AP}\cdot\mathbf{j}}{|\mathbf{AP}|} = (y - y_1)/\sqrt{\{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\}}$

and $n = \frac{\mathbf{AP}\cdot\mathbf{k}}{|\mathbf{AP}|} = (z - z_1)/\sqrt{\{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\}}$

It then clearly follows from these three relations that

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

which is the required equation of the straight line.

**Conclusion**

This paper is an extension of the novel concept of vectorial coordinates introduced in [12] for dealing with the real world problems. Remaining within the frame work of vectorial coordinate geometry, it demonstrates that the techniques of derivation of the standard results of the long-running three dimensional Cartesian coordinate geometry could be translated to Vector algebra. Merits of the present work must be sought in the following.

(i) The present work increases the range of applicability of vector algebra.

(ii) Vectorial treatments of derivation offered in the present study for dealing with the three dimensional cases are most unlikely to be found in the traditional literature.

(iii) The present study confirms the validity of the fact that Vector algebra is a more powerful mathematical tool from the view point of applicability.

(iv) The vectorial treatments offered are simple, straight forward and novel. As a result they will enrich as well as enhance the relevant literature.

(v) The present work will provide sufficient feedback for deepening of thought and understanding regarding Vector algebra and its applications to Cartesian coordinate geometry.

On account of space limitations, discussions are kept limited to some restricted cases. Efforts, may however, be made to cover up the derivation for the rest of the cases.

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