α-Dot Interval Valued Fuzzy New Ideal of Pu-Algebra

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Abstract

In this paper, our aim is to introduce and study the notion of an α-dot interval valued fuzzy new-ideal of a PU-algebra. The homomorphic images (pre images) of α-dot interval valued fuzzy new-ideal under homomorphism of a PU-algebras have been obtained and some related results have been derived. Finally, we give the properties of the concept of Cartesian product of an α-dot interval valued fuzzy new-ideal of a PU-algebra.

Keywords. PU-Algebra, A-Dot Interval Valued Fuzzy New-Ideal, The Homomorphic Images (Pre Images) Of A-Dot Interval Valued Fuzzy New -Ideal.

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1- Introduction

In 1966, Imai and Iseki [3, 4, 5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [13], Neggers and Kim introduced the notion of d-algebras, which is a generalization of BCK-algebras and investigated a relation between d-algebras and BCK-algebras. Neggers et al. [14] introduced the notion of Q-algebras, which is a generalization of BCH/BCI/BCK-algebras. Megalai and Tamilarasi [7] introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented. Mostafa et al. [12] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated several basic properties. Moreover they derived new view of several ideals on PU-algebra and studied some properties of them. The concept of fuzzy sets was introduced by Zadeh [17]. In 1991, Xi [16] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has been developed in many directions and is finding applications in a wide variety of fields [6, 8, 9, 10, 11, 15]. Here in this paper, we modify the ideas of Xi [16], to introduce the notion of an α-dot interval valued fuzzy new-ideal of a PU-algebra. The homomorphic image (pre image) of α-dot interval valued fuzzy new-ideal of a PU-algebra under homomorphism of a PU-algebras are discussed. The examples based on tables are not convincing. Examples of concrete algebras of matrices, operators, functions and concrete operations over them would be welcome.

2- Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article.

Definition 2.1 [12] A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(i) $0 * x = x$,

(ii) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X$.

On X we can define a binary relation "$\leq$" by: $x \leq y$ if and only if $y * x = 0$.

Example 2.2 [12] Let X = {0, 1, 2, 3, 4} in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then $(X, *, 0)$ is a PU-algebra.

Proposition 2.3 [12] In a PU-algebra $(X, *, 0)$ the following hold for all $x, y, z, u \in X$:

(a) $x * x = 0$.

(b) $(x * z) * z = x$.

(c) $x * (y * z) = y * (x * z)$.

(d) $x * (y * x) = y * 0$.

(e) $(x * y) * 0 = y * x$.

(f) If $x \leq y$, then $x * 0 = y * 0$.

(g) $(x * y) * 0 = (x * z) * (y * z)$.

(h) $x * y \leq z$ if and only if $z * y \leq x$.

(i) $x \leq y$ if and only if $y * z \leq x * z$.

(j) In a PU-algebra $(X, *, 0)$, the following are equivalent:

(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$.

(k) The right and the left cancellation laws hold in X.

(l) $(z * x) * (z * y) = x * y$,

(m) $(x * y) * z = (z * y) * x$.

(n) $(x * y) * (z * u) = (x * z) * (y * u)$.

Definition 2.4 [12] A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a sub-algebra of X if $x * y \in I$ whenever $x, y \in I$.

Definition 2.5 [12] A non-empty subset I of a PU-algebra (X, $*$, 0) is called a new-ideal of X if,

(i) $0 \in I$,

(ii) $((a * (b * x)) * x) \in I$, for all a, b $\in$I and x$\in$ X.

Example 2.6 [12] Let X = {0, a, b, c} in which $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X,$*$, 0) is a PU-algebra. It is easy to show that I_1 = {0, a}, I_2 = {0, b}, I_3 = {0, c} are new-ideals of X.

Lemma 2.7 [12] If (X, $*$, 0) is a PU-algebra, then (x * (y * z)) * z = (y * 0) * x for all x, y, z$\in$X.

Theorem 2.8 Any sub-algebra S of a PU-algebra X is a new-ideal of X.

Definition 2.9 [12] Let (X, $*$, 0) and (X`, $*$`, 0`) be PU-algebras. A map *f*: X $\rightarrow$ X` is called a homomorphism if *f*(x $*$ y) = *f*(x) $*$` *f*(y) for all x, y$\in$X.

Proposition 2.10 Let (X, $*$, 0) and (X`, $*$`, 0`) be PU-algebras and *f*: X $\rightarrow$ X` be a homomorphism, then Ker *f* is a new-ideal of X.

3- α-Dot interval valued Fuzzy new-ideal of PU-algebra

In this section, we will discuss and investigate a new notion called α-dot interval valued fuzzy new-ideal of a PU-algebra and study several basic properties which related to α-dot interval valued fuzzy new-ideal.

Definition 3.1 [17] Let X be a non-empty set, a fuzzy subset μ in X is a function μ: X $\rightarrow$ [0, 1].

Definition 3.2 [16] An interval-valued fuzzy subset (briefly i-v fuzzy subset) A defined in the set X is given by A = {(x, [μ_A^L(x), μ_A^U(x)])}, for all x$\in$X (briefly, it is denoted by A = [μ_A^L(x), μ_A^U(x)] where μ_A^L(x) and μ_A^U(x) are any two fuzzy subsets in X such that μ_A^L(x) $\le$ μ_A^U (x) for all x$\in$X. Let $\tilde{\mu}_A$(x)= [μ_A^L(x), μ_A^U(x)], for all x$\in$X and let D[0, 1] be denotes the family of all closed sub-intervals of [0, 1]. It is clear that if μ_A^L(x) = μ_A^U(x) = c, where 0 $\le$ c $\le$ 1, then $\tilde{\mu}_A$(x)=[c, c] in D[0, 1], then $\tilde{\mu}_A$(x)$\in$D[0, 1], for all x$\in$X. Therefore the i-v fuzzy subset A is given by:

$$A = \{(x, \tilde{\mu}_A)\}, \text{ for all } x\in X, \text{ where } \tilde{\mu}_A: X \rightarrow D[0,1].$$

Now we define the refined minimum (briefly r min) and order "≤" on elements D_1= [a_1, b_1] and D_2= [a_2, b_2] of D[0, 1] as follows:

$$\text{rmin}(D_1, D_2)= [\min\{a_1, a_2\}, \min\{b_1, b_2\}],\ D_1 \le D_2 \Leftrightarrow a_1 \le a_2 \text{ and } b_1 \le b_2.$$

Similarly we can define (≥) and (=).

Also we can define $D_1 + D_2 = [a_1 + a_2, b_1 + b2]$, and if $c \in [0,1]$, then $cD_1 = [ca_1, cb_1]$. Also if $D_i = [a_i, b_i]$, $i \in I$ then we define

$$\text{rsup}\,(D_i) = [\sup a_i, \sup b_i] \text{ and } \text{rinf}\,(Di) = [\inf a_i, \inf b_i].$$

We will consider that $\widetilde{1}=[1, 1]$ and $\widetilde{0}=[0, 0]$.

In the sequel, let X denotes a PU-algebra unless otherwise specified, we begin with the following definition.

Definition 3.3 Let X be a PU-algebra. An interval valued fuzzy subset $\tilde{\mu}$ in X is called an interval valued fuzzy sub-algebra of X if $\tilde{\mu}(x * y) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$

for all x, y∈X.

Definition 3.4 Let $\tilde{\mu}$ be an interval valued fuzzy subset of a PU-algebra X. Let $\alpha \in [0, 1]$. Then the interval valued fuzzy set $\tilde{\mu}^{\alpha}$ of X is called the α-dot interval valued fuzzy subset of X (w. r. t. interval valued fuzzy set $\tilde{\mu}$) and is defined by:
$\tilde{\mu}^{\alpha}(x) = \tilde{\mu}(x) \bullet \alpha$, for all $x \in X$, $\alpha \in [0, 1]$.

Remark 3.5 $\tilde{\mu}^{1} = \tilde{\mu}$ and $\tilde{\mu}^{0} = \tilde{0}$.

Lemma 3.6 If $\tilde{\mu}$ is an interval valued fuzzy sub-algebra of a PU-algebra X and $\alpha \in [0, 1]$, then $\tilde{\mu}^{\alpha}(x * y) \geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}$, for all x, y ∈X.

Proof. Let X be a PU-algebra and $\alpha \in [0,1]$. Then by Definition 3.4, we have that

$$\tilde{\mu}^{\alpha}((x * y) = \tilde{\mu}\big((x * y)\big) \bullet \alpha$$

$$\geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \bullet \alpha$$

$$\geq r\min\{\tilde{\mu}(x) \bullet \alpha, \tilde{\mu}(y) \bullet \alpha\}$$

$$\geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}, \text{ for all x, y, z} \in X.$$

Definition 3.7 Let X be a PU-algebra. An interval valued fuzzy subset $\tilde{\mu}^{\alpha}$ in X is called an α-dot interval valued fuzzy sub-algebra of X if $\tilde{\mu}^{\alpha}(x * y) \geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}$, for all x, y ∈X.

It is clear that an α-dot interval valued fuzzy sub-algebra of a PU-algebra X is a generalization of an interval valued fuzzy sub-algebra of X and an interval valued fuzzy sub-algebra of X is an α-dot interval valued fuzzy sub-algebra of X in case of α=1.

Definition 3.8 Let (X, ∗, 0) be a PU-algebra, an interval valued fuzzy subset $\tilde{\mu}$ in X is called an interval valued fuzzy new-ideal of X if it satisfies the following conditions:

$(\widetilde{F_1})$ $\tilde{\mu}(0) \geq \tilde{\mu}(x)$,

$(\widetilde{F_2})$ $\tilde{\mu}((x * (y * z) * z) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$, for all x, y, z∈X.

Lemma 3.9 If $\tilde{\mu}$ is an interval valued fuzzy new-ideal of a PU-algebra X and $\alpha \in [0,1]$. Then

$(\widetilde{F_1}^{\tilde{\alpha}})$ $\tilde{\mu}^{\alpha}(0) \geq \tilde{\mu}^{\alpha}(x)$,

$(\widetilde{F_2}^{\widetilde{\alpha}})$ $\tilde{\mu}^{\alpha}((x * (y * z) * z) \geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}$, for all x, y, z∈X.

Proof. Let X be a PU-algebra and α∈[0,1]. Then by Definition 3.4 and Definition 3.8, we have that:

$(\widetilde{F_1}^{\widetilde{\alpha}})$ $\tilde{\mu}^{\alpha}(0) = \tilde{\mu}(0) \bullet \alpha \geq \tilde{\mu}(x) \bullet \alpha = \tilde{\mu}^{\alpha}(x), for\ all\ x \in X.$

$(\widetilde{F_2}^{\widetilde{\alpha}})$ $\tilde{\mu}^{\alpha}((x * (y * z)) * z) = \tilde{\mu}((x * (y * z)) * z)) \bullet \alpha$

$$\geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \bullet \alpha$$

$$\geq r\min\{\tilde{\mu}(x) \bullet \alpha, \tilde{\mu}(y) \bullet \alpha\}$$

$$\geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}$$, for all x, y, z∈X.

Definition 3.10 Let (X, ∗, 0) be a PU-algebra, an interval valued fuzzy subset $\tilde{\mu}^{\alpha}$ in X is called an α-dot interval valued fuzzy new-ideal of X if it satisfies the following conditions:

$(\widetilde{F_1}^{\widetilde{\alpha}})$ $\tilde{\mu}^{\alpha}(0) \geq \tilde{\mu}^{\alpha}(x)$,

$(\widetilde{F_2}^{\widetilde{\alpha}})$ $\tilde{\mu}^{\alpha}((x * (y * z) * z) \geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(y)\}$, for all x, y, z∈X.

It is clear that an α-dot interval valued fuzzy new-ideal of a PU-algebra X is a generalization of an interval valued fuzzy new-ideal of X and an interval valued fuzzy new-ideal of X is special case, when α=1

Example 3.11 Let X = {0, 1, 2, 3} in which ∗ is defined by the following table:

∗	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X, ∗, 0) is a PU-algebra.

Define an $\tilde{\alpha}$-interval valued fuzzy subset $\tilde{\mu}^{\alpha}$: X → D[0,1] by

$$\tilde{\mu}^{\alpha}(x) = \begin{cases} \alpha \bullet [0.3, 0.9]\} & if \quad x \in \{0,1\} \\ \alpha \bullet [0.1, 0.6]\} & otherwise \end{cases}$$

Routine calculation gives that $\tilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X.

Lemma 3.12 Let $\tilde{\mu}^{\alpha}$ be an α-dot interval valued fuzzy new-ideal of a PU-algebra X. If the inequality x * y ≤ z holds in X, then $\tilde{\mu}^{\alpha}(y) \geq r\min\{\tilde{\mu}^{\alpha}(x), \tilde{\mu}^{\alpha}(z)\}$.

Proof. Assume that the inequality x * y ≤ z holds in X, then z * (x * y) = 0 and by $(\widetilde{F}_2^{\widetilde{\alpha}})$ $\widetilde{\mu}^{\alpha}((z * (x * y)) * y) \geq r\min\{\widetilde{\mu}^{\alpha}(x), \widetilde{\mu}^{\alpha}(z)\}$. Since $\widetilde{\mu}^{\alpha}(y) = \widetilde{\mu}^{\alpha}(0 * y)$, then we have that $\widetilde{\mu}^{\alpha}(y) \geq r\min\{\widetilde{\mu}^{\alpha}(x), \widetilde{\mu}^{\alpha}(z)\}$.

Corollary 3.13 Let $\widetilde{\mu}$ be an interval valued fuzzy new-ideal of a PU-algebra X. If the inequality x * y ≤ z holds in X, then $\widetilde{\mu}$ (y) ≥ r min {$\widetilde{\mu}$(x), $\widetilde{\mu}$(z)}.

Lemma 3.14 If $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy subset of a PU-algebra X and if x≤y, then $\widetilde{\mu}^{\alpha}$(x)= $\widetilde{\mu}^{\alpha}$(y).

Proof. If x≤y, then y * x = 0. Hence by the definition of PU-algebra and its properties, we have that

$$\widetilde{\mu}^{\alpha}(x) = \widetilde{\mu}(x)\bullet\alpha = \widetilde{\mu}(0 * x)\bullet\alpha = \widetilde{\mu}((y * x) * x)\bullet\alpha = \widetilde{\mu}(y)\bullet\alpha = \widetilde{\mu}^{\alpha}(y).$$

Corollary 3.15 If $\widetilde{\mu}$ is an interval valued fuzzy subset of a PU-algebra X and if x≤y, then $\widetilde{\mu}$(x)= $\widetilde{\mu}$ (y).

Definition 3.16 Let $\widetilde{\mu}^{\alpha}$ be an α-dot interval valued fuzzy new-ideal of a PU-algebra X and let x be an element of X. We define $(\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha}) = r\inf\,(\widetilde{\mu}_i^{\alpha}(x))_{i\in I}$

Proposition 3.17 The intersection of any set of α-dot interval valued fuzzy new-ideals of a PU-algebra X is also an α-dot interval valued fuzzy new-ideal of X.

Proof. Let $\{\widetilde{\mu}_i^{\alpha}\}_{i\in I}$ be a family of α-dot interval valued fuzzy new-ideals of a PU-algebra X, then for any x, y, z∈X, $(\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha})(0) = r\inf(\widetilde{\mu}_i^{\alpha}(0))_{i\in I} \geq r\inf(\widetilde{\mu}_i^{\alpha}(x))_{i\in I} = (\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha})(x)$

and

$$(\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha})((x*(y*z))*z) = r\inf(\widetilde{\mu}_i^{\alpha}((x*(y*z))*z)_{i\in I}$$

$$\geq r\inf(r\min\{\widetilde{\mu}_i^{\alpha}(x), \widetilde{\mu}_i^{\alpha}(y)\})_{i\in I}$$

$$= r\min\{r\inf(\widetilde{\mu}_i^{\alpha}(x))_{i\in I}, r\inf(\widetilde{\mu}_i^{\alpha}(y))_{i\in I}\}$$

$$= r\min\{(\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha})(x), (\bigcap_{i\in I}\widetilde{\mu}_i^{\alpha})(y)\}.$$

This completes the proof.

Theorem 3.18 Let $\widetilde{\mu}^{\alpha}$ be an α-dot interval valued fuzzy subset of a PU-algebra X. Then $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X if and only if it satisfies:
$(\forall\tilde{\mathcal{E}} \in D[0,1])\ (U(\widetilde{\mu}^{\alpha};\ \tilde{\mathcal{E}}) \neq \emptyset \Longrightarrow\ U(\widetilde{\mu}^{\alpha};\ \tilde{\mathcal{E}})$ is a new-ideal of X), where
$U(\widetilde{\mu}^{\alpha};\ \tilde{\mathcal{E}}) = \{x \in X : \widetilde{\mu}^{\alpha}(x) \geq \tilde{\mathcal{E}}\}$.

Proof. Assume that $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X. Let $\widetilde{\varepsilon} \in D[0,1]$ be such that $U(\widetilde{\mu}^{\widetilde{\alpha}};\widetilde{\varepsilon}) \neq \phi$. Let $x \in U(\widetilde{\mu}^{\alpha};\widetilde{\varepsilon})$, then $\widetilde{\mu}^{\alpha}(x) \geq \widetilde{\varepsilon}$. Since $\widetilde{\mu}^{\alpha}(0) \geq \widetilde{\mu}^{\alpha}(x)$ for all x∈X, then $\widetilde{\mu}^{\alpha}(0) \geq \widetilde{\varepsilon}$. Thus $0 \in U(\widetilde{\mu}^{\alpha};\widetilde{\varepsilon})$. Let x∈X and $a, b \in U(\widetilde{\mu}^{\alpha};\widetilde{\varepsilon})$, then $\widetilde{\mu}^{\alpha}(a) \geq \widetilde{\varepsilon}$ and $\widetilde{\mu}^{\alpha}(b) \geq \widetilde{\varepsilon}$. It follows by the definition of α-dot interval valued fuzzy new-ideal that $\tilde{\mu}^{\alpha}((a*(b*x))*x) \geq r\min\{\tilde{\mu}^{\alpha}(a), \tilde{\mu}^{\alpha}(b)\} \geq \tilde{\varepsilon}$, so that $(a*(b*x))*x \in U(\tilde{\mu}^{\alpha}; \tilde{\varepsilon})$. Hence $U(\tilde{\mu}^{\alpha}; \tilde{\varepsilon})$ is a new-ideal of X.

Conversely, suppose that $(\forall \tilde{\varepsilon} \in D[0,1]), (U(\tilde{\mu}, \tilde{\varepsilon}) \neq \emptyset \Rightarrow U(\tilde{\mu}, \tilde{\varepsilon})$ is a new-ideal of X), where $U(\tilde{\mu}^{\alpha}; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}^{\alpha}(x) \geq \tilde{\varepsilon}\}$. If $\tilde{\mu}^{\tilde{\alpha}}(0) < \tilde{\mu}^{\tilde{\alpha}}(x)$ for some x∈X, then $\tilde{\mu}^{\alpha}(0) < \tilde{\varepsilon}_0 < \tilde{\mu}^{\alpha}(x)$ by taking $\tilde{\varepsilon}_0 = (\tilde{\mu}^{\alpha}(0) + \tilde{\mu}^{\alpha}(x))/2$. Hence $0 \notin U(\tilde{\mu}^{\alpha}; \tilde{\varepsilon}_0)$, which is a contradiction.

Let a, b, c∈X be such that $\tilde{\mu}^{\alpha}((a*(b*c))*c) < r\min\{\tilde{\mu}^{\alpha}(a), \tilde{\mu}^{\alpha}(b)\}$. Taking $\tilde{\varepsilon}_1 = (\tilde{\mu}^{\alpha}(a*(b*c)*c) + r\min\{\tilde{\mu}^{\alpha}(a), \tilde{\mu}^{\alpha}(b)\})/2$, we have $\tilde{\varepsilon}_1 \in D[0,1]$ and $\tilde{\mu}^{\alpha}((a*(b*c))*c) < \tilde{\varepsilon}_1 < r\min\{\tilde{\mu}^{\alpha}(a), \tilde{\mu}^{\alpha}(b)\}$. It follows that $a, b \in U(\tilde{\mu}^{\tilde{\alpha}}; \tilde{\varepsilon}_1)$ and $(a*(b*c))*c \notin U(\tilde{\mu}^{\alpha}; \tilde{\varepsilon}_1)$. This is a contradiction, and therefore $\tilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X.

Corollary 3.19 Let $\tilde{\mu}$ be an interval valued fuzzy subset of a PU-algebra X. Then $\tilde{\mu}$ is an interval valued fuzzy new-ideal of X if and only if it satisfies:

$(\forall \tilde{\varepsilon} \in D[0,1]), (U(\tilde{\mu}, \tilde{\varepsilon}) \neq \emptyset \Rightarrow U(\tilde{\mu}, \tilde{\varepsilon})$ is a new-ideal of X), where $U(\tilde{\mu}; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}(x) \geq \tilde{\varepsilon}\}$.

Definition 3.20 Let *f* be a mapping from X to Y. If $\tilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy subset of X, then the α-dot interval valued fuzzy subset $\tilde{\beta}^{\alpha}$ of Y defined by

$$f(\tilde{\mu}^{\alpha})(y) = \tilde{\beta}^{\alpha}(y) = \begin{cases} r\sup\limits_{x \in f^{-1}(y)} \tilde{\mu}^{\alpha}(x) & if \quad f^{-1}(y) \neq \phi \\ \tilde{0} & otherwise \end{cases}$$

is said to be the image of $\tilde{\mu}^{\alpha}$ under *f*.

Similarly if $\tilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy subset of Y, then the α-dot interval valued fuzzy subset $\tilde{\mu}^{\alpha}$=($\tilde{\beta}^{\alpha} \circ f$) of X (i.e. the α-dot interval valued fuzzy subset defined by $\tilde{\mu}^{\alpha}$(x) = $\tilde{\beta}^{\alpha}$(*f*(x)) for all x∈X) is called the pre-image of $\tilde{\beta}^{\alpha}$ under *f*.

Theorem 3.21 Let (X, ∗, 0) and (X`, ∗`, 0`) be PU-algebras and *f*: X → X` be a homomorphism. If $\tilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X` and $\tilde{\mu}^{\alpha}$ is the pre-image of $\tilde{\beta}^{\alpha}$ under f, then $\tilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X.

Proof. Since $\tilde{\mu}^{\alpha}$ is the pre-image of $\tilde{\beta}^{\alpha}$ under *f*, then $\tilde{\mu}^{\alpha}(x) = \tilde{\beta}^{\alpha}(f(x))$ for all x∈X. Let x∈X, then $\tilde{\mu}^{\alpha}(0) = \tilde{\beta}^{\alpha}(f(0)) \geq \tilde{\beta}^{\alpha}(f(x)) = \tilde{\mu}^{\alpha}(x)$. Now let x, y, z∈X then $\tilde{\mu}^{\alpha}((x*(y*z))*z) =$
$\tilde{\beta}^{\alpha}(f((x*(y*z))*z))$

$$= \tilde{\beta}^{\alpha}(f(x*(y*z))*^{\backslash} f(z))$$

$$= \tilde{\beta}^{\alpha}((f(x)*^{\backslash} f(y*z))*^{\backslash} f(z))$$

$$= \tilde{\beta}^{\alpha}((f(x)*^{\backslash}(f(y)*^{\backslash} f(z)))*^{\backslash} f(z))$$

$$\geq r\min\{\widetilde{\beta}^{\widetilde{\alpha}}(f(x)), \widetilde{\beta}^{\widetilde{\alpha}}(f(y))\}$$

$= r\min\{\widetilde{\mu}^{\alpha}(x), \widetilde{\mu}^{\alpha}(y)\}$, then the proof is completed.

Theorem 3.22 Let (X, ∗, 0) and (Y, ∗\`, 0\`) be PU-algebras. Let f: X → Y be a homomorphism, $\widetilde{\mu}^{\alpha}$ be an α-dot interval valued fuzzy subset of X and $\widetilde{\beta}^{\alpha}$ be the image of $\widetilde{\mu}^{\alpha}$ under f. If $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X, then $\tilde{\beta}^{\alpha}$

is an α-dot interval valued fuzzy new-ideal of Y.

Proof. Since $0 \in f^{-1}(\dot{0})$, then $f^{-1}(\dot{0}) \neq \emptyset$. It follows that $\widetilde{\beta}^{\alpha}(\dot{0}) = r\ sup_{t\in f^{-1}(\dot{0})}\widetilde{\mu}^{\alpha}(t) = \widetilde{\mu}^{\alpha}(0) \geq \widetilde{\mu}^{\alpha}(x), \forall x \in X$.

Thus $\widetilde{\beta}^{\alpha}(\dot{0}) = r\ sup_{t\in f^{-1}(\dot{x})}\widetilde{\mu}^{\alpha}(t), \forall \dot{x} \in Y$. Hence $\tilde{\beta}^{\alpha}(\dot{0}) \geq \tilde{\beta}^{\alpha}(\dot{x}), \forall \dot{x} \in Y$.

For any $\dot{x}, \dot{y}, \dot{z} \in Y$, if $f^{-1}(\dot{x}) \neq \emptyset$ or $f^{-1}(\dot{y}) \neq \emptyset$, then $\widetilde{\beta}^{\alpha}(\dot{x}) = \widetilde{0}$ or $\widetilde{\beta}^{\alpha}(\dot{y}) = \widetilde{0}$. It follows that $r\ min\{\tilde{\beta}^{\alpha}(\dot{x}), \tilde{\beta}^{\alpha}(\dot{y})\} = \tilde{0}$ and hence $\widetilde{\beta}^{\alpha}((\dot{x} \dot{*} (\dot{y} \dot{*} \dot{z})) \dot{*} \dot{z}) \geq r\ min\{\widetilde{\beta}^{\alpha}(\dot{x}), \widetilde{\beta}^{\alpha}(\dot{y})\}$.

If $f^{-1}(\dot{x}) \neq \emptyset$ and $f^{-1}(\dot{y}) \neq \emptyset$, let $x_0 \in f^{-1}(\dot{x})$, $y_0 \in f^{-1}(\dot{y})$ be such that $\widetilde{\mu}^{\alpha}(x_0) = r\ sup_{t\in f^{-1}(\dot{x})}\widetilde{\mu}^{\alpha}(t)$ and $\widetilde{\mu}^{\alpha}(y_0) = r\ sup_{t\in f^{-1}(\dot{y})}\widetilde{\mu}^{\alpha}(t)$. It follows by given and properties of PU-algebra that

$$\widetilde{\beta}^{\alpha}\left((\dot{x} \dot{*} (\dot{y} \dot{*} \dot{z})) \dot{*} \dot{z}\right) = \widetilde{\beta}^{\alpha}((z` *`(y` *`z`)) *`x`)$$

$$= \tilde{\beta}^{\alpha}\left((\dot{y} \dot{*} (\dot{z} \dot{*} \dot{z})) \dot{*} \dot{x}\right) = \tilde{\beta}^{\alpha}((y` \dot{*} 0`) *`x`)$$

$$= \tilde{\beta}^{\alpha}((f(y_0) \dot{*} f(o)) \dot{*} f(x_0)) = \tilde{\beta}^{\alpha}(f(y_0 * 0) * x_0)$$

$$= \tilde{\mu}^{\alpha}((y_0 * 0) * x_0) = \tilde{\mu}^{\alpha}((y_0 * (z_0 * z_0)) * x_0)$$

$$= \tilde{\mu}^{\alpha}((z_0 * (y_0 * z_0)) * x_0) = \tilde{\mu}^{\alpha}((x_0 * (y_0 * z_0)) * z_0)$$

$$\geq r\ min\{\tilde{\mu}^{\alpha}(x_0), \tilde{\mu}^{\alpha}(y_0)\} = r\ min\{r\sup_{t\in f^{-1}(x)}\tilde{\mu}^{\alpha}(t), r\sup_{t\in f^{-1}(y)}\tilde{\mu}^{\alpha}(t)\}$$

$$= r\ min\{\tilde{\beta}^{\alpha}(\dot{x}), \tilde{\beta}^{\alpha}(\dot{y})\}.$$

Hence $\tilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of Y.

Corollary 3.23 Let (X, ∗, 0) and (Y, ∗\`, 0\`) be PU-algebras, f: X → Y be a homomorphism, $\widetilde{\mu}$ be an interval valued fuzzy subset of X, $\widetilde{\beta}$ be the image of $\widetilde{\mu}$ under f. If $\widetilde{\mu}$ is an interval valued fuzzy new-ideal of X, then $\widetilde{\beta}$ is an interval valued fuzzy new-ideal of Y.

4- Cartesian Product of α-dot Interval valued Fuzzy new-ideals of PU-algebras

In this section, we introduce the concept of Cartesian product of an α-dot interval valued fuzzy new-ideal of a PU-algebra.

Definition 4.1 An α-dot interval valued fuzzy relation on any set S is an α-dot interval valued fuzzy subset $\widetilde{\mu}^{\alpha}$: S×S → D[0, 1].

Definition 4.2 If $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy relation on a set S and $\widetilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy subset of S, then $\widetilde{\mu}^{\alpha}$ is an α-dot interval valued fuzzy relation on $\widetilde{\beta}^{\alpha}$ if $\widetilde{\mu}^{\alpha}$ (x, y) ≤ r min {$\widetilde{\beta}^{\alpha}$ (x), $\widetilde{\beta}^{\alpha}$(y)}, for all x, y∈S.

Definition 4.3 If $\widetilde{\beta}^{\alpha}$ is an $\widetilde{\alpha}$-interval valued fuzzy subset of a set S, the strongest $\widetilde{\alpha}$-interval valued fuzzy relation on S that is an α-dot interval valued fuzzy relation on $\widetilde{\beta}^{\alpha}$ is $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}$ given by

$\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x,y) = r\ min\left\{\widetilde{\beta}^{\alpha}(x),\widetilde{\beta}^{\alpha}(y)\right\}$ for all x, y∈S.

Definition 4.4 We define the binary operation * on the Cartesian product X×X as follows:

$$(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2) \text{ for all } (x_1, x_2), (y_1, y_2)\in X\times X.$$

Lemma 4.5 If (X, ∗, 0) is a PU-algebra, then (X×X, *, (0, 0)) is a PU-algebra, where

$$(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2) \text{ for all } (x_1, x_2), (y_1, y_2)\in X\times X.$$

Proof. Clear.

Theorem 4.6 Let $\widetilde{\beta}^{\alpha}$ be an α-dot interval valued fuzzy subset of a PU-algebra X and $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}$ be the strongest α-dot interval valued fuzzy relation on X, then $\widetilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X if and only if $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}$ is an α-dot interval valued fuzzy new-ideal of X×X.

Proof. (⇒): Assume that $\widetilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X, we note from $(\widetilde{F}_1^{\alpha})$ that:

$$\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(0,0) = r\min\{\widetilde{\beta}^{\alpha}(0),\widetilde{\beta}^{\alpha}(0)\} \geq r\min\{\widetilde{\beta}^{\alpha}(x),\widetilde{\beta}^{\alpha}(y)\} = \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x,y) \text{ for all } x, y\in X$$

Now, for any $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$, we have from $(\widetilde{F}_2^{\alpha})$:

$$\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2)) =$$

$$\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2)) = \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}((x_1*(y_1*z_1),x_2*(y_2*z_2))*(z_1,z_2)) =$$

$$\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2) = r\min\{\widetilde{\beta}^{\alpha}((x_1*(y_1*z_1))*z_1),\widetilde{\beta}^{\alpha}((x_2*(y_2*z_2))*z_2)\} \geq$$
$$r\min\{r\min\{\widetilde{\beta}^{\alpha}(x_1),\widetilde{\beta}^{\alpha}(y_1)\},r\min\{\widetilde{\beta}^{\alpha}(x_2),\widetilde{\beta}^{\alpha}(y_2)\}\} =$$
$$r\min\{r\min\{\widetilde{\beta}^{\alpha}(x_1),\widetilde{\beta}^{\alpha}(x_2)\},r\min\{\widetilde{\beta}^{\alpha}(y_1),\widetilde{\beta}^{\alpha}(y_2)\}\} =$$

$$r\min\{\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x_1,x_2),\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(y_1,y_2)\}.$$

Hence $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}$is an α-dot interval valued fuzzy new-ideal of $X\times X$.

(⇐): For all (x, x)∈X×X, we have $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(0,0) = r\ min\left\{\widetilde{\beta}^{\alpha}(0),\widetilde{\beta}^{\alpha}(0)\right\} \geq \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x,x)$.

Then $\widetilde{\beta}^{\widetilde{\alpha}}(0) = r\min\{\widetilde{\beta}^{\alpha}(0), \widetilde{\beta}^{\alpha}(0)\} \geq r\min\{\widetilde{\beta}^{\alpha}(x), \widetilde{\beta}^{\alpha}(x)\} = \widetilde{\beta}^{\alpha}(x)$ for all x∈X

Now, for all x, y, z∈X we have

$$\widetilde{\beta}^{\alpha}((x*(y*z))*z) = r\min\{\widetilde{\beta}^{\alpha}((x*(y*z))*z), \widetilde{\beta}^{\alpha}((x*(y*z))*z)\}$$

$$= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}((x*(y*z))*z, (x*(y*z))*z)$$

$$= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}((x*(y*z), x*(y*z))*(z,z))$$

$$= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x,x)*((y*z),(y*z)))*(z,z))$$

$$= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x,x)*((y,y)*(z,z)))*(z,z))$$

$$\geq r\min\{\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x,x), \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(y,y)\}$$

$$= r\min\{r\min\{\widetilde{\beta}^{\alpha}(x), \widetilde{\beta}^{\alpha}(x)\}, r\min\{\widetilde{\beta}^{\alpha}(y), \widetilde{\beta}^{\alpha}(y)\}\}$$

$$= r\min\{\widetilde{\beta}^{\alpha}(x), \widetilde{\beta}^{\alpha}(y)\}.$$

Hence $\widetilde{\beta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal of X.

Definition 4.7 Let $\widetilde{\mu}$ and $\widetilde{\delta}$ be the interval valued fuzzy subsets in X. The Cartesian product $\widetilde{\mu}\times\widetilde{\delta}$: X×X → D[0, 1] is defined by

$$(\widetilde{\mu}\times\widetilde{\delta})(x, y)=r\min\{(\widetilde{\mu})(x), (\widetilde{\delta})(y)\}, \text{ for all } x, y\in X$$

Definition 4.8 Let $\widetilde{\mu}^{\alpha}$ and $\widetilde{\delta}^{\alpha}$ be the α-dot interval valued fuzzy subsets in X. The Cartesian product $\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha}$: X×X → D[0, 1] is defined by

$$(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(x, y)=r\min\{(\widetilde{\mu}^{\alpha})(x), (\widetilde{\delta}^{\alpha})(y)\}, \text{ for all } x, y\in X.$$

Theorem 4.9 If $\widetilde{\mu}^{\alpha}$ and $\widetilde{\delta}^{\alpha}$ are α-dot interval valued fuzzy new-ideals in a PU-algebra X, then $\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal in X × X.

Proof. $(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(0,0) = r\min\{\widetilde{\mu}^{\alpha}(0), \widetilde{\delta}^{\alpha}(0)\} \geq r\min\{\widetilde{\mu}^{\alpha}(x_1), \widetilde{\delta}^{\alpha}(x_2)\}$

= $(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(x_1, x_2)$ for all $(x_1, x_2)\in X \times X$.

Let (x_1, x_2), (y_1, y_2), $(z_1, z_2)\in X \times X$. Then we have that $(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2)) = (\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(((x_1,x_2)*(y_1*z_1, y_2*z_2))*(z_1,z_2)) =$

$(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})((x_1*(y_1*z_1), x_2*(y_2*z_2))*(z_1,z_2)) =$

$$(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)=$$
$$r\min\{\widetilde{\mu}^{\alpha}(x_1*(y_1*z_1))*z_1),\widetilde{\delta}^{\alpha}(x_2*(y_2*z_2))*z_2)\}\geq$$
$$r\min\{r\min\{\widetilde{\mu}^{\alpha}(x_1),\widetilde{\mu}^{\alpha}(y_1)\},r\min\{\widetilde{\delta}^{\alpha}(x_2),\widetilde{\delta}^{\alpha}(y_2)\}\}=r\min\{(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(x_1,x_2),(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(y_1,y_2)\}.$$

Therefore $\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha}$ is an α-dot interval valued fuzzy new-ideal in X × X.

5- Conclusions

In the present paper, we have introduced the concept of $\widetilde{\alpha}$-interval valued fuzzy new-ideal of PU-algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS -algebras, Q-algebras, SU-algebras, IS-algebras, β-algebras and semirings. It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of PU-algebras, may be the following topics can be considered:

(1) To establish the interval value, bipolar and intuitionistic α-fuzzy new-ideal in PU-algebras.

(2) To consider the structure of ($\widetilde{\tau}$, $\widetilde{\rho}$)-interval valued α-fuzzy new-ideal of PU-algebras.

(3) To get more results in $\widetilde{\tau}$-cubic α-fuzzy new-ideal of PU-algebras and it's application.

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Algorithms for PU-algebra

Input (X: set with 0 element, *: Binary operation)

Output ("X is a PU-algebra or not")

If X =∅ then;

Go to (1.)

End if

If 0∉X then go to (1.);

End If

Stop: = false

i = 1;

While i ≤|X| and not (Stop) do

If 0 * $x_i \neq x_i$, then

Stop: = true

End if

j = 1;

While j ≤|X| and not (stop) do

k = 1

While k ≤|X| and not (stop) do

If $(x_i * x_k)* (x_j * x_k) \neq xj * x_i$, then

Stop: = true

End if

End while

End if

End while

If stop then

Output ("X is a PU-algebra")

Else

(1.) Output ("X is not a PU-algebra")

End if

End.

Algorithms for PU-ideal in PU-algebra

Input (X: PU-algebra, I: subset of X)

Output ("I is a PU-ideal of X or not")

If I=∅ then

Go to (1.);

End if

If 0∉ I then

Go to (1.);

End if

Stop: = false

i = 1;

While i $\leq|X|$ and not (stop) do

j = 1

While j $\leq|X|$ and not (stop) do

k = 1

While k $\leq|X|$ and not (stop) do

If $x_j * x_i \in I$, and $x_i * x_k \in I$ then

If $x_j * x_k \notin I$ then

Stop: = false

End if

End while

End while

End while

If stop then

Output ("I is a PU-ideal of X")

Else

(1.) Output ("I is not ("I is a PU-ideal of X")

End if

End.

Algorithm for fuzzy subsets

Input (X: PU-algebra, A: $X \rightarrow [0. 1]$);

Output (“A is a fuzzy subset of X or not”)

Begin

Stop: =false;

i=1;

While i $\leq|X|$ and not (Stop) do

If $(A(x_i) < 0)$ or $(A(x_i) > 1)$ then

Stop: = true;

End If

End While

If Stop then

Output ("A is a fuzzy subset of X")

Else

Output ("A is not a fuzzy subset of X")

End If

End.

Algorithm for fuzzy subsets Algorithm fuzzy new-ideal

Input (X : PU-algebra,I subset of X);

Output ("I is an new-ideal of X or not");

Begin

If I= ∅ then go to (1.);

End If

If $0 \notin I$ then go to (1.);

End If

Stop: =false;

i = 1;

While i ≤|X| and not (stop) do

j = 1

While j ≤|X| and not (stop) do

k = 1

While k ≤|X| and not (stop) do

If x_i , $x_j \in I$, and $x_k \in X$ then

If $(x_j *(x_j * x_k)) * x_k \notin I$ then

Stop: = true;

End If

End If

End While

End While

End While

If Stop then

Output (“I is new-ideal of X”)

Else (1.) Output (“I is not new-ideal of X”)

End If

End

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