Product of Polycyclic-by-Finite Groups(PPFG)

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Abstract:

In this paper we show that If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

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Introduction:

In 1955 N.Itô (see [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (see[21]) and L.Redei (1950)(see [22]) considered products of cyclic groups, and around 1965 O.H.Kegel (See [30] & [31]) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [20]&[1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See [20])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1], [2],[3],[4] and [6]) , N.S. Chernikov (see [5]), S. Franciosi, F. de Giovanni (see [3],[6],[32],[33],[34],[35], and [36]), O.H.Kegel (see [8]), J.C.Lennox (see [12]) , D.J.S. Robinson(see [9] and [15]), J.E.

Roseblade(see [13]), Y.P.Sysak(see [37],[38],[39]and[40]), J.S.Wilson (see [41]), and D.I.Zaitsev(see [11] and [18]).

Now, In this paper we show that If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

2. Priliminaries : (elementary properties and theorems.)

2.1. Difinition: Recall that the FC-centre of a group G is the subgroup of all elements of G with a finite number of conjugates. A group is an FC-group if it coincides with its FC-centre.

2.2.Lemma: Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized . Moreover, every term of the upper central series of G is factorized.

Proof: Since S is factorized, we have that S=$(A \cap S)(B \cap S)$. Let x=ab be an element of S, where a is in $A \cap S$ and b is in $B \cap S$ .If c=a_1b_1 is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1,x]=[a_1,ab]=[a_1,b]=\left[cb_1^{-1},b\right]=[c,b]^{b_1^{-1}}=1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of [4]. In particular, the center of G is factorized. It follows from Lemma 1.1.2 of [4] that also every term of the upper central series of G is factorized.

2.3. Lemma: Let the group G=AB be the product of two subgroups A and B. If A_1, B_1, and F are the FC-centers of A, B, and C, respectively, then $F=A_1F \cap B_1F$. In particular, if A and B are FC-groups, the FC-centre of G is factorized subgroup.

Proof : Let x be an element of $A_1F \cap B_1F$, and write x=au where a is in A_1 and u is in F. Since the centralizers $C_A(a)$ and $C_A(u)$ have finite index in A, the index $|A: C_A(x)|$ is also finite. Similarly, $C_B(x)$ has finite index in B. Therefore $|G:\langle C_A(x),C_B(x)\rangle|$ is finite by Lemma 1.2.5 of [4]. It follows that $C_G(x)$ has finite index in G and hence x belongs to F. Thus $F=A_1F \cap B_1F$.

2.4.Lemma: (See [7]) Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof : Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.11 have $Z(G)=(A \cap Z(G))(B \cap Z(G))=1.$ The centralizer $C=C_G(A \cap C_G(G'))$ *contains* AG', and so is normal in G. Since $B \cap (AZ(C)) \leq Z(G)=1,$ it follows that $AZ(C)=A(B \cap AZ(C))=A.$ This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.9, we have $A \cap G' \leq A \cap C_G(G') \leq Z(C)=1.$

Similarly $B \cap G' \leq B \cap C_G(G') \leq Z(C)=1.$ The factorizer $X=X(G')$ has the triple factorization $X=A^*B^*=A^*G'=B^*G'$, Where $A^*=A \cap BG'$ *and* $B^*=B \cap AG'.$ Thus X is nilpotent by Corollary 2.8, so that

$$Z(X)=(A \cap Z(X))(B \cap Z(X))$$

is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G'] \leq N \cap G' \leq A \cap G'=1.$ Therefore we obtain the contradiction $N \leq A \cap G_G(G')=1.$

2.5.Corrollary: Let the finite group $G=A_1 \ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

Proof. Let p be a prime, and for every i=1…,t let P_1 be the unique Sylow

p-complement of A_i. If $i \neq j$, the subgroup A_iA_j is soluble by Theorem 2.4.3 of [4]. Hence it follows from Lemma 2.6,that P_iP_j is a Sylow p-complement of A_iA_j. Thuse the subgroups $P_1,\ldots,P_t$ pairwise permute, and the product $P_1P_2\ldots P_t$ is a Sylow p-complement of G. Since G has a Sylow p-complement for every prime p, it is soluble.

2.6.Theorem(See [8]&[10])**:** If the finite group G=AB is the product of two nilpotent subgroups A and B, then G is soluble.

Proof: See [4] ,(Theorem 2.4.3).

2.7.Lemma : Let A and B be subgroups of a group G, and let A_1 and B_1 be subgroups of A and B, respectively, such that $|A:A_1|\le m$ and $|B:B_1|\le n$. Then $|A\cap B:A_1\cap B_1|\le mn.$

Proof : To each left coset $x(A_1\cap B_1)$ of $A_1\cap B_1$ in $A\cap B$ assign the pair of left cosets (xA_1,xB_1). Clearly this defines an injective map from the set of left cosets of $A_1\cap B_1$ in $A\cap B$ into the cartesian product of the set of left cosets of A_1 in A and the set of left cosets of B_1 in B. The lemma is proved.

2.8.Lemma(See [11])**:** Let the finitely generated group G=AB=AK=BK be the product of two ablian-by-finite subgroups A and B and an abelian normal subgroup K of G. Then G is nilpotent-by-finite.

Proof: Let A_1 and B_1 be abelian subgroups of finite index of A and B, respectively, and let n be a positive integer such that $|A:A_1|\le n$ AND $|B:B_1|\le n$. Since G is finitely generated, it has only finitely many subgroups of each finite index, and hence the intersection H of all subgroups of G with index at most n^4 also has finite index in G. In particular H is finitely generated.

Consider a finite homomorphic image H/N of H. Then N has finite index in G, and hence also its core N_G has finite index in G. Let $p_1,\ldots,p_t$ be the prime divisors of the order of the finite abelian group $K/(K\cap N_G)$. For each $j\le t$, let $K_j/(K\cap N_G)$ be the $p'_j-component$ of $K/(K\cap N_G)$. Clearly each K, is normal in G and $\cap_{j=1}^{t}K_j=K\cap N_G$. The factor group $\overline{G}=G/K,$ has the triple factorization $\overline{G}=\overline{A}\overline{B}=\overline{A}\overline{K}=\overline{B}\overline{K}$,where $\overline{K}$ is a finite normal $p_j-subgroup$ of $\overline{G}$. Clearly

$$\begin{aligned}|\overline{G}:\overline{A}\cap\overline{B}|&=|\overline{G}:\overline{A}|.|\overline{A}:\overline{A}\cap\overline{B}|=|\overline{G}:\overline{A}|.|\overline{G}:\overline{B}|\\&=|\overline{K}:\overline{A}\cap\overline{K}|.|\overline{K}:\overline{B}\cap\overline{K}|=p_j^k\end{aligned}$$

for some non-negative integer k. On the other hand, $|\overline{A}\cap\overline{B}:\overline{A}_1\cap\overline{B}_1|\le n^2$ by Lemma 2.16, so that $|\overline{G}:\overline{A}_1\cap\overline{B}_1|\le p_j^k n^2$. As $\overline{A}_1$ and $\overline{B}_1$ are abelian, the intersection $\overline{A}_1\cap\overline{B}_1$ is contained in the centre of $\langle\overline{A}_1,\overline{B}_1\rangle$, and the factor group $\langle\overline{A}_1,\overline{B}_1\rangle/(\overline{A}_1\cap\overline{B}_1)$ has order at most $p_j^k n^2$. Let $\overline{P}/(\overline{A}_1\cap\overline{B}_1)$ be a Sylow p_j-subgroup of $\langle\overline{A}_1,\overline{B}_1\rangle/(\overline{A}_1\cap\overline{B}_1)$. Then $|\langle\overline{A}_1,\overline{B}_1\rangle:\overline{P}|\le n^2$, and since $|\overline{G}:\langle\overline{A}_1,\overline{B}_1\rangle|\le n^2$ by Lemma 2.2, we obtain $|\overline{G}:\overline{P}|\le n^4$. Therefore HK_j/K_j is contained in $\overline{P}$. As an extension of the central subgroup $\overline{A}_1\cap\overline{B}_1$ by a finite p_j-group, $\overline{P}$ is nilpotent, so that $H/(H\cap K_j)\simeq HK_j/K_j$ is also nilpotent for each j. Hence.

$H\Big/\left(\bigcap_{j=1}^{t}(H\cap K_j)\right)=H/(K\cap N_G)$ is nilpotent. We have shown that each finite homomorphic image of H is nilpotent . As K is abelian, H is soluble, and hence even nilpotent (Robinson 1972,Part 2,Theorem 10.51).Therefore G is nilpotent-by-finite.

2.9.Difinition: A group G has finite Prüfer rank r=r(G) if every finitly generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this properly. Clearly subgroups and homomorphic images of groups with finite Prüfer rank also have finite Prüfer rank.

2.10.Lemma: (See [13]) If N is a maximal abelian normal dubgroup of a finite p-group G, then $r(G)\le \frac{1}{2}r(N)(5\,r(N)+1).$

Proof : Since $C_G(N)=N,$ the factor group G/N is isomorphism with a p-group of automorphism of N. Thus G/N has perüfer rank at most $\frac{1}{2}r(N)(5r(N)-1)$ (See [15], part2, lemma 7.44), and hence $r(G)\le\frac{1}{2}r(N)(5r(N)+1).$

2.11. Theorem: (See [9] and [11]) If the locally soluble group G=AB with finite Prüfer rank is the product of two subgroups A and B, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B.

Proof : First, let G be a finite p-group for some prime p. If N is a maximal abelian normal subgroup of G, by Lemma 2.18 we have $r(G)\le\frac{1}{2}r(N)(5r(N)+1).$ Hence it is enough to prove that r=r(N) is bounded by a function of the maximum s of r(A) and r(B). The socle S of N is an elementary abelian group of order p'. Clearly it is sufficient to prove the theorem for the factorizer X(S) of S. Therefore we may suppose that the group G has a triple factorization G=AB=AK=BK, where K is an elementary abelian normal subgroup of G of order p'.

Let e be the least positive integer such that A^{p^e} is contained in B. By Lemma 4.3.3 of [4], we have $|A:A\cap B|\le|A:A^{p^e}|\le p^{eg(s)-s^2}$ Where $g(s)=\frac{1}{2}s(3s+1).$ *Since*

$$|G|=\frac{|A|.|B|}{|A\cap B|}=\frac{|B|.|K|}{|B\cap K|},$$

It follows that $|K|=|A:A\cap B|.|B\cap K|\le p^{eg(s)-s^2}p^s=p^{eg(s)-s^2+s}$. Hence $r\le eg(s)-s^2+s\le eg(s)$. Therefore it is enough to show that $e\le g(s)+3.$ Therefore it is enough to show that $e\le g(s)+3$.

Clearly we may suppose that e>1. Let a be an element of A such that $a^{p^{e-1}}$ is not in B, and write $a^{p^{e-1}}$ =xb, with x in K and b in B. Then $[x,a^{p^{e-2}}]\neq 1,$ because otherwise

$$b^p=(x^{-1}a^{p^{e-2}})^p=x^{-p}a^{p^{e-1}}=a^{p^{e-1}},$$

contrary to the choice of a. As K has exponent p, it follows from the usual commutator laws that .

$$[x,a^{p^{e-2}}]=\prod_{i=1}^{p^{e-2}}[x,{}_ia]^{(p^{e_i-2})}=[x,p^e.2a].$$

Thus $[K,\underbrace{G,\ldots,G}_{\leftarrow p^{e-2} \rightarrow}] \neq 1,$ and so |K|> $p^{p^{e-2}}$ since G isa finite p-group. Therefore $p^{p-2} < r \leq eg(s).$ If $e \geq g(s)+4$, then $p^{e-2} \geq 2^{e-2} > (e+1)(e-4) \geq (e+1)g(s) > eg(s)$..

This contradiction shows that $e \leq g(s)+3.$

Suppose now that G=AB is an arbitrary finite soluble group. For each prime p, by Corollary 2.7 there exist Sylow p-subgroups A_p of A and B_p of B such that $G_p=A_pB_p$ is a Sylow p-subgroup of G. As was shown above, $r(G_p)$ is bounded by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by at most f(s) elements. Application of Theorem 4.2.1 of [4] yields that every subgroup of G can be generated by at most f(s)+1 elements, and hence the Prüfer rank of G is bounded by f(s)+1. This proves the theorem is the finite case.

Let G=AB be an arbitrary locally soluble group with finite Prüfer rank. If N is a finite normal subgroup of G, and X=X(N) is its factorizer, then the index $|X : A \cap B|$ *is finite by Lemma 1.1.5. Let Y be the core of* $A \cap B$ *in X* Since the factorized group X/Y is finite, it follows from the first part of the proof that the Prüfer rank of X/Y is bounded by a function of the Prüfer ranks of A and B. As $r(N) \leq r(X) \leq r(Y) + r(X/Y) \leq r(A) + r(X/Y)$(e.g.see Robinson 1972, Part 1,Lemma 1.44) we obtain that there exists a function h such that $r(N) \leq h(r(A),r(B)) = k,$ for every finite normal subgroup N of G. Clearly the same holds for every finite normal section of G.

Let T be the maximum periodic normal subgroup of G. If p is a prime, the group $\overline{T} = T/O_{p'}(T)$ is Chernikov by Lemma 3.2.5 of [4] (See also [16]). Let $\overline{J}$ be the finite residual of $\overline{T}$, and $\overline{S}$ the socle of $\overline{J}$. Since $\overline{S}$ and $\overline{T}/\overline{J}$ are finite, it follows that $r(\overline{T}) \leq r(\overline{J}) + r(\overline{T}/\overline{J}) = r(\overline{S}) + r(\overline{T}/\overline{J}) \leq 2k.$

As the Sylow p-subgroups of T can be embedded in $\overline{T}$, they have Prüfer rank at most 2k. Application of Theorem 4.2.1 of [4] (See also [14]). yields that every finite subgroup of T can be generated by atmost 2k+1 elements. Hence r(T) $\leq 2k+1.$

The group G/T is soluble (See[15]), Part 2, Lemma 10.39), and so the setoff primes $\pi(G/T)$ is finite by Lemma 4.1.5 of [5](See also [15]). It follows from Lemma 4.1.4 of [4] (See also [15]) that there exists in G a normal series of finite length $T \leq G_1 \leq G_2 \leq G,$ where G_1/T is torsion-free nilpotent, G_2/G_1 is torsion-free abelian, and G/G_2 is finite. Therefore

$$\begin{aligned} r(G) &\leq r(T) + r(G_1/T) + r(G_2/G_1) + r(G/G_2) \\ &\leq r(T) + r_0(G) + r(G/G_2) \\ &\leq r_0(G) + 3k + 1. \end{aligned}$$

By theorem 4.1.8 of [4] (See also [3]) we have that $r_0(G) \leq r_0(A) + r_0(B).$

Moreover, $r_0(A) \leq r(A)$ and $r_0(B) \leq r(B)$ by Lemma 4.3.4 of [4] (See also [9]). Therefore $r(G) \leq r(A)+r(B)+3k+1$. The theorem is proved.

2.12.Lemma(See [17]**:** Every finitely generated abelian-by- polycyclic Group is residually finite.

Proof : See ([4], Lemma 4.4.1)

3.MAIN Theorem:

3.1. Theorem: If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Proof : Assume that G it not polycyclic-by-finite. Then G contains an abelian normal section U/V which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of U/V in G/V is also a counterexample. Hence we may suppose that G has a triple factorization G=AB=AK=BK, Where K is an abelian normal subgroup of G which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group G satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup M which is maximal with respect to the condition that G/M is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of G is polycylic-by-finite.

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