Minimizing and Evaluating Weighted Means of Special Mappings

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Abstract

A constrained optimization problem is solved, as an application of minimum principle for a sum of strictly concave continuous functions, subject to a linear constraint, firstly for finite sums of elementary such functions. The motivation of solving such problems is minimizing and evaluating the (unknown) mean of a random variable, in terms of the (known) mean of another related random variable. The corresponding result for infinite sums of such type of functions follows as a consequence, passing to the limit. Note that in our statements and proofs the condition $\sum_j p_j = 1$ on the positive numbers p_j is not essential for the interesting part of the results. So, our work refers not only to means of random variables, but to more general weighted means. A related example is given. A corresponding result for special concave mappings taking values into an order-complete Banach lattice of self-adjoint operators is also proved. Namely, one finds a lower bound for a sum of special concave mappings with ranges in the above mention order-complete Banach lattice, under a suitable linear constraint.

Keywords: Constrained Minimization; Concave Objective-Mappings; Minimum Principle; Optimization/Evaluation of Weighted Means; Operator-Valued Concave Mappings

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1 Introduction

Constrained optimization problems are basic in solving real-life problems, as discussed in [1]. Further results on this subject have been revealed recently in [2], [3]. The background of the present work is contained in some chapters of [4]-[8]. In a way, we continue our study started in [9]. For constraint optimization problems related to Markov moment problem see [10], [11]. Other aspects of optimization theory have been recently recalled in [12]. All these results concern mainly one dimensional valued objective functions. For optimization of vector valued mappings see [13], [14], where applications to real-life problems are pointed out. Finally, in [15], the geometric meaning of the maximum principle is used in order to generalize the notion of a convex function. Using Carathéodory's theorem [5], it is known that the minimum of any continuous concave function on a compact convex finite dimensional subset is attained at an extreme point of that set. A similar result remains true for unbounded closed convex finite dimensional subsets A having extreme points, and for concave continuous functions on A which attain their minimum on A at a point of A, as discussed in [6]. Applying such type results and other preliminaries, we solve a constrained optimization problem involving finite sums of special strictly concave elementary functions, subject to a linear constraint. The case of infinite sums is deduced as well. In the end of Section 2, a self-adjoint operator-valued case for special concave mappings and its particular case of symmetric matrix-valued mappings is discussed. The latter discrete case is closely related to computational problems. For the background related to the operator-valued case see [7], [8]. The rest of the paper is organized as follows. In Section 2 minimizing and evaluating finite and infinite sums of special elementary strictly concave mappings, under a liner constraint is discussed. In the end of this section, an operator-valued case is pointed out. In particular, one can deduce the corresponding result for concave mappings taking values in an order-complete Banach lattice of symmetric matrices with real entries. To apply such a result, specific computations involving positive definite symmetric matrices with real entries are useful. Section 3 presents detailed proofs for the above results and reveals the corresponding methods. Section 4 concludes the paper.

2 Constrained minimization problems: results

Lemma 2.1.

The function $\varphi(t) := t(1-e^{-1/t}), t>0,$ *is strictly increasing and strictly concave on* $(0,\infty)$.

Let X be an arbitrary discrete random variable which takes a finite number of values $x_j \in \mathbb{R}_+$ with the corresponding probabilities $p_j > 0, j = 1,2,\dots,n, (\sum_{j=1}^n p_j = 1)$. Denote by m_n the mean of the r. v. X: $m_n := \sum_{j=1}^n p_j x_j$. Consider now another random variable Y, which takes the values $1-e^{-x_j}$ with the probabilities $p_j, j = 1,2,\dots,n$, where $p_j, j = 1,\dots,n$ are the probabilities involved in the random variable X. Note that we have assumed that all probabilities $p_j, j = 1,\dots,n$ are positive numbers. The next problem is to minimize the mean

$M_n := \sum_{j=1}^n p_j (1-e^{-x_j})$ subject to $\sum_{j=1}^n p_j x_j := m_n = const.$

Theorem 2.1.

Using the above notations and hypothesis, we have

$$m_n \ge M_n = \sum_{j=1}^n p_j(1-e^{-x_j}) \ge \left(\min_{1\le k\le n} p_k\right)\left(1-e^{-\frac{m_n}{\min_{1\le k\le n} p_k}}\right) \quad (2.1)$$

Equality occurs in the last inequality (2.1) if and only if $x_j = 0$ *for* $j \ne j_m$ *where* $p_{j_m} = \min_{1\le k\le n} p_k, x_{j_m} = \frac{m_n}{p_{j_m}}, \sum_{j=1}^n p_j x_j = m_n = const.$

Observe that the index j_m appearing above might be not unique (the minimum could be attained at several points). A similar result holds true when we replace the equality constraint $\sum_{j=1}^n p_j x_j = m_n = const.$ by the inequality constraint $\sum_{j=1}^n p_j x_j \ge m_n = const.$ Observe also that the last inequality (2.1) holds true without using hypothesis $\sum_{j=1}^n p_j = 1$ (see the proof in Section 3).

Theorem 2.2.

With the above notations, the same last inequality (2.1) holds, subject to $x_j \ge 0, p_j > 0, j = 1,\dots,n,$ $\sum_{j=1}^n p_j x_j \ge m_n$*, and equality occurs in the same case as that of Theorem 2.1.*

The next result is similar to that of Theorem 2.1, where the numbers $p_n, n \ge 1$, have not the significance of probabilities, because of: $\sum_{n=1}^\infty p_n = \infty$.

Theorem 2.3.

Assume that $\infty > \sup_{n\ge1} p_n \ge \inf_{n\ge1} p_n > 0, \sum_{n\ge1} x_n < \infty, x_n \ge 0\ \forall n \ge 1, n \in \mathbb{N}$. *The following inequalities hold true*

$$\infty > m \ge \sum_{n=1}^\infty p_n(1-e^{-x_n}) \ge \left(\inf_{n\ge1} p_n\right)\left(1-e^{-\frac{m}{\inf_{n\ge1} p_n}}\right), \quad (2.2)$$

where

$$m := \sum_{n=1}^{\infty} p_n x_n = const.$$

Example 2.1.

Denote

$$p_n = \varepsilon + \frac{1}{n}, x_n = \frac{1}{2^n}, n \in \mathbb{N}, n \geq 1,$$

for some $\varepsilon > 0$. Then the last relation (2.2) can be written as

$$\sum_{n=1}^{\infty} \left(\varepsilon + \frac{1}{n}\right)\left(1 - e^{-1/2^n}\right) \geq \varepsilon\left(1 - exp\left(-\frac{m}{\varepsilon}\right)\right) \quad (2.2')$$

Here

$$m := \sum_{n=1}^{\infty} \left(\varepsilon + \frac{1}{n}\right)\frac{1}{2^n} = \varepsilon\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) + \left(\sum_{n=1}^{\infty} \frac{y^n}{n}\right)|_{y=1/2} =$$

$$\varepsilon + \int_0^{1/2} \left(\sum_{n=1}^{\infty} t^{n-1}\right) dt = \varepsilon - ln(1-t)|_{t=1/2} = \varepsilon + ln(2)$$

Thus the right hand side member of (2.2') is

$$\varepsilon\left(1 - exp\left(-\frac{\varepsilon + ln(2)}{\varepsilon}\right)\right) = \varepsilon\left(1 - \frac{1}{e}\cdot\frac{1}{2^{1/\varepsilon}}\right)$$

The conclusion is that (2.2') one writes as

$$\sum_{n=1}^{\infty} \left(\varepsilon + \frac{1}{n}\right)\left(1 - e^{-1/2^n}\right) \geq \varepsilon\left(1 - \frac{1}{e}\cdot\frac{1}{2^{1/\varepsilon}}\right)$$

Remark 2.1.

From programming viewpoint related to Theorem 2.1, finding the minimum and minimum point(s) for $\sum_{j=1}^{n} p_j \left(1 - e^{-x_j}\right)$ subject to $\sum_{j=1}^{n} p_j x_j := m_n = const.$ it is sufficient to determine $\min_{1 \leq k \leq n} p_k$ (see also the proof Theorem 2.1 in section 3). In case of Theorem 2.3, to determine minimum of the function

$$\sum_{n=1}^{\infty} p_n \left(1 - e^{-x_n}\right) s.t. \sum_{n=1}^{\infty} p_n x_n = const.$$

we should find (or estimate) $\inf_{n \geq 1} p_n = \inf q_n$, where $q_n := \min_{1 \leq k \leq n} p_k, n \in \mathbb{N}, n \geq 1.$ Contrary to the case of the sequence $(p_n)_{n \geq 1}$, the sequence $(q_n)_{n \geq 1}$ is nonincreasing

Let H be an arbitrary complex or real Hilbert space and $\mathcal{A}$ the real vector space of all self-adjoint operators acting on H. Recall that the natural order relation on the space $\mathcal{A}$ is defined by

$$U \leq V \Leftrightarrow < U(h), h > \leq < V(h), h > \quad \forall h \in H, \qquad U, V \in \mathcal{A}$$

Observe that the operation of multiplication of elements of $\mathcal{A}$ is not commutative. Also, for arbitrary $U,V \in \mathcal{A}$, the elements $inf\{U,V\}, sup\{U,V\}$ do not exist ($\mathcal{A}$ is not a vector lattice). Also, the space $\mathcal{A}$ is not an order-complete (Dedekind complete) vector lattice. Therefore, one uses the following construction [8].

Theorem 2.4.

Let $H,\mathcal{A}$ be as above, $U \in \mathcal{A}, Y_1 = Y_1(U) := \{W \in \mathcal{A}; WU = UW\}, Y = Y(U) := \{V \in Y_1; VW = WV \quad \forall W \in Y_1\}$. Then Y is a commutative (real) Banach algebra and an order-complete Banach lattice.

The next result is devoted to an assertion similar to that of Theorem 2.3, but for concave mappings taking values into a commutative real algebra and an order-complete Banach lattice of self-adjoint operators pointed out in Theorem 2.4.

Theorem 2.5.

Let U be a self-adjoint operator acting on a complex or real Hilbert space H, with the spectrum $\sigma(U) \subset (0,\infty)$. In the space $Y = Y(U)$ defined in Theorem 2.4, consider a sequence $(T_n)_{n\geq 1}$, such that the spectrums $\sigma(T_n) \subset [a,b] \subset (0,\infty)$ for all $n \in \mathbb{N}, n \geq 1$. Let $(x_n)_{n\geq 1}$ be an arbitrary sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} x_n = 1$. Then the following inequality holds

$$bU \geq \sum_{n=1}^{\infty} T_n\left(I - exp(-x_n U)\right) \geq \left(\inf_{n\geq 1} T_n\right)\left(I - exp(-U)\right)$$

Corollary 2.1.

Under the hypothesis and using the notations of Theorem 2.5, the following inequality holds

$$\sum_{n=1}^{\infty} T_n\left(I - exp(-x_n U)\right) \geq a(1 - e^{-\omega})I,$$

where $\omega := inf\left(\sigma(U)\right) = \inf_{\|h\|=1} < U(h), h > \; > 0$, and $I: H \to H$ is the identity operator.

Remark 2.2.

In the very particular case when $H = \mathbb{R}^p/\mathbb{R}, p \in \mathbb{N}, p \geq 2$, a positive invertible self-adjoint operator U acting on H can be represented by a positive definite symmetric matrix with real entries (which will be denoted by M_U). In this case, if $\lambda_1, \dots, \lambda_p$ are the (not necessarily distinct) proper values of the matrix M_U, we have

$$\sigma(U) = \{\lambda_1, \dots, \lambda_p\}, \omega := inf\left(\sigma(U)\right) = \min_{1\leq j\leq p} \lambda_j > 0, \|U\| = sup\left(\sigma(U)\right) = \max_{1\leq j\leq p} \lambda_j$$

In this case, verifying the hypothesis and writing the conclusion of Theorem 2.5 and of its Corollary 2.1 involve computational operations. This is the simplest (discrete) finite dimensional case of Theorem 2.5 (and Corollary 2.1). Hence the latter result may be formulated in terms of (positive definite) commuting symmetric matrixes and their spectrums.

3 Proofs and related methods

Proof of Lemma 2.1. Let $\varphi: (0,\infty) \to (0,\infty), \varphi(t) := t\left(1 - e^{-1/t}\right), t > 0$. Then the following computational results hold true

$$\varphi^{'}(t) = 1 - e^{-1/t} + t\left(-e^{-1/t}\frac{1}{t^2}\right) = 1 - e^{-1/t} - \frac{1}{t}e^{-1/t},$$

$$\varphi''(t) = -e^{-1/t}\frac{1}{t^2} + \frac{1}{t^2}e^{-1/t} - \frac{1}{t}e^{-1/t}\frac{1}{t^2} = -\frac{1}{t^3}e^{-1/t} < 0 \ \forall t > 0$$

Thus φ is strictly concave on $(0,\infty)$, or, equivalently, the first derivative φ' is strictly decreasing on the same interval. Therefore, it results

$$\varphi'(t) > \varphi'(\infty) := \lim_{x\to\infty} \varphi'(x) = \lim_{x\to\infty} \left(1 - e^{-1/x} - \frac{1}{x}e^{-1/x}\right) = 1 - 1 - 0 = 0 \ \forall t \in (0,\infty)$$

Hence φ is also strictly increasing on $(0,\infty)$. This concludes the proof. □

Proof of Theorem 2.1. The first inequality (2.1) is almost obvious. Indeed, using the elementary relation $e^u \geq 1 + u \ \forall u \in \mathbb{R}$ we get $e^{-x_j} \geq 1 - x_j \Leftrightarrow 1 - e^{-x_j} \leq x_j, j \in \{1,\dots,n\} \Rightarrow$

$$\sum_{j=1}^{n} p_j(1 - e^{-x_j}) \leq \sum_{j=1}^{n} p_j x_j = m_n$$

To prove the last inequality (2.1), according to notations and hypothesis, we have to minimize

$$M_n(x) = M_n(x_1,\dots,x_n) := \sum_{j=1}^{n} p_j(1 - e^{-x_j})$$

subject to

$x_j \geq 0,\ \sum_{j=1}^{n} p_j x_j = m_n = const.$,

where $p_j > 0, j = 1,2,\dots,n,\ \sum_{j=1}^{n} p_j = 1$. Obviously, the function M_n is strictly concave on the simplex defined by the constraints on $x_j, p_j, j = 1,\dots n$, as a sum of strictly concave functions. Denote by K_{n-1} the $n-1$ dimensional simplex

$$K_{n-1} := \left\{x = (x_1,\dots,x_n) \in \mathbb{R}^n;\ x_j \geq 0,\ \sum\nolimits_{j=1}^{n} p_j x_j = m_n = const.\right\}$$

It is easy to show that the set $Ex(K_{n-1})$ of all extreme points of K_{n-1} is given by

$$Ex(K_{n-1}) = \{e_1,\dots,e_n\},$$

$$e_1 = \left(\frac{m_n}{p_1}, 0, \dots, 0\right), \dots, e_n = \left(0, \dots, 0, \frac{m_n}{p_n}\right)$$

Thanks to minimum principle for concave functions, for any $x \in K_{n-1}$, we have

$$M_n(x) \geq min\{M_n(e_1),\dots,M_n(e_n)\} =$$

$$\min_{1\leq j\leq n} p_j\left(1 - e^{-m_n/p_j}\right) = m_n \min_{1\leq j\leq n} \frac{p_j}{m_n}\left(1 - e^{-m_n/p_j}\right) = m_n \min_{1\leq j\leq n} \varphi\left(\frac{p_j}{m_n}\right),$$

because of $M_n(e_j) = p_j\left(1 - e^{-m_n/p_j}\right) = m_n\varphi\left(\frac{p_j}{m_n}\right), j = 1,\dots,n$. Here φ is the function from Lemma 2.1. On the other hand, by increasing monotony of φ, we obviously have

$$\frac{p_j}{m_n} \geq \frac{\min_{1\leq k\leq n} p_k}{m_n}, j = 1,\dots,n \Rightarrow \min_{1\leq j\leq n} \varphi\left(\frac{p_j}{m_n}\right) = \varphi\left(\frac{\min_{1\leq k\leq n} p_k}{m_n}\right)$$

From the last relations, it results

$$M_n(x) \geq m_n\varphi\left(\frac{\min\limits_{1\leq k\leq n} p_k}{m_n}\right) = \left(\min_{1\leq k\leq n} p_k\right)\left(1-e^{-\frac{m_n}{\min\limits_{1\leq k\leq n} p_k}}\right),$$

as claimed. Moreover, we have

$$M_n\left(e_{j_m}\right) = m_n\varphi\left(\frac{p_{j_m}}{m_n}\right) = min\{M_n(e_1),\dots,M_n(e_n)\} = \inf_{x\in K_{n-1}} M_n(x)$$

if and only if $p_{j_m} = \min\limits_{1\leq k\leq n} p_k$, according to Lemma 2.1. Namely, φ is increasing, so its minimum value on the finite subset $\left\{\frac{p_1}{m_n},\dots,\frac{p_n}{m_n}\right\}$ is attained at the smallest element of this set, which is $\frac{p_{j_m}}{m_n}$. On the other hand, from minimum principle for concave continuous functions, it results, as we have seen, that $\inf\limits_{x\in K_{n-1}} M_n(x) = min\{M_n(e_1),\dots,M_n(e_n)\}$. Note also that M_n is strictly concave, as a finite sum of such functions. Application of Carathéodory's and Jensen's inequalities, having in mind when equality occurs, lead to the fact that any minimum point of M_n over K_{n-1} must be an extreme point of this simplex. This concludes the proof. □

Proof of Theorem 2.2. In this case the constraints $x_j \geq 0, j = 1,\dots,n, \sum_{j=1}^{n} p_j x_j \geq m_n$ define a closed convex unbounded subset C of $\mathbb{R}^n$. The idea of the proof is to reduce the problem to that in Theorem 2.1, by writing

$$C = \bigcup_{\rho\geq 0}\left\{x = (x_1,\dots,x_n)\in\mathbb{R}_+^n; \sum_{j=1}^{n} p_j x_j = m_n + \rho\right\} = \bigcup_{\rho\geq 0} K_{n-1,\rho},$$

where $K_{n-1,\rho} := \left\{x = (x_1,\dots,x_n)\in\mathbb{R}_+^n; \sum_{j=1}^{n} p_j x_j = m_n + \rho\right\}$ is a convex compact simplex of the type used in the proof of Theorem 2.1. Applying the latter theorem, one obtains

$$\inf_{x\in K_{n-1,\rho}} M_n(x) = \left(\min_{1\leq k\leq n} p_k\right)\left(1-e^{-\frac{m_n+\rho}{\min\limits_{1\leq k\leq n} p_k}}\right),$$

where

$$M_n(x) = M_n(x_1,\dots,x_n) := \sum_{j=1}^{n} p_j\left(1-e^{-x_j}\right)$$

It results

$$\inf_{x\in C} M_n(x) = \inf_{\rho\geq 0}\inf_{x\in K_{n-1,\rho}} M_n(x) = \inf_{\rho\geq 0}\left(\min_{1\leq k\leq n} p_k\right)\left(1-e^{-\frac{m_n+\rho}{\min\limits_{1\leq k\leq n} p_k}}\right) =$$

$$\left(\min_{1\leq k\leq n} p_k\right)\left(1-e^{-\frac{m_n}{\min\limits_{1\leq k\leq n} p_k}}\right)$$

This concludes the proof. □

Proof of Theorem 2.3. The first inequalities (2.2) are similar to the first inequalities (2.1). Indeed, we have

$$\sum_{n=1}^{\infty} p_n\left(1-e^{-x_n}\right) \leq \sum_{n=1}^{\infty} p_n x_n := m \leq \left(\sup_{n\geq 1} p_n\right)\left(\sum_{n\geq 1} x_n\right) < \infty$$

To prove the last inequality (2.2), consider an increasing sequence $(m_n)_{n\geq 1}$ converging to m and for each n let K_{n-1} be the $n-1$ dimensional simplex defined in the proof of Theorem 2.1. According to the last inequality of the latter theorem one has

$$\inf_{x\in K_{n-1}} M_n(x) = \left(\min_{1\leq k\leq n} p_k\right)\left(1-e^{-\frac{m_n}{\min_{1\leq k\leq n} p_k}}\right)$$

Now the proof of Theorem 2.3 follows passing to the limit. Namely, it results:

$$x_n \geq 0, n\geq 1, \sum_{n=1}^{\infty} p_n x_n = m \Rightarrow$$

$$\sum_{n=1}^{\infty} p_n\left(1-e^{-x_n}\right) = \lim_{n\to\infty}\sum_{j=1}^{n} p_j\left(1-e^{-x_j}\right) \geq$$

$$\lim_{n\to\infty}\left(\min_{1\leq j\leq n} p_j\right)\left(1-e^{-\frac{m_n}{\min_{1\leq k\leq n} p_k}}\right) = \left(\lim_{n\to\infty}\left(\min_{1\leq j\leq n} p_j\right)\right)\left(1-e^{-\lim_n \frac{m_n}{\min_{1\leq j\leq n} p_j}}\right) =$$

$$\left(\inf_{n\geq 1} p_n\right)\left(1-e^{-\frac{m}{\inf_{n\geq 1} p_n}}\right)$$

This concludes the proof. □

Remark 3.1. From the preceding proof we see that $\sum_{n=1}^{\infty} p_n\left(1-e^{-x_n}\right)$ is minimized by a product of two sequences (of positive numbers), one of which is decreasing, while the other one is increasing with n. It follows that such type results are not trivial.

Proof of Theorem 2.5. Using the elementary inequality $e^r \geq 1+r$ for all $r\in\mathbb{R}$, we infer that for any natural number $n\geq 1$ the following relations hold

$$e^{-x_n u} \geq 1-x_n u \Leftrightarrow 0 \leq 1-e^{-x_n u} \leq x_n u, \qquad \forall u\in\sigma(U) \Rightarrow I-e^{-x_n U} \leq x_n U$$

On the other hand, conditions $\sigma(T_n)\subset[a,b]\subset(0,\infty)$ leads to $0<T_n\leq bI, n\in\mathbb{N}, n\geq 1.$ It results

$$\sum_{n=1}^{\infty} T_n\left(I-exp(-x_n U)\right) \leq \left(\sum_{n=1}^{\infty} x_n T_n\right) U \leq bI\left(\sum_{n=1}^{\infty} x_n\right) U = bU \in Y_+$$

This proves the first inequality in the statement. Let fix a natural number $n\geq 1$ and for an arbitrary $u\in\sigma(U)$ and $j\in\{1,\dots,n\}$ define

$$\varphi_{u.j}(x) = \varphi_{u,j}(x_1,\dots,x_n) := 1-e^{-ux_j},$$

$$x\in K_{n-1} := \left\{x\in\mathbb{R}^n;\ x_j\geq 0, j=1,\dots,n, \sum_{j=1}^{n} x_j = m_n\right\}$$

where $m_n \uparrow 1, n\uparrow\infty$. Obviously, $\varphi_{u.j}$ is concave as a function of $x=(x_1,\dots,x_n), j\in\{1,\dots,n\}$, so that for any $\lambda\in[0,1], u\in\sigma(U)$, we have

$$\varphi_{u.j}\left((1-\lambda)x+\lambda y\right) = 1-e^{-u\left((1-\lambda)\right)x_j+\lambda y_j)} \geq (1-\lambda)\varphi_{u.j}(x)+\lambda\varphi_{u.j}(y) =$$

$$(1-\lambda)\left(1-e^{-ux_j}\right)+\lambda\left(1-e^{-uy_j}\right), u\in\sigma(U) \Rightarrow$$

$$I-e^{-((1-\lambda))x_j+\lambda y_j)U} \geq (1-\lambda)(1-e^{-Ux_j})+\lambda(1-e^{-Uy_j}), j\in\{1,\dots,n\}$$

Since T_j are positive (commuting) operators in Y, it follows that $x \to T_j(I-e^{-x_jU})$ is concave on the $n-1$-dimensional simplex $K_{n-1}\subset\mathbb{R}^n$. Here we take as p_j from theorems 2.1, 2.3, $p_j=1, j\geq 1 \Rightarrow \sum_{j=1}^n p_j = n\neq 1$. One can see that condition $\sum_{j=1}^n p_j=1$ is not used in the last inequality (2.1) (see the proof of Theorem 2.1). On the other side, the finite sum of concave operators is concave. According to the minimum principle for concave mappings (based on Carathéodory's theorem) and Jensen's inequality (discrete form), it results

$$\sum_{j=1}^n T_j\left(I-e^{-x_jU}\right) \geq \inf_{1\leq k\leq n}\left(\sum_{j=1}^n T_j\left(I-e^{-x_jU}\right)\right)(e_k) = \inf_{1\leq k\leq n} T_k(I-e^{-m_nU}) =$$

$$(I-e^{-m_nU})\left(\inf_{1\leq k\leq n} T_k\right)$$

where $Ex(K_{n-1})=\{e_1,\dots,e_n\}=\{(m_n,0\dots,0),\dots,(0,\dots,0,m_n\}$ is the set of all extreme points of K_{n-1}. On the other side,

$$\|(I-e^{-m_nU})-(I-e^{-U})\| = \sup_{u\in\sigma(U)}|e^{-u}-e^{-m_nu}| \leq$$

$$\sup_{u\in\sigma(U)}|e^{-\tau u}(1-m_n)u| \leq (1-m_n)\|U\|\to 0, n\to\infty,$$

where τ is between m_n and 1. Thus

$$m_n\uparrow 1 \Rightarrow \lim_n(I-e^{-m_nU}) = I-e^{-U}$$

(here the convergence holds with respect to operatorial norm on Y and is uniform with respect to $u\in\sigma(U)$). To conclude the proof, we pass to the limit, using the last equality from above and elementary theorems on self-adjoint operators [7], [8] (such as pointwise convergence of the sequence $\left(\inf_{1\leq k\leq n} T_k\right)_{n\geq 1}$ (see [7] and Theorem 2.4 from above)). It results

$$\sum_{n=1}^\infty T_n\left(I-exp(-x_nU)\right) = \lim_{n\to\infty}\sum_{j=1}^n T_j\left(I-e^{-x_jU}\right) \geq$$

$$\lim_{n\to\infty}(I-e^{-m_nU})\left(\inf_{1\leq k\leq n} T_k\right) = (I-e^{-U})\left(\inf_{n\geq 1} T_n\right)$$

The convergence is pointwise. This concludes the proof. □

Proof of Corollary 2.1. Since $\omega := inf(\sigma(U))$, for all $u\in\sigma(U)$ we have $1-e^{-u}\geq 1-e^{-\omega}$. Integrating with respect to the (positive) spectral measure dE_U, on the spectrum $\sigma(U)$, it results

$$I-exp(-U) = \int_{\sigma(U)}(1-e^{-u})dE_U \geq (1-e^{-\omega})\int_{\sigma(U)} dE_U = (1-e^{-\omega})I$$

On the other hand, the hypothesis $\sigma(T_n)\subset[a,b]\subset(0,\infty)$ *for all* $n\in\mathbb{N}, n\geq 1$ leads to

$$T_n \geq aI \;\; \forall n\geq 1, n\in\mathbb{N} \Rightarrow \inf_{n\geq 1} T_n \geq aI$$

Thus $(I-e^{-U})\left(\inf_{n\geq 1} T_n\right)\geq(1-e^{-\omega})aI$. The conclusion follows thanks to Theorem 2.5. □

4 Conclusions

Section 2 is devoted to constrained optimization (or finding lower bounds) of finite and infinite sums of concave elementary transcendent functions. An example related to Theorem 2.3 is sketched. In the end, an operator-valued mapping constrained minimization problem is solved (Theorem 2.5). The first aim of the paper is to minimize and evaluate the unknown mean of a random variable, in terms of the given (known) mean of a related random variable. Most of the results can be completed by adding the corresponding programming and computational methods. Numerical methods and examples related to Corollary 2.1 could illustrate the applicability of Theorem 2.5. Such results are closely related to computing the greatest and the smallest eigenvalue of a positive definite symmetric matrix with real entries.

Conflicts of Interest

The authors declare that there is no conflict of interest in publishing this paper.

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