Chelyshkov's Collocation Method for Solving Three-Dimensional Linear Fredholm Integral Equations

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Abstract

The main purpose of this work is to use the Chelyshkov-collocation method for the solution of three- dimensional Fredholm integral equations. The method is based on the approximate solution in terms of Chelyshkov polynomials with unknown coefficients. This method transforms the integral equation to a system of linear algebraic equations by means of collocation points. Finally, numerical results are included to show the validity and applicability of the method and comparisons are made with existing results.

Keywords: Three-Dimensional Integral Equations, Chelyshkov Polynomials, Collocation Points.

Mathematics Subject Classification: 45L05, 65R10 , 45A05.

1. Introduction

Consider the following three-dimensional linear Fredholm integral equation of the second kind:

$$u(x,y,z) = f(x,y,z) + \lambda \int_e^f \int_c^d \int_a^b K(x,y,z,r,s,t)\ u(r,s,t)drdsdt, \qquad (x,y,z) \in D \tag{1}$$

where f and K are known functions, $u(x,y,z)$ is the unknown function to be determined, λ is a constant and D is a cubic domain.

Equation (1) has been studied in [1] by Radial basis functions method, in [2] by Adomian decomposition method, in [3] by Jacobi polynomials method. In this paper we will study the approximate solution of equation (1) by using a new orthogonal polynomial (Chelyshkov polynomials).

The orthogonal Chelyshkov polynomials are introduced in [4]. These polynomials have been used in the solution integral equations of one and two-dimensional in [5,6,7], differential equations in [8]. Recently, Sezer et al. [9] derived the matrix method based on Chelyshkov polynomials for solving a class of mixed functional integro-differential equations. On the other hand, Talaei in [10] proposed the Chelyskov matrix formulation based on collocation method for numerical solution of the multi-order fractional differential equations.

The main purpose of this paper is to extend the application of the Chelyshkov polynomials to solve three-dimensional Fredholm integral equation (1).

2.Some properties of Chelyshkov polynomials

Chelyshkov [4] has introduced sequences of polynomials which are orthogonal in the interval [0, 1] with the weight function 1. These polynomials are explicitly defined by:

$$C_{N,n}(x) = \sum_{j=0}^{N-n} (-1)^j \binom{N-n}{j} \binom{N+n+j+1}{N-n} x^{n+j}, \qquad n = 0,1,\ldots,N. \tag{2}$$

This gives the Rodrigues formula

$$C_{N,n}(x) = \frac{1}{(N-n)!}\ \frac{1}{x^{n+1}}\ \frac{d^{N-n}}{dx^{N-n}}\ [x^{N+n+1}\ (1-x)^{N-n}], \qquad n = 0, 1, \ldots, N, \tag{3}$$

and the orthogonality condition of Chelyshkov polynomials [4] is

$$\int_0^1 C_{N,j}(x)\ C_{N,k}(x)\ dx = \begin{cases} 0, & k \neq j \\ \frac{1}{j+k+1}, \quad k = j, & k = 0, 1, \ldots, N, N+1, \end{cases} \tag{4}$$

it follows from (3) that

$$\int_0^1 C_{N,n}(x)\ dx = \int_0^1 x^n dx = \frac{1}{n+1}. \tag{5}$$

Making use of formula (3) and Cauchy integral formula for derivatives of an analytic function, we can get the integral relation:

$$C_{N,n}(x) = \frac{1}{2\pi i}\ \frac{1}{x^{n+2}}\ \int_C \frac{z^{-(N+n+2)}\ (1-z)^{N-n}}{(z-x^{-1})^{N-n+1}}\ dz, \tag{6}$$

where C is a closed curve, which inclose the point $z = x^{-1}$.

3.Solution of Three-dimensional Integral Equation

In this section, we solve three-dimensional Fredholm integral equation of the second kind of the form (1) by using Chelyshkov polynomials.

A three-dimensional function $u(x, y, z)$ can be approximated by the expansion:

$$u(x,y,z) \approx u_N(x,y,z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} a_{i(N+1)^2+j(N+1)+k}\ C_{N,i}(x)\ C_{N,j}(y)\ C_{N,k}(z), \tag{7}$$

where $i, j, k = 0, 1, \ldots, N, a_{i(N+1)^2+j(N+1)+k}$ are unknown Chelyshkov coefficients and $C_{N,i}(x),\ C_{N,j}(y),\ C_{N,k}(z)$ are Chelyshkov orthogonal polynomials of degree N.

Inserting (7) into equation (1) we obtain :

$$\sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} a_{i(N+1)^2+j(N+1)+k}\ \ [C_{N,i}(x)\ C_{N,j}(y)\ C_{N,k}(z) -$$

$$- \int_0^1 \int_0^1 \int_0^1 k(x,y,z,r,s,t)\ \ C_{N,i}(r)\ C_{N,j}(s)\ C_{N,k}(t) dr ds dt] = f(x,y,z)\ . \tag{8}$$

By using the collocation points

$$x_l = \frac{2l-1}{2N+1}, \qquad y_m = \frac{2m-1}{2N+1}, \qquad z_n = \frac{2n-1}{2N+1}, \quad l, m, n = 1, 2, \ldots, N+1. \tag{9}$$

We get

$$\sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} a_{i(N+1)^2+j(N+1)+k}\ \ [C_{N,i}(x_l)\ C_{N,j}(y_m)\ C_{N,k}(z_n) -$$

$$- \lambda \int_e^f \int_c^d \int_a^b K(x_l, y_m, z_n, r, s, t) \;\; C_{N,i}(r) \; C_{N,j}(s) \; C_{N,k}(t) dr ds dt] = f(x_l, y_m, z_n) \; . \qquad (10)$$

Clearly, the obtained system of linear algebraic equations contains $(N+1)^3$ equations in the same number as unknowns. Solving this system we obtain the value of the constants $a_{i(N+1)^2+j(N+1)+k}$ such that $i, j, k = 0, \ldots, N$.

4. Numerical Examples

In this section, we illustrate the presented method by giving some examples. The results are compared with the exact solutions by calculating the following absolute error

$$e(x, y, z) = |u(x, y, z) - u_N(x, y, z)|$$

where, $u(x, y, z)$ denotes the exact solution of the given examples, and $u_N(x, y, z)$ is the approximate solution by the presented method. It should be mentioned that all calculations are done by Maple.

Example 1 [1]:

Consider the following three dimensional Fredholm integral equation :

$$u(x, y, z) = xyz \; e^{-x^2-y^2-z^2} \; - \frac{1}{16} \; (1 - e^{-1})^3 \; xyz + \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 xyz \; u(r, s, t) \; dr \; ds \; dt, \qquad (11)$$

with the exact solution $u(x, y, z) = xyz \; e^{-x^2-y^2-z^2}$.

Applying Chelyshkov polynomials for equation (11) when $N = 2$, and by using the collocation points (9) we obtain a system of linear algebraic equations contains 27 equations in the same number as unknowns. Solving this system we obtain the value of the constants, and we obtain the approximate solution which is given by:

$$\begin{gathered}
u_N(x, y, z) = 0.0007873128x + 0.0007873126z + 0.944605983xy^2z^2 - 1.37306981x^2yz + 0.94460599x^2yz^2 + \\
0.94460599x^2y^2z - 0.64984347x^2y^2z^2 - 1.373069821xyz^2 - 1.37306981xy^2z + 0.0007873129y + 0.027270882x^2y \\
-0.018761055x^2y^2 + 0.027270882xy^2 - 0.039640682xz + 0.027270882xz^2 + 0.0272708857x^2z - \\
0.018761055x^2z^2 - 0.039640682yz + 0.027270882yz^2 + 0.0272708816y^2z - 0.018761055y^2z^2 - 0.0000156371 \\
+1.99583869xyz - 0.0005416333y^2 - 0.0005416339z^2 - 0.000541633x^2 - 0.039640683xy.
\end{gathered}$$

The Absolute error of example 1 given in Table 1, Figs 1,2 and 3 represented the exact solution, the approximate solution and the Absolute error respectively when $z = 0.2$.

Table 1. Numerical results of example 1

(x, y, z)	Exact solution	Approx solution	Abs.Error
(0,0,0)	0	-0.0000156371	1.56371×10^{-5}
(0.1,0.1,0.1)	0.0009704455335	0.0007846903197	$1.857552148 \times 10^{-4}$
(0.2,0.2,0.2)	0.007095363492	0.007095028152	3.35340×10^{-7}
(0.3,0.3,0.3)	0.02061124635	0.02074019839	1.2895204×10^{-4}
(0.4,0.4,0.4)	0.03960213707	0.03931124813	2.9088894×10^{-4}
(0.5,0.5,0.5)	0.05904581910	0.05839631262	6.4950651×10^{-4}
(0.6,0.6,0.6)	0.07335263354	0.07334359182	9.04169×10^{-6}
(0.7,0.7,0.7)	0.07886444140	0.08055643859	$1.69199721 \times 10^{-3}$
(0.8,0.8,0.8)	0.07506276461	0.07832055994	$3.25779549 \times 10^{-3}$
(0.9,0.9,0.9)	0.06417885095	0.06716333197	$2.98448145 \times 10^{-3}$
(1,1,1)	0.04978706837	0.0497452263	4.184167×10^{-5}

Example 2 [1]:
Consider the following three dimensional Fredholm integral equation :

$$u(x, y, z) = f(x, y, z) + \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \frac{xyz}{1 + x + y + z} \; u(r, s, t) \; dr \; ds \; dt, \tag{12}$$

where $f(x, y, z) = sin^2x \; sin^2y \; sin^2z + \frac{1}{2} \; \frac{xyz \; (cos(1) \; sin(1)-1)^3}{1+x+y+z}$

with the exact solution $u(x, y, z) = sin^2x \; sin^2y \; sin^2z$.

Similarly as in example 1 applying Chelyshkov polynomials for equation (12) when $N = 2$, and by using the collocation points (9) we obtain a system of linear algebraic equations contains 27 equations in the same number as unknowns. Solving this system we obtain the value of the constants, and we obtain the approximate solution which is given by:

$$u_2(x, y, z) = 0.001429524699x + 0.0014295247z + 0.001429524672y - 0.0112888601xy - 0.00820966506xy^2 - 0.008209663633x^2y - 0.007180000558x^2y^2 - 0.01128885993yz - 0.008209665051yz^2 - 0.008209664561y^2z - 0.007180000y^2z^2 - 0.011288860xz - 0.008209664920xz^2 - 0.008209664x^2z - 0.007180000461x^2z^2 - 0.0002179117715 + 0.001230252165z^2 + 0.0012302521y^2 + 0.0012302519x^2 + 0.0318514693xyz + 0.0739536993xyz^2 + 0.07395369738xy^2z + 0.04511383xy^2z^2 + 0.07395367978x^2yz + 0.04511384x^2yz^2 + 0.04511384x^2y^2z + 0.04278394949x^2y^2z^2.$$

The Absolute error of example 2 given in Table 2, Figs 4 and 5 represented the exact solution and the Absolute error respectively when $z = 0.1$.

Table 2. Numerical results of example 2

(x, y, z)	Exact solution	Approx solution	Abs.Error
(0,0,0)	0	-0.0002179117715	$2.179117715 \times 10^{-4}$
(0.1,0.1,0.1)	$9.900465317 \times 10^{-7}$	-0.00008679081435	$8.778086086 \times 10^{-5}$
(0.2,0.2,0.2)	0.00006148723635	-0.0003399208785	$4.014081148 \times 10^{-4}$
(0.3,0.3,0.3)	0.0006660749352	0.02074019839	$8.005445385 \times 10^{-4}$
(0.4,0.4,0.4)	0.003487373171	-0.0001344696027	$1.242644630 \times 10^{-3}$
(0.5,0.5,0.5)	0.01214302780	0.002244728541	$2.518434630 \times 10^{-3}$
(0.6,0.6,0.6)	0.03240718143	0.009624593168	$6.19315807 \times 10^{-3}$
(0.7,0.7,0.7)	0.07886444140	0.02621402335	$1.357693612 \times 10^{-2}$
(0.8,0.8,0.8)	0.1362726624	0.1126040572	$2.366860522 \times 10^{-2}$
(0.9,0.9,0.9)	0.2310246264	0.2005956467	$3.04289798 \times 10^{-2}$
(1,1,1)	0.3550053292	0.3349348555	$2.00704739 \times 10^{-2}$

Example 3 [1]:

Consider the following three dimensional Fredholm integral equation :

$$u(x,y,z) = x^2y^2z^2 - \frac{8}{27}\, sin(xyz) + \frac{1}{2}\int_0^2\int_0^1\int_{-1}^1 sin(xyz)\ u(r,s,t)\ dr\ ds\ dt, \tag{13}$$

with the exact solution $u(x,y,z) = x^2y^2z^2$.

Applying Chelyshkov polynomials for equation (13) when $N = 2$, and by using the collocation points (9) we obtain a system of linear algebraic equations contains 27 equations in the same number as unknowns. Solving this system we obtain the value of the constants as follows:

$$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = 0, a_{11} = 0, a_{12} = 0, a_{13} = 0,$$
$$a_{14} = 0, a_{15} = 0, a_{16} = 0, a_{17} = 0, a_{18} = 0, a_{19} = 0, a_{20} = 0, a_{21} = 0, a_{22} = 0, a_{23} = 0, a_{24} = 0, a_{25} = 0, a_{26} = 1,$$

and we obtain the approximate solution which is the same as the exact solution.

Example 4 [2]:

Consider the following three dimensional Fredholm integral equation :

$$u(x,y,z) = \frac{1}{180} + xy(z-y) + \int_0^1\int_0^1\int_0^1 \frac{1}{5}\ r\ u(r,s,t)\ dr\ ds\ dt, \tag{14}$$

with the exact solution $u(x,y,z) = xy(z-y)$.

Applying Chelyshkov polynomials for equation (14) when $N = 2$, and by using the collocation points (9) we obtain a system of linear algebraic equations contains 27 equations in the same number as unknowns. Solving this system we obtain the value of the constants as follows:

$$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = 0, a_{11} = 0, a_{12} = 0, a_{13} = 1/64,$$
$$a_{14} = 5/64, a_{15} = -1/12, a_{16} = -11/64, a_{17} = -5/192, a_{18} = 0, a_{19} = 0, a_{20} = 0, a_{21} = 0, a_{22} = 5/64,$$
$$a_{23} = 25/64, \quad a_{24} = -5/12, \quad a_{25} = -55/64, \quad a_{26} = -25/192,$$

and we obtain the approximate solution which is the same as the exact solution.

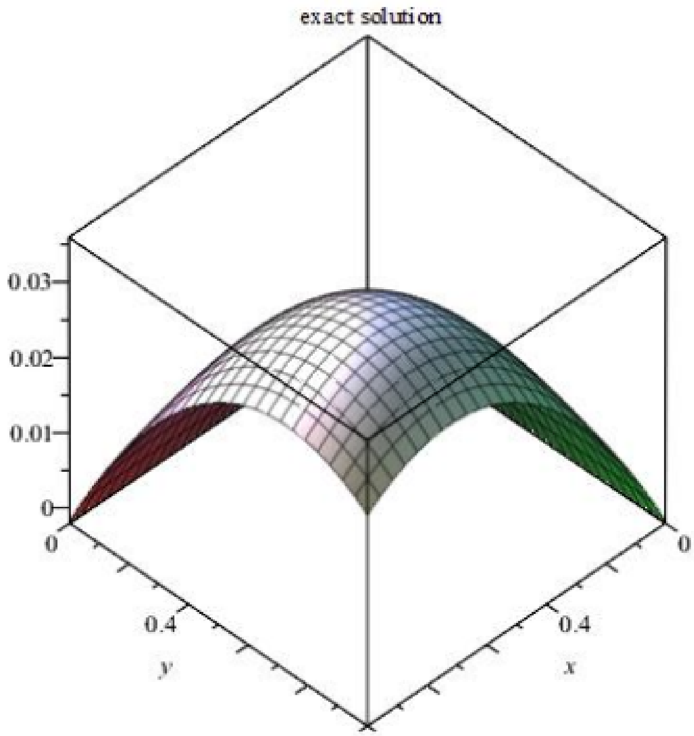

Figure 1: Exact solution of example 1

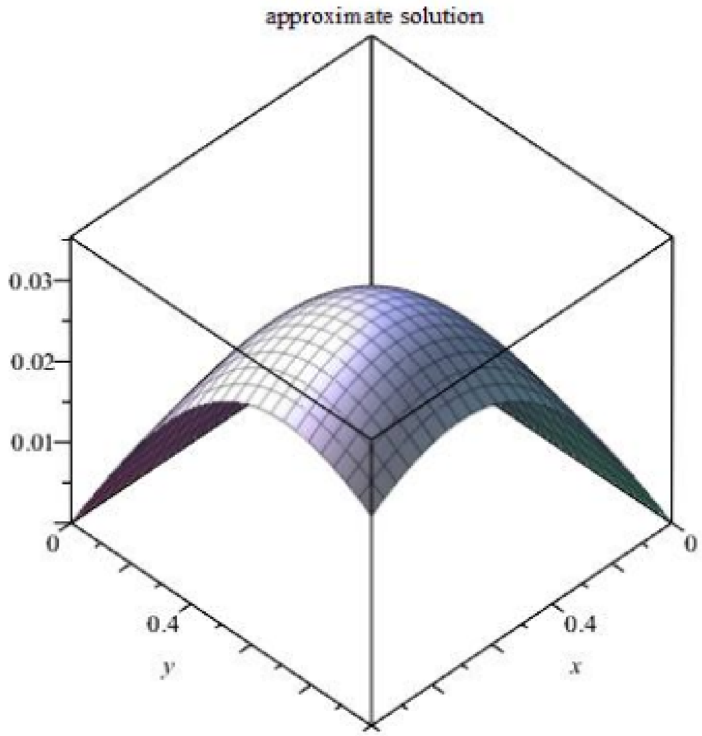

Figure 2: Approximate solution of example 1

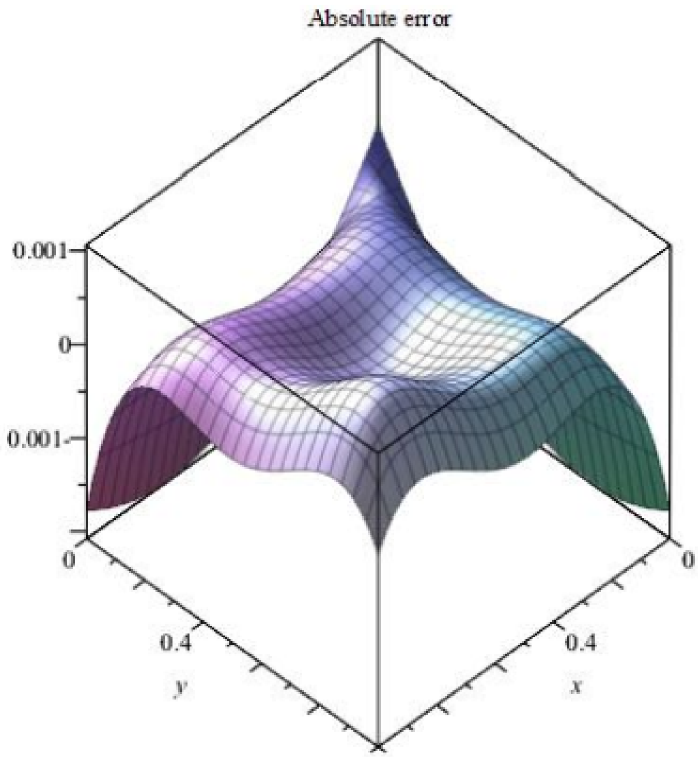

Figure 3: Absolute Error of example 1

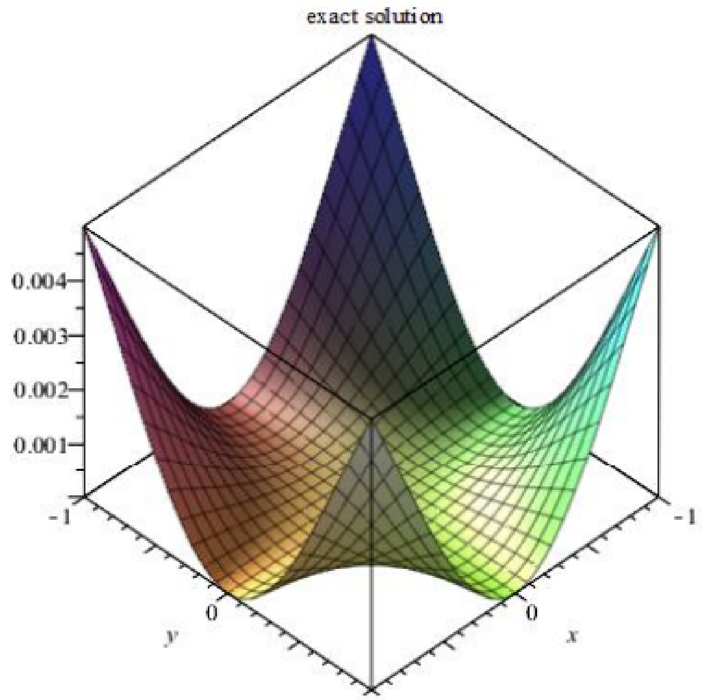

Figure 4: Exact solution of example 2

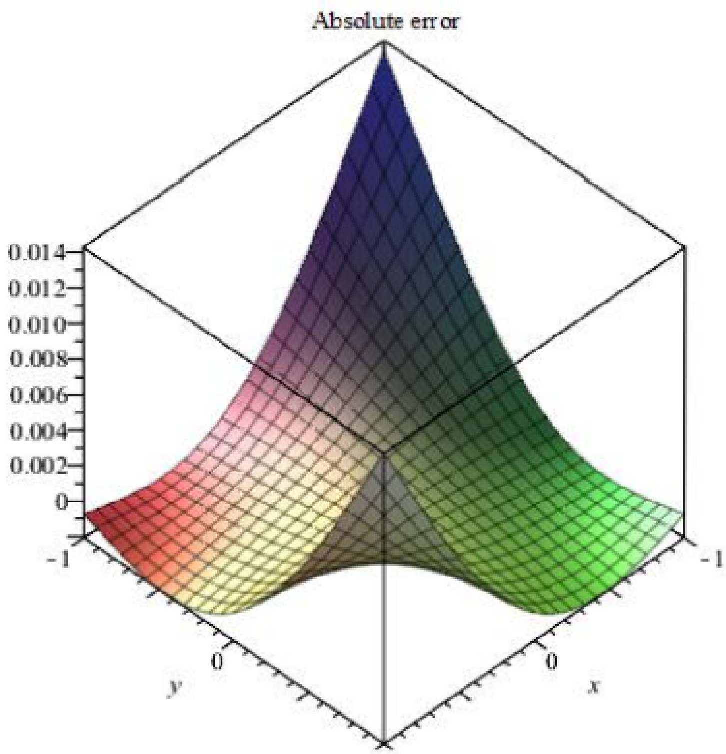

Figure 5: Absolute Error of example 2

Conclusion

This paper concerns the approximate solution of three- dimensional Fredholm integral equations of the second kind by using Chelyshkov polynomial method. The authors in [1] study the same examples 1,2 and 3 by using Radial basis functions method when $N = 27,\ N = 64,\ N = 125,\ N = 216$, but we apply our method for $N = 2$. We show that the results of our method is better than the results of the method which used in [1]. Also example 4 has been studied in [2] by using Adomian decomposition method and we obtain the same results.

References

[1] M. Esmaeilbeigi, F. Mirzaee and M. Moazami; Radial basis functions method for solving three-dimensional linear Fredholm integral equations on the cubic domains. Iranian Journal of Numerical Analysis and Optimization. Vol. 7, No. 2, (2017), 15-37.

[2] M. Almousa; The Solutions of Three Dimensional Fredholm Integral Equations using Adomian Decomposition Method. AIP Conf. Proc. 1739, 020053-1020053-6; (2016),1-7.

[3] K. Sadri, A. Aminic and C. Chenge; Low cost numerical solution for three-dimensional linear and nonlinear integral equations via three-dimensional Jacobi polynomials. Journal of Computational and Applied Mathematics. 319, (2017), 493-513.

[4] V. S. Chelyshkov; Alternative Orthogonal Polynomials and quadratures. Electronic Transactions on Numerical Analysis. Volume 25, (2006), 17-26.

[5] J. Saffar Ardabili and Y. Talaei; Chelyshkov Collocation Method for Solving the Two-Dimensional Fredholm-Volterra Integral Equations. Int. J. Appl. Comput. Math. (2018),1-13.

[6] J. Ahmadi Shali , P. Darania and A.A. Akbarfam; Collocation method for nonlinear Volterra-Fredholm integral equations. J Appl Sci 2:(2012),115-121.

[7] M. Rasty and M. Hadizadeh ; A Product integration approach on new orthogonal polynomials for nonlinear weakly singular integral equations. Acta Appl Math. 109:(2010), 861-873.

[8] Z. Soori and A. Aminataei; The spectral method for solving Sine Gordon equation using a new orthogonal polynomial. Appl Math 2012:(2012),1-12.

[9] C. Oguza and M. Sezer. Chelyshkov collocation method for a class of mixed functional integro-differential equations. Appl Math Comput 259:(2015),943-954.

[10] Y. Talaei and M. Asgari; An operational matrix based on Chelyshkov polynomials for solving multi-order fractional differential equations. Neural Comput and Applic. (2017), 1-8.