On modified Picard−S−AK Hybrid iterative algorithm for approximating fixed point of Banach contraction map

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Abstract

The purpose of this work is to introduce a new iteration called the modified Picard−S−AK hybrid iterative scheme for approximating fixed point for Banach contractive maps. We show that our scheme converges to a unique fixed point p at a rate faster than the recent AK iterative scheme for Banach contractive maps. Furthermore, using Java programming language, we give some numerical examples to justify our claim. Stability and data dependence of the proposed scheme are also explored.

1 Introduction

Among the various methods of finding the solution for non-linear equations is the fixed point iterative procedure. Over the years, we have seen how the theory of fixed point has been used to solve problems in other areas of research including but not limited to Applied Mathematics, Biology, Chemistry, Economics, Engineering, Game theory etc. Sequel to the latter, many literature has sprang up, searching for different iterations of finding the fixed points of several types of equations and also the fastest method to arrive at the fixed point.
Let X be a Banach space, and C be a non-empty convex subset of X and $T : X \to X$ be a mapping, a point $x \in X$ is called a fixed point of T if $T(x) = x$, and F_T represents the set of all fixed points of a mapping T.

It is well known that Banach S in 1922 [6] used the Picard's iterative scheme of the form

$$x_{n+1} = Tx_n \tag{1}$$

to approximate the unique fixed point for maps that satisfies the inequality

$$d(Tx, Ty) \leq Ld(x, y), \quad L \in (0, 1). \tag{2}$$

After Banach (1922), different schemes for approximating fixed point for several types of contractive maps sprang up, we will only mention few of the works that are directly connected to the proposed scheme.
In 1953, Mann [16] introduced the Mann iterative scheme

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \alpha_n \in [0, 1), \quad n = 1, 2, \ldots, \end{cases} \tag{3}$$

to obtain convergence of nonexpansive maps where Picard's iteration scheme cannot be applied. Observe that if $\alpha_n = 1$, then the scheme in equation (3) is reduced to Picard's iterative scheme.

Ishikawa [12], generalized the result of Mann by introducing a two step iterative scheme defined as:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad \alpha_n, \beta_n \in [0,1) \ \ n = 1, 2, \ldots, \end{cases} \tag{4}$$

to obtain convergence of a Lipschitzian pseudo-contractive map where Mann iterative algorithm fails to converge. If $\beta_n = 1$ for each n in equation (4), the Ishikawa iterative scheme is reduced to Mann iterative scheme [16]. Similarly, if $\alpha_n = \beta_n = 1$, then equation (4) is reduced to the equation (1).

Later in 2000, Noor [17] introduced a three step iterative scheme (also known as the Noor Iteration) which extends the results of [6], [16] and [12]. The scheme is defined as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad \alpha_n, \beta_n, \gamma_n \in [0,1) \ \ n = 1, 2, \ldots. \end{cases} \tag{5}$$

Similarly, if $\gamma_n = 1$ for all n, equation (5) is reduced to (4). Recently, other iterative schemes were introduced, and they can be found in Abbas et al. [2], Agarwal et al. [3], Akewe et al. [4], Berinde [8], Karahan et al. [13], Khan [15].

Gursoy and Karakaya [9] introduced the Picard-S iterative scheme, defined as

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T y_n, \\ y_n = (1-\beta_n)T x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad \beta_n, \gamma_n \in [0,1) \ \ n = 1, 2, \ldots. \end{cases} \tag{6}$$

Gursoy et al. result showed that Picard-S iterative scheme converges strongly to the fixed point and that it is faster that those of Picard, Mann, Ishikawa, Noor, SP, CR, etc

Karakaya et al. [14] introduced another form of iteration known as Vatan Two step iterative scheme(we will call it the

VTS iterative scheme in this paper), defined as follows,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T((1-\alpha_n)y_n + \alpha_n T y_n), \\ y_n = T\big((1-\beta_n)x_n + \beta_n T x_n\big], \quad \alpha_n, \beta_n \in [0,1) \;\; n = 1, 2, \ldots. \end{cases} \tag{7}$$

and they showed that the VTS iterative scheme is faster than the Picard$-S$ iterative scheme, hence, faster than the other known iterations.

Recently, Thakur et al [28], presented a three step iterative scheme defined as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T y_n, \\ y_n = T\big[(1-\alpha_n)x_n + \alpha_n z_n\big], \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \quad \alpha_n, \beta_n \in [0,1) \;\; n = 1, 2, \ldots. \end{cases} \tag{8}$$

using numerical example, they showed that their method is faster than Picard, Mann, Ishikawa, Noor, Agarwal, Abbas iterative schemes.

In the same year, Ullah and Arshad [29] introduced another three step iteration, which they referred to as the AK iteration, defined as

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T y_n, \\ y_n = T\big[(1-\alpha_n)z_n + \alpha_n T z_n\big], \\ z_n = T\big[(1-\beta_n)x_n + \beta_n T x_n\big], \quad \alpha_n, \beta_n \in [0,1) \;\; n = 1, 2, \ldots. \end{cases} \tag{9}$$

They showed that the AK iteration converges faster that those of Picard-S in equation (6), Vatan Two step in equation (7) and that of Thakur et al in equation (8)

The AK iterative scheme motivate us to introduce a modified Picard$-S-$AK hybrid iterative scheme and show that it converges at a faster rate to its fixed point p than that of AK and Picard$-S$ iteration defined in (6) and (9) respectively. Using both the analytical definition cum numerical example to prove the latter, we also show that our iteration is $T-$Stable and data dependent.

2 Preliminary

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called contraction if there exists $\delta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\|. \tag{10}$$

At this point let us introduce a new three step modified Picard−S−AK hybrid iterative scheme, which we define as follows,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T(Ty_n), \\ y_n = T[(1-\beta_n)Tx_n + \beta_n Tz_n], \\ z_n = T[(1-\gamma_n)x_n + \gamma_n Tx_n], \quad \alpha_n, \beta_n, \gamma_n \in [0,1) \;\; n = 1, 2, \ldots. \end{cases} \tag{11}$$

Let now state some definitions and lemmas that will be useful in the coming theories.

Lemma 2.1 (Sandwich Theorem) *Let ε_n, η_n be sequences of real numbers such that $\varepsilon_n, \eta_n \to l(n \to \infty)$, and let ϱ_n be any sequence such that $\eta_n \leq \varrho_n \leq \varepsilon_n$ then $\varrho_n \to l(n \to \infty)$.*

A special case of Lemma 2.1, gives rise to Corollary 2.1 below,

Corollary 2.1 *Let ε_n be a sequence of real number such that $\varepsilon_n \to 0(n \to \infty)$, and let ϱ_n be an arbitrary sequence such that $0 \leq \varrho_n \leq \varepsilon_n$ then $\varrho_n \to 0(n \to \infty)$.*

Lemma 2.2 (see [7]) *Let δ be a real number such that $0 \leq \delta < 1$ and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{\varrho\}_{n=0}^{\infty}$ satisfying*

$$\varrho_{n+1} \leq \delta\varrho_n + \varepsilon_n, \quad n = 0, 1, 2, \ldots$$

we have $\lim_{n\to\infty} \varrho_n = 0$.

Lemma 2.3 (see [30]) *Let η_n be a nonnegative sequence for which one supposes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the following inequality:*

$$\eta_{n+1} \leq (1 - \vartheta_n)\eta_n + \vartheta_n\varphi_n \tag{12}$$

where $\vartheta_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \vartheta_n = \infty$, and $\varphi_n \geq 0$ for all $n \in \mathbb{N}$. Then,

$$0 \leq \lim_{n\to\infty} \sup \eta_n \leq \lim_{n\to\infty} \sup \varphi_n. \tag{13}$$

Definition 2.1 (see Berinde [8]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real convergent sequences that converge to a and b respectively, then we say that $\{a_n\}_{n=0}^{\infty}$ converge faster than $\{b_n\}_{n=0}^{\infty}$ if*

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$$

Definition 2.2 (see Berinde [8]) *Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same fixed point p. If $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \in \mathbb{N}$, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers(converging to zero). Then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p if $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$.*

Definition 2.3 (see [8]) *Let $T, \widetilde{T} : C \to C$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for a fixed $\epsilon > 0$ we have*

$$\|Tx - \widetilde{T}\widetilde{x}\| \leq \epsilon \tag{14}$$

After the advent of computational mathematics, the iterative aspect of fixed point theory received a precedented recognition. Sequel to the above, mathematicians needed to know how stable a method is before using it to approximate the fixed point of any operator. The first result on $T-$ stability was introduced by Ostrowski [24] in 1967. His result was followed by Harder and Hicks in [10] in 1988, by Rhoades [24,25], other notable results on stability are those of Osilike [21] in 1995, Osilike and Udemene [23] in 1999, and Berinde [7] in 2002, who give a clear explanation of the meaning of stability and gave a more simpler approach than that of Harder and Hicks [10]. The following definition is credited to Harder and Hicks [10].

Definition 2.4 (see [7]) *Let X be a Banach space and, $T : X \to X$ a self map, $x_0 \in X$ and the iteration procedure defined by*

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots \tag{15}$$

such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and the set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\| \quad \textit{for } n = 0, 1, 2, 3, \ldots,$$

then the iteration process (15) is said to be $T-$ stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

3 Main Results

3.1 Convergence Result

Theorem 3.1 *Let C be a non-empty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping satisfying (10). Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (11) with sequences of real numbers $\beta_n, \gamma_n \in (0, 1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$ then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of T.*

Proof:

Let $p \in F_T$, we show that $x_n \to p (n \to \infty)$ and that $F_T = \{p\}$. From (11) we have

$$\begin{aligned} \|z_n - p\| &= \|T[(1-\gamma_n)x_n + \gamma_n T x_n] - p\| \\ &\leq \delta\|(1-\gamma_n)x_n + \gamma_n T x_n - (1-\gamma_n+\gamma_n)p\| \end{aligned}$$

$$\begin{aligned} &\leq \delta(1-\gamma_n)\|x_n - p\| + \delta^2\gamma_n\|x_n - p)\| \\ &= \delta\|x_n - p\|[1-\gamma_n(1-\delta)], \end{aligned} \tag{16}$$

$$\begin{aligned} \|y_n - p\| &= \|T[(1-\beta_n)Tx_n + \beta_n Tz_n] - p\| \\ &\leq \delta\|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\ &\leq \delta(1-\beta_n)\|Tx_n - p\| + \delta\beta_n\|Tz_n - p\| \\ &\leq \delta^2(1-\beta_n)\|x_n - p\| + \delta^2\beta_n\|z_n - p\|. \end{aligned} \tag{17}$$

Substituting (16) into (17) we have

$$\|y_n - p\| \leq \delta^2\Big(1-(1-\delta)(\beta_n + \delta\beta_n\gamma_n)\Big)\|x_n - p\| \tag{18}$$

$$\|x_{n+1} - p\| = \|T^2 y_n - p\| \leq \delta^2\|y_n - p\|. \tag{19}$$

Substituting (18) into (19), we have

$$\|x_{n+1} - p\| \leq \delta^4\Big(1-(1-\delta)(\beta_n + \delta\beta_n\gamma_n)\Big)\|x_n - p\|. \tag{20}$$

From (20) we can derive the following inequalities

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta^4\Big(1-(1-\delta)(\beta_n + \delta\beta_n\gamma_n)\Big)\|x_n - p\| \\ \|x_n - p\| &\leq \delta^4\Big(1-(1-\delta)(\beta_{n-1} + \delta\beta_{n-1}\gamma_{n-1})\Big)\|x_{n-1} - p\| \\ \|x_{n-1} - p\| &\leq \delta^4\Big(1-(1-\delta)(\beta_{n-2} + \delta\beta_{n-2}\gamma_{n-2})\Big)\|x_{n-2} - p\| \\ &\vdots \\ \|x_1 - p\| &\leq \delta^4\Big(1-(1-\delta)(\beta_0 + \delta\beta_0\gamma_0)\Big)\|x_0 - p\|. \end{aligned} \tag{21}$$

From (21) we can conclude that,

$$\|x_{n+1} - p\| \leq \|x_0 - p\|\delta^{4(n+1)}\prod_{k=0}^{n}\Big(1-(1-\delta)(\beta_k + \delta\beta_k\gamma_k)\Big) \tag{22}$$

Since $\Big(1-(1-\delta)(\beta_k + \delta\beta_k\gamma_k)\Big) = \Big[1-\beta_n\big(1-\delta(1-\gamma_n(1-\delta))\big)\Big] < 1$ and from classical analysis we know that for $x \in (0,1)$ $1 - x < e^{-x}$, (22) becomes

$$\|x_{n+1} - p\| \leq \|x_0 - p\|\delta^{4(n+1)}e^{-(1-\delta)\sum\limits_{k=0}^{n}(\beta_k + \delta\beta_k\gamma_k)}. \tag{23}$$

Since, $\delta \in (0,1)$ and $\sum\limits_{n=0}^{\infty}\beta_n = \infty$ we know that $\delta^n \to 0 (n \to \infty)$ and $e^{-\infty} = 0$, therefore,

$$\delta^{4(n+1)}e^{-(1-\delta)\sum_{k=0}^{n}(\beta_k + \delta\beta_k\gamma_k)} \to 0 (n \to \infty).$$

Hence by Corollary 2.1, we conclude that $\|x_n - p\| \to 0 (n \to \infty)$.

Next, we show that p is a unique fixed point of T. Let p^* be another fixed point of T such that $p \neq p^*$.

$$0 \leq \|p - p^*\| = \|Tp - Tp^*\| \leq \delta\|p - p^*\|,$$

therefore, $p = p^*$. □

3.2 Stability Result

Theorem 3.2 *Let X be a Banach space $T : X \to X$ be a self mapping with fixed point p satisfying (10). For arbitrary $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty}$ be an iterative scheme defined by (11), with real sequences $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then the iterative process (11) is $T-$stable.*

Proof: Let v_n be in X and $\varepsilon_n = \|v_{n+1} - f(T, v_n)\|$. Suppose that $\varepsilon_n \to 0(n \to \infty)$, we show that $v_n \to p(n \to \infty)$. Consider the following

$$\begin{aligned} \|v_{n+1} - p\| &= \|v_{n+1} - f(T, v_n) + f(T, v_n) - p\| \\ &= \|v_{n+1} - f(T, v_n)\| + \|f(T, v_n) - p\| \\ &\leq \varepsilon_n + \|f(T, v_n) - p\| \leq \varepsilon_n + \|T^2 y_n - p\| \\ &\leq \varepsilon_n + \delta^2 \|y_n - p\|, \end{aligned} \tag{24}$$

$$\begin{aligned} \|y_n - p\| &\leq \left\|T\Big[(1-\beta_n)Tv_n + \beta_n Tz_n\Big] - p\right\| \leq \delta\|(1-\beta_n)Tv_n + \beta_n Tz_n - p\| \\ &\leq \delta^2(1-\beta_n)\|v_n - p\| + \delta^2\beta_n\|z_n - p\| \end{aligned} \tag{25}$$

$$\begin{aligned} \|z_n - p\| &= \|T\big((1-\gamma_n)v_n + \gamma_n Tv_n\big] - p\| \\ &\leq \delta(1-\gamma_n)\|v_n - p\| + \delta^2\gamma_n\|v_n - p\| \\ &= \|v_n - p\|\delta(1 - \gamma_n(1-\delta)). \end{aligned} \tag{26}$$

Using (24),(25) and (26), we have

$$\|v_{n+1} - p\| \leq \|v_n - p\|\delta^4\Big(1 - (1-\delta)(\beta_n + \delta\beta_n\gamma_n)\Big) + \varepsilon_n \tag{27}$$

Since $\Big(1 - (1-\delta)(\beta_n + \delta\beta_n\gamma_n)\Big) < 1$, by Lemma 2.2, we can conclude that $v_n \to p(n \to \infty)$.
Conversely, suppose $v_n \to p(n \to \infty)$ we show that $\varepsilon_n \to 0$.

$$\begin{aligned} \varepsilon_n &= \|v_{n+1} - f(T, v_n)\| = \|v_{n+1} - p + p - f(T, v_n)\| \\ &\leq \|v_{n+1} - p\| + \|f(T, v_n) - p\| \\ &\leq \|v_{n+1} - p\| + \|v_n - p\|\delta^2\big(1 - (1-\delta)(\beta_n + \delta\gamma_n\beta_n)\big] \to 0(n \to \infty) \end{aligned} \tag{28}$$

Therefore, $\varepsilon_n \to 0(n \to \infty)$, hence, the iterative scheme of equation (11) is $T-$stable. □

3.3 Rate of Convergence

Fixed point theory was designed to solve equations that arises from physical problem using a computer; to save computing time and cost, researchers search for the fastest way to get the fixed point by using different iterative scheme. It is our aim in this section to show that the proposed scheme is faster and better than the earlier existing schemes.

Theorem 3.3 *Let C be a non empty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with fixed point p. Given that $x_0 = v_0 \in C$, consider the iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ defined by (11) and (9) respectively, also let β_n, γ_n be sequences in $[0,1]$ such that $\beta \leq \beta_n < 1$ and $\gamma \leq \delta\beta_n\gamma_n < 1$ where $\beta, \gamma > 0$. Then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converge to p faster then $\{v_n\}_{n=0}^{\infty}$ does.*

Proof: From the inequality in (22), we have

$$\|x_{n+1}-p\| \leq \|x_0-p\|\delta^{4(n+1)}\prod_{k=0}^{n}\Big(1-(1-\delta)(\beta_k+\delta\beta_k\gamma_k)\Big) \tag{29}$$

From our assumption that $\beta \leq \beta_n$ and $\gamma \leq \delta\beta_n\gamma_n$ for all $n \in \mathbb{N}$ where $\beta, \gamma > 0$, and (29), we have

$$\begin{aligned}\|x_{n+1}-p\| &\leq \|x_0-p\|\delta^{4(n+1)}\Big(1-(1-\delta)(\beta+\gamma)\Big)^{n+1}\\ &\leq \|x_0-p\|\delta^{4(n+1)}\prod_{k=0}^{n}\Big(1-(1-\delta)\beta\Big)^{n+1}.\end{aligned} \tag{30}$$

From [28, (9) of Theorem 2], we have

$$\|v_{n+1}-p\| \leq \|v_0-p\|\delta^{3(n+1)}\prod_{k=0}^{n}\Big(1-(1-\delta)\beta_k\Big). \tag{31}$$

Applying the assumption to (31) yields

$$\|x_{n+1}-p\| \leq \|v_0-p\|\delta^{3(n+1)}\Big(1-(1-\delta)\beta\Big)^{n+1}. \tag{32}$$

Now, following the procedure from Definition 2.2, we let

$$a_n = \|x_0-p\|\delta^{4(n+1)}\Big(1-(1-\delta)\beta\Big)^{n+1}$$

and

$$b_n = \|v_0-p\|\delta^{3(n+1)}\Big(1-(1-\delta)\beta\Big)^{n+1}.$$

It suffices to show that $\lim_{n\to\infty}\frac{a_n}{b_n}=0$. First, computing $\frac{a_n}{b_n}$ gives

$$\frac{a_n}{b_n} = \frac{\|x_0-p\|\delta^{4(n+1)}\Big(1-(1-\delta)\beta\Big)^{n+1}}{\|v_0-p\|\delta^{3(n+1)}\Big(1-(1-\delta)\beta\Big)^{n+1}} = \delta^{n+1}, \tag{33}$$

since, $\delta \in (0,1)$, and from classical analysis we know that if $\delta < 1$ then $\delta^n \to 0 (n \to \infty)$. Therefore, from (33) we can conclude that,

$$\lim_{n\to\infty}\frac{\|x_n-p\|}{\|v_n-p\|} = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\delta^{n+1} = 0, \tag{34}$$

hence by Definition 2.1 we say that $\{x_n\}_{n\to 0}^{\infty}$ is faster than $\{v_n\}_{n\to 0}^{\infty}$. Therefore, our iteration(modified Picard−S−AK Hybrid Iterative scheme) in (11) is faster and better than AK iteration in (9), thereby faster than other types of iterations. □

Using Java programing Language we present some numerical examples to support the analytical proof of Theorem 3.3.

Example 3.1 *Let $T : [0,2] \to [0,2]$ be a mapping defined by $Tx = \frac{x}{4}$. We can check that the mapping T is a contraction mapping whose fixed point is $p = 0$. We take $\alpha_n = \beta_n = \gamma_n = 1/3$, the initial value $x_0 = 1.5$. In Table 1 we show with a numerical example that the modified Picard−S−AK hybrid iteration converge faster than Picard−S (6), Vatan two step (7), and AK (9).*

Table 1: Numerical Example for the map $Tx = \frac{x}{4}$ to compare the rate of convergence for the Modified Picard$-S-$AK, AK, VTS and Thakur iterations

S/N	**Modified Picard$-S-$AK**	AK	VTS	Picard$-$S
0	1.5	1.5	1.5	1.5
1	0.004272	0.013184	0.052734	0.273438
2	1.22E-05	0.000116	0.001854	0.049845
3	3.47E-08	1.02E-06	6.52E-05	0.009086
4	9.87E-11	8.95E-09	2.29E-06	0.001656
5	2.81E-13	7.87E-11	8.06E-08	0.000302
6	8.01E-16	6.91E-13	2.83E-09	5.50E-05
7	2.28E-18	6.08E-15	9.96E-11	1.00E-05
8	6.50E-21	5.34E-17	3.50E-12	1.83E-06
9	1.85E-23	4.69E-19	1.23E-13	3.33E-07
10	5.27E-26	4.13E-21	4.33E-15	6.08E-08

Using the same example considered by Ullah et al. (2016) [29]

Example 3.2 *Define a mapping* $T : [0, 4] \to [0, 4]$ *by* $Tx = (x + 2)^{1/3}$. *T is a contraction map whose fixed point is* $p = 1.5213797068045676$. *Take* $\alpha_n = \beta_n = \gamma_n = 1/4$, *the initial value* $x_0 = 1.99$. *In Table 2 we show with a numerical example that the modified Picard–S–AK hybrid iteration converge faster than Picard–S (6), VTS (7), and AK (9).*

Example 3.3 *Define a mapping* $T : [0, 2] \to [0, 2]$ *by* $Tx = (4x - 1)^{1/2}$. *T is a contraction map whose fixed point is* $p = 3.7320508075688770$. *Take* $\alpha_n = \beta_n = \gamma_n = 1/3$, *the initial value* $x_0 = 2$. *In Table 3 we show with a numerical example that the modified Picard–S–AK hybrid iteration converge faster than Picard–S (6), VTS (7), and AK (9).*

Example 3.4 *Define a mapping* $T : [-1, 1] \to [-1, 1]$ *by* $Tx = \cos x$. *T is a contraction map whose fixed point is* $p = 0.7390851332151607$. *take* $0 < \alpha_n = \beta_n = \gamma_n \leq 1/10$, *the initial value* $x_0 = -0.3$. *In Table 4 we show with a numerical example that the modified Picard–S−AK hybrid iteration converge faster than Picard–S (6), VTS (7), and AK (9).*

From Table 1 through Table 4, we can conclude that our iteration (11) is faster than AK iterative scheme.

Table 2: Numerical Example for the map $Tx = (2+x)^{1/3}$ to compare the rate of convergence for the Modified Picard$-S-$AK, AK, VTS and Thakur iterations

S/N	**Picard$-S-$AK**	AK	VTS	Thakur
0	1.99	1.99	1.99	1.99
1	1.5215292883611031	1.5222105961579007	1.5287181133965206	1.5286532306858767
2	1.5213797568807825	1.5213812399046278	1.5214992564560188	1.5214970406053234
3	1.5213797068213322	1.5213797096335470	1.5213816556397390	1.5213816007700969
4	1.5213797068045731	1.5213797068097878	1.5213797385737817	1.5213797373766744
5	1.5213797068045676	1.5213797068045771	1.5213797073224580	1.5213797072980580
6	1.5213797068045676	1.5213797068045676	1.5213797068130100	1.5213797068125334
7	1.5213797068045676	1.5213797068045676	1.5213797068047050	1.5213797068046961
8	1.5213797068045676	1.5213797068045676	1.5213797068045698	1.5213797068045696
9	1.5213797068045676	1.5213797068045676	1.5213797068045676	1.5213797068045676
10	1.5213797068045676	1.5213797068045676	1.5213797068045676	1.5213797068045676
⋮				
15	1.5213797068045676	1.5213797068045676	1.5213797068045676	1.5213797068045676

3.4 Data Dependence

Theorem 3.4 *Let $\widetilde{T}$ be an approximate operator of T. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (11) for T and define an iterative sequence $\{\widetilde{x_n}\}_{n=0}^{\infty}$ as follows*

$$\begin{cases} \widetilde{x_0} \in C, \\ \widetilde{x}_{n+1} = \widetilde{T}(\widetilde{T}\widetilde{y}_n), \\ \widetilde{y}_n = \widetilde{T}[(1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{z}_n], \\ \widetilde{z}_n = \widetilde{T}[(1-\gamma_n)\widetilde{x}_n + \gamma_n \widetilde{T}\widetilde{x}_n], \quad \beta_n, \gamma_n \in [0,1) \;\; n = 1, 2, \dots, \end{cases} \tag{35}$$

where $\beta_n, \gamma_n \in [0,1]$ are real sequences satisfying (i) $1/2 \le \beta_n$, for all $n \in \mathbb{N}$, and (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$. If $Tp = p$ and $\widetilde{T}\widetilde{p} = \widetilde{p}$ such that $\widetilde{x}_n \to \widetilde{p}(n \to \infty)$, then we have

$$\|p - \widetilde{p}\| \le \frac{10\epsilon}{1-\delta} \tag{36}$$

for fixed $\epsilon > 0$. Recall that $\delta \in (0,1)$

Proof: It follows from (11) and (35) that

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_{n+1}\| &= \|T(Ty_n) - \widetilde{T}(\widetilde{T}\widetilde{y}_n)\| = \|T(Ty_n) - T(\widetilde{T}\widetilde{y_n}) + T(\widetilde{T}\widetilde{y_n}) - \widetilde{T}(\widetilde{T}\widetilde{y}_n)\| \\ &\le \delta\|Ty_n - \widetilde{T}\widetilde{y_n}\| + \epsilon = \delta\Big(\|Ty_n - T\widetilde{y}_n\| + \|T\widetilde{y}_n - \widetilde{T}\widetilde{y}_n\|\Big) + \epsilon \\ &\le \delta^2\|y_n - \widetilde{y}_n\| + \delta\epsilon + \epsilon \end{aligned} \tag{37}$$

Table 3: Numerical Example for the map $Tx = (4x - 1)^{1/2}$ to compare the rate of convergence for the Modified Picard$-S-$AK, AK, VTS and Thakur iterations

S/N	**Picard$-S-$AK**	AK	VTS	Thakur
0	2.0000000000000000	2.0000000000000000	2.0000000000000000	2.0000000000000000
1	3.5732777508068327	3.4796012314488170	3.1960899886816800	3.2193585442051633
2	3.7211052942003800	3.7033712616872220	3.5936101086277454	3.6046231067887002
3	3.7313115629602813	3.7288855974576460	3.6979265700426110	3.7017220880648662
4	3.7320009493297834	3.7317025962270440	3.7237360209411080	3.7249063182141935
5	3.7320474452026750	3.7320125136035327	3.7300304959408828	3.7303718685274690
⋮				
10	3.7320508075641876	3.7320508069530270	3.7320491037582713	3.7320496085242962
11	3.7320508075685610	3.7320508075011530	3.7320503939533003	3.7320505260058527
⋮				
14	3.7320508075688770	3.7320508075687875	3.7320508016515800	3.7320508039230478
⋮				
17	3.7320508075688770	3.7320508075688770	3.7320508074842230	3.7320508075216690
⋮				
20	3.7320508075688770	3.7320508075688770	3.7320508075676660	3.7320508075682660
⋮				
26	3.7320508075688770	3.7320508075688770	3.7320508075688770	3.7320508075688770
27	3.7320508075688770	3.7320508075688770	3.7320508075688770	3.7320508075688770
⋮				
30	3.7320508075688770	3.7320508075688770	3.7320508075688770	3.7320508075688770

$$\begin{aligned}
\|y_n - \widetilde{y}_n\| &= \Big\| T\big((1-\beta_n)Tx_n + \beta_n Tz_n\big] - \widetilde{T}\big((1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{z}_n\big] \Big\| \\
&= \Big\| T\big((1-\beta_n)Tx_n + \beta_n Tz_n\big] - T\big((1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{z}_n\big] + \\
&\quad T\big((1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{z}_n\big] - \widetilde{T}\big((1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{z}_n\big] \Big\| \\
&\leq \delta \Big\| (1-\beta_n)Tx_n + \beta_n Tz_n - (1-\beta_n)\widetilde{T}\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{z}_n \Big\| + \epsilon \\
&= \delta\left(\Big\| (1-\beta_n)(Tx_n - \widetilde{T}\widetilde{x}_n) + \beta_n(Tz_n - \widetilde{T}\widetilde{z}_n) \Big\| \right) + \epsilon \\
&= \delta(1-\beta_n)\|Tx_n - \widetilde{T}\widetilde{x}_n\| + \delta\beta_n\|Tz_n - \widetilde{T}\widetilde{z}_n\| + \epsilon \\
&= \delta(1-\beta_n)\Big(\|Tx_n - T\widetilde{x}_n + T\widetilde{x}_n - \widetilde{T}\widetilde{x}_n\|\Big) + \delta\beta_n\Big(\|Tz_n - T\widetilde{z}_n + T\widetilde{z}_n\widetilde{T} - \widetilde{z}_n\|\Big) + \epsilon \\
&\leq \delta^2(1-\beta_n)\|x_n - \widetilde{x}_n\| + \delta(1-\beta_n)\epsilon + \delta^2\beta_n\|z_n - \widetilde{z}_n\| + \delta\beta_n\epsilon + \epsilon \qquad (38)
\end{aligned}$$

$$\begin{aligned}
\|z_n - \widetilde{z}_n\| &= \Big\| T\big[(1-\gamma_n)x_n + \gamma_n Tx_n\big] - \widetilde{T}\big[(1-\gamma_n)\widetilde{x}_n + \gamma_n\widetilde{T}\widetilde{x}_n\big] \Big\| \\
&\leq \Big\| T\big[(1-\gamma_n)x_n + \gamma_n Tx_n\big] - T\big[(1-\gamma_n)\widetilde{x}_n + \gamma_n\widetilde{T}\widetilde{x}_n\big] \Big\| + \\
&\quad \Big\| T\big[(1-\gamma_n)\widetilde{x}_n + \gamma_n\widetilde{T}\widetilde{x}_n\big] - \widetilde{T}\big[(1-\gamma_n)\widetilde{x}_n + \gamma_n\widetilde{T}\widetilde{x}_n\big] \Big\| \\
&\leq \delta \Big\| (1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n)\widetilde{x}_n - \gamma_n\widetilde{T}\widetilde{x}_n \Big\| + \epsilon \\
&\leq \delta(1-\gamma_n)\|x_n - \widetilde{x}_n\| + \delta\gamma_n \Big\| Tx_n - \widetilde{T}\widetilde{x}_n \Big\| + \epsilon \\
&\leq \delta(1-\gamma_n)\|x_n - \widetilde{x}_n\| + \delta\gamma_n\|Tx_n - T\widetilde{x}_n\| + \delta\gamma_n\|T\widetilde{x}_n - \widetilde{T}\widetilde{x}_n\| + \epsilon \\
&\leq \delta(1-\gamma_n)\|x_n - \widetilde{x}_n\| + \delta^2\gamma_n\|x_n - \widetilde{x}_n\| + \delta\gamma_n\epsilon + \epsilon \\
&= \big(\delta(1-\gamma_n) + \delta^2\gamma_n\big)\|x_n - \widetilde{x}_n\| + \delta\gamma_n\epsilon + \epsilon \qquad (39)
\end{aligned}$$

Combining inequality (37),(38) and (39), we have

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}_{n+1}\| &\leq \Big[\delta^4(1-\beta_n) + \delta^4\beta_n(\delta - \delta\gamma_n + \delta^2\gamma_n)\Big]\|x_n - \widetilde{x}_n\| + \\
&\quad \delta^3\epsilon(1-\beta_n) + \delta^5\beta_n\gamma_n\epsilon + \delta^4\beta_n\epsilon + \delta^3\beta_n\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon \\
&\leq \big[1 - (1-\delta)\beta_n\big]\|x_n - \widetilde{x}_n\| + \delta^3\epsilon(1-\beta_n) + \delta^5\beta_n\gamma_n\epsilon + \delta^4\beta_n\epsilon \\
&\quad + \delta^3\beta_n\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon \qquad (40)
\end{aligned}$$

From the assumption (i) $1 - \beta_n \leq \beta_n$, hence, we have the following

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}_{n+1}\| &\leq \big[1 - (1-\delta)\beta_n\big]\|x_n - \widetilde{x}_n\| + 4\beta_n\epsilon + 3\epsilon \\
&= \big[1 - (1-\delta)\beta_n\big]\|x_n - \widetilde{x}_n\| + 4\beta_n\epsilon + 3\epsilon(1 - \beta_n + \beta_n) \\
&\leq \big[1 - (1-\delta)\beta_n\big]\|x_n - \widetilde{x}_n\| + \beta_n(1-\delta)\frac{10\epsilon}{(1-\delta)} \qquad (41)
\end{aligned}$$

Now, let $\eta_n = \|x_n - \widetilde{x}_n\|, \vartheta_n = (1-\delta)\beta_n$ and $\varphi_n = \frac{10\epsilon}{1-\delta}$. Hence, using Lemma 2.3 on (41) we have

$$0 \leq \lim_{n\to\infty} \sup \|x_n - \widetilde{x}_n\| \leq \lim_{n\to\infty} \sup \frac{10\epsilon}{1-\delta}. \qquad (42)$$

From Theorem 1, $x_n \to p(n \to \infty)$ and from the assumption that $\widetilde{x}_n \to \widetilde{p}(n \to \infty)$ combining them with (42) we have $\|p - \widetilde{p}\| \leq \frac{10\epsilon}{1-\delta}$. Therefore, our method is data dependent. □

Table 4: Numerical Example for the map $Tx = \cos x$ to compare the rate of convergence for the Modified Picard$-S-$AK, AK, VTS and Thakur iteration

L	**Picard$-S-$AK**	AK	VTS	Thakur
0	-0.3000000000000000	-0.3000000000000000	-0.3000000000000000	-0.3000000000000000
1	0.6702719493486955	0.8378263440771122	0.5743627193346635	0.5743542852078412
2	0.7254561590812675	0.7088635848549613	0.6688801170029883	0.6688708950367270
3	0.7363361243609965	0.7478976100281707	0.7082366483424817	0.7082295243225575
4	0.7385289558618012	0.7364714135954801	0.7254031074375175	0.7253984733603084
5	0.7389725411549252	0.7398566550956421	0.7329958773044617	0.7329931362464921
⋮				
23	0.7390851332151607	0.7390851332153885	0.7390851302329063	0.7390851302254091
24	0.7390851332151607	0.7390851332150933	0.7390851318846007	0.7390851318811028
⋮				
30	0.7390851332151607	0.7390851332151607	0.7390851332046660	0.7390851332046310
⋮				
45	0.7390851332151607	0.7390851332151607	0.7390851332151606	0.7390851332151606
46	0.7390851332151607	0.7390851332151607	0.7390851332151607	0.7390851332151607
⋮				

Conclusion

In this work we introduce a new iterative scheme called modified Picard$-S-$AK Hybrid iterative scheme, we showed that our algorithm converge faster then other methods, and we gave some example to verify our claim.

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