$\mathcal{I}_{\ddot{g}}$-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract

Characterizations and properties of $\mathcal{I}_{\ddot{g}}$-closed sets and $\mathcal{I}_{\ddot{g}}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\ddot{g}}$-open sets. Also, it is established that an $\mathcal{I}_{\ddot{g}}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact. We introduced the concepts of sg -$\mathcal{I}$-locally closed sets, $\wedge_{sg}$-sets and ζ_{sg}-$\mathcal{I}$-closed sets. We introduced $\mathcal{I}_{\ddot{g}}$-continuous, $\mathcal{I}_{\ddot{g}}$-irresolute, sg-$\mathcal{I}$-LC-continuous, ζ_{sg}-$\mathcal{I}$-continuous and to obtain decompositions of $\star$-continuity in ideal topological spaces.

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1 Introduction

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A $\in \mathcal{I}$ and B$\subset$ A $\Rightarrow$ B $\in \mathcal{I}$ and (ii) A $\in \mathcal{I}$ and B $\in \mathcal{I} \Rightarrow$ A $\cup$ B $\in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X)$\rightarrow\wp$(X), called a local function [12] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A $\subseteq$ X, A^* ($\mathcal{I}$, τ)=\{x $\in$ X | U $\cap$ A $\notin \mathcal{I}$ for every U $\in \tau$(x)\} where τ(x) = \{U $\in \tau$ | x $\in$ U\}. We will make use of the basic facts about the local functions [[10], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl^*(.) for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology, finer than τ is defined by cl^*(A) = A $\cup$ $A^*(\mathcal{I}, \tau)$ [22]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$.

2 Preliminaries

If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\star$-closed [10] (resp. $\star$-dense in itself [8]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [2] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open. In this paper, we characterize $\mathcal{I}_{\ddot{g}}$-closed sets and discuss their properties. Also, we characterize normal spaces in terms of $\mathcal{I}_{\ddot{g}}$-open sets. Finally, we obtain decompositions of $\star$-continuity. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and $\text{int}^*(A)$ will denote the interior of A in (X, τ^*). A subset A of a space (X, τ) is an α-open [19] (resp. semi-open [13], preopen [16]) set if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subseteq \text{cl}(\text{int}(A))$, $A \subseteq \text{int}(\text{cl}(A))$). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by $\text{cl}_\alpha(A)$.

2.1 Definition

A subset A of a space (X, τ) is called:

1. g-closed [14] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of g-closed set is called g-open set.
2. a sg-closed set [1] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X, τ). The complement of sg-closed set is called sg-open set.
3. a $\ddot{g}$-closed set [4] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ). The complement of $\ddot{g}$-closed set is called $\ddot{g}$-open set.

The family of all sg-open sets in (X, τ) is a topology on X. The sg-closure [1] of a subset A of X, denoted by sgcl(A), is defined to be the intersection of all sg-closed sets containing A.

2.2 Definition

An ideal $\mathcal{I}$ is said to be

1. codense [3] or τ-boundary [18] if $\tau \cap \mathcal{I} = \{\emptyset\}$,
2. completely codense [3] if $PO(X) \cap \mathcal{I} = \{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ).

2.3 Lemma

Every completely codense ideal is codense but not the converse [3].

2.4 Lemma

Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = \text{cl}(A^*) = \text{cl}(A)$
$= \text{cl}^*(A)$ [[21], Theorem 5].

2.5 Lemma

Let (X, τ, $\mathcal{I}$) be an ideal space. Then $\mathcal{I}$ is codense if and only if G $\subseteq$ G* for every semi-open set G in X [[21], Theorem 3].

2.6 Lemma

Let (X, τ, $\mathcal{I}$) be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$ [[21], Theorem 6].

2.7 Definition

An ideal topological space (X, τ, $\mathcal{I}$) is said to be a T$_{\mathcal{I}}$-space [2] if every $\mathcal{I}_g$-closed subset of X is a $\star$-closed set.

2.8 Lemma

If (X, τ, $\mathcal{I}$) is a T$_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[17], Corollary 2.2].

2.9 Lemma

Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[2], Theorem 2.1].

2.10 Remark

If (X, τ) is a topological space the following properties hold:

1. Every closed set is sg-closed but not conversely [1].
2. Every closed set is $\ddot{g}$-closed but not conversely [4].
3. Every $\ddot{g}$-closed set is g-closed but not conversely [4].

2.11 Definition

[11] A subset a of ideal topological space (X,τ,$\mathcal{I}$) is called a weakly $\mathcal{I}$-locally closed set(briefly weakly $\mathcal{I}$-LC) if A = M $\cap$ N where M is open and N is $\star$-closed.

2.12 Definition

A function f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) is said to be $\star$-continuous [9] (resp. $\mathcal{I}_g$-continuous [9], weakly $\mathcal{I}$-LC-continuous [11]) if f^{-1}(A) is $\star$-closed (resp. $\mathcal{I}_g$-closed, weakly $\mathcal{I}$-LC-set) in (X, τ, $\mathcal{I}$) for every closed set A of (Y, σ).

3 $\mathcal{I}_{\ddot{g}}$-closed sets

3.1 Definition

A subset A of an ideal space (X,τ,$\mathcal{I}$) is said to be

1. $\mathcal{I}_{\ddot{g}}$-closed if $A^* \subseteq U$ whenever $A\subseteq U$ and U is sg-open,
2. $\mathcal{I}_{\ddot{g}}$-open if its complement is $\mathcal{I}_{\ddot{g}}$-closed.

3.2 Theorem

If (X, τ, $\mathcal{I}$) is any ideal space,

1. Every closed set is $\star$-closed but not conversely.
2. Every $\mathcal{I}_{\ddot{g}}$-closed set is $\mathcal{I}_g$-closed but not conversely.

Proof (1) This is obvious.
(2) It follows from the fact that every open set is sg-open. □

3.3 Example

Let X={5, 6, 7, 8}, τ = {∅, X, {5}, {5, 6}, {5, 6, 7}} and $\mathcal{I}$={∅, {5}}. Then $\star$-closed sets are ∅, X, {5}, {8}, {5, 8}, {7, 8}, {5, 7, 8}, {6, 7, 8}, $\mathcal{I}_{\ddot{g}}$-closed sets are ∅, X, {5}, {8}, {5, 8}, {7, 8}, {5, 7, 8}, {6, 7, 8} and $\mathcal{I}_g$-closed sets are ∅, X, {5}, {8}, {5, 8}, {6, 8}, {7, 8}, {5, 6, 8}, {5, 7, 8}, {6, 7, 8}. It is clear that {6, 8} is $\mathcal{I}_g$-closed but it is not $\mathcal{I}_{\ddot{g}}$-closed.

The following theorem gives characterizations of $\mathcal{I}_{\ddot{g}}$-closed sets.

3.4 Theorem

If (X, τ, $\mathcal{I}$) is any ideal space and $A \subseteq X$, then the following are equivalent.

1. A is $\mathcal{I}_{\ddot{g}}$-closed,
2. $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in X,
3. For all $x \in cl^*(A)$, $sgcl(\{x\}) \cap A \neq \emptyset$.
4. $cl^*(A) -$ A contains no nonempty sg-closed set,
5. $A^* - A$ contains no nonempty sg-closed set.

Proof (1) ⇒ (2) If A is $\mathcal{I}_{\ddot{g}}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is sg-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is sg-open in X. This proves (2).

(2) $\Rightarrow$ (3) Suppose x $\in$cl*(A). If sgcl({x}) $\cap$ A $=$ $\emptyset$, then A $\subseteq$ X $-$ sgcl({x}). By (2), cl*(A) $\subseteq$ X $-$sgcl({x}), a contradiction, since x $\in$ cl*(A).

(3) $\Rightarrow$ (4) Suppose F $\subseteq$ cl*(A) $-$ A, F is sg-closed and x $\in$ F. Since F $\subseteq$ X $-$ A and F is sg-closed, then A $\subseteq$ X $-$ F and F is sg-closed, sgcl({x}) $\cap$ A $=$ $\emptyset$. Since x $\in$ cl*(A) by (3), sgcl({x}) $\cap$ A $\neq$ $\emptyset$. Therefore cl*(A) $-$ A contains no nonempty sg-closed set.

(4) $\Rightarrow$ (5) Since cl*(A) $-$ A $=$ (A $\cup$ A*) $-$ A $=$ (A $\cup$ A*) $\cap$ A^c $=$ (A $\cap$ A^c) $\cup$ (A* $\cap$ A^c) $=$A* $\cap$ A^c $=$ A* $-$ A. Therefore A* $-$ A contains no nonempty sg-closed set.

(5) $\Rightarrow$ (1) Let A $\subseteq$ U where U is sg-open set. Therefore X $-$ U $\subseteq$ X $-$ A and so A* $\cap$ (X $-$ U) $\subseteq$ A* $\cap$ (X $-$ A) $=$A* $-$ A. Therefore A * $\cap$(X $-$ U) $\subseteq$ A* $-$ A. Since A* is always a closed set, so A* is a sg-closed set and so A* $\cap$(X $-$ U) is a sg-closed set contained in A* $-$ A. Therefore A* $\cap$(X $-$ U) $=$ $\emptyset$ and hence A* $\subseteq$ U. Therefore A is $\mathcal{I}_{\ddot{g}}$-closed. □

3.5 Theorem

Every $\star$-closed set is $\mathcal{I}_{\ddot{g}}$-closed but not conversely.

Proof Let A be a $\star$-closed, then A* $\subseteq$ A. Let A $\subset$ U where U is sg-open. Hence A* $\subseteq$ U whenever A $\subseteq$ U and U is sg-open. Therefore A is $\mathcal{I}_{\ddot{g}}$-closed. □

3.6 Example

Let X={5, 6, 7, 8}, τ={ϕ, X, {5, 7}, {5, 6, 7}} and $\mathcal{I}$ $=$ {$\emptyset$, {8}}. Then $\mathcal{I}_{\ddot{g}}$-closed sets are ϕ, X, {8}, {6, 8}, {5, 6, 8}, {6, 7, 8} and $\star$-closed sets are ϕ, X, {8}, {6, 8}. It is clear that {5, 6, 8} is $\mathcal{I}_{\ddot{g}}$-closed set but it is not $\star$-closed.

3.7 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. For every A $\in$ $\mathcal{I}$, A is $\mathcal{I}_{\ddot{g}}$-closed.

Proof Let A $\subseteq$ U where U is a sg-open set. Since A* $=$ $\emptyset$ for every A $\in$ $\mathcal{I}$, then cl*(A) $=$ A $\cup$ A* $=$ A $\subseteq$ U. Therefore, by Theorem 3.4, A is $\mathcal{I}_{\ddot{g}}$-closed. □

3.8 Theorem

If (X, τ, $\mathcal{I}$) is an ideal space, then A* is always $\mathcal{I}_{\ddot{g}}$-closed for every subset A of X.

Proof Let A* $\subseteq$ U where U is a sg-open. Since $(A^*)^*$ $\subseteq$ A* [10], we have $(A^*)^*$ $\subseteq$ U whenever A* $\subseteq$ U and U is a sg-open. Hence A* is $\mathcal{I}_{\ddot{g}}$-closed. □

3.9 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $\mathcal{I}_{\ddot{g}}$-closed, sg-open set is a $\star$-closed set.

Proof Since A is $\mathcal{I}_{\ddot{g}}$-closed and sg-open. Then A* $\subseteq$ A whenever A $\subseteq$ A and A is a sg-open. Hence A is a $\star$-closed. □

3.10 Corollary

If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_{\ddot{g}}$-closed set, then A is a $\star$-closed set.

Proof By assumption A is $\mathcal{I}_{\ddot{g}}$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 3.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 2.7, A is $\star$-closed. □

3.11 Corollary

Let (X, τ, $\mathcal{I}$) be an ideal space and A be an $\mathcal{I}_{\ddot{g}}$-closed set. Then the following are equivalent.

1. A is a $\star$-closed set,
2. $cl^*(A)-$ A is a sg-closed set,
3. $A^* - A$ is a sg-closed set.

Proof (1) $\Rightarrow$ (2) If A is $\star$-closed, then $A^* \subseteq A$ and so $cl^*(A) - A = (A \cup A^*) - A = \emptyset$. Hence $cl^*(A) - A$ is sg-closed set.

(2) $\Rightarrow$ (3) Since $cl^*(A) - A = A^* - A$ and so $A^* - A$ is sg-closed set.

(3) $\Rightarrow$ (1) If $A^* - A$ is a sg-closed set, since A is an $\mathcal{I}_{\ddot{g}}$-closed set, by Theorem 3.4 (5), $A^* - A = \emptyset$ and so A is $\star$-closed. □

3.12 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. Then every $\ddot{g}$-closed set is an $\mathcal{I}_{\ddot{g}}$-closed set but not conversely.

Proof Let A be a $\ddot{g}$-closed set. Then $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. We have $cl^*(A) \subseteq cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. Hence A is $\mathcal{I}_{\ddot{g}}$-closed. □

3.13 Example

Let X, τ and $\mathcal{I}$ be as in the Example 3.3. Then $\ddot{g}$-closed sets are ϕ, X, {8}, {7, 8}, {6, 7, 8} It is clear that {5} is an $\mathcal{I}_{\ddot{g}}$-closed set but it is not $\ddot{g}$-closed.

3.14 Theorem

If (X, τ, $\mathcal{I}$) is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}_{\ddot{g}}$-closed subset of X, then A is $\ddot{g}$-closed.

Proof Suppose A is a $\star$-dense in itself, $\mathcal{I}_{\ddot{g}}$-closed subset of X. Let $A \subseteq U$ where U is sg-open. Then by Theorem 3.4 (2), $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. Since A is $\star$-dense in itself, by Lemma 2.4, $cl(A) = cl^*(A)$. Therefore $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. Hence A is $\ddot{g}$-closed. □

3.15 Corollary

If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$=$\{\emptyset\}$, then A is $\mathcal{I}_{\ddot{g}}$-closed if and only if A is $\ddot{g}$-closed.

Proof The proof follows from the fact that for $\mathcal{I} = \{\emptyset\}$, $A^* = cl(A) \supseteq A$. Therefore A is $\star$-dense in itself. Since A is $\mathcal{I}_{\ddot{g}}$-closed, by Theorem 3.14, A is $\ddot{g}$-closed.

Conversely, by Theorem 3.12, every $\ddot{g}$-closed set is $\mathcal{I}_{\ddot{g}}$-closed set. □

3.16 Corollary

If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$ is codense and A is a semi-open, $\mathcal{I}_{\ddot{g}}$-closed subset of X, then A is $\ddot{g}$-closed.

Proof The proof follows Lemma 2.5, A is $\star$-dense in itself. By Theorem 3.14, A is $\ddot{g}$-closed. □

3.17 Remark

g-closed sets and $\mathcal{I}_{\ddot{g}}$-closed sets are independent.

3.18 Example

Let X, τ and $\mathcal{I}$ be as in the Example 3.3. Then g-closed sets are ϕ, X, $\{8\}$, $\{5, 8\}$, $\{6, 8\}$, $\{7, 8\}$, $\{5, 6, 8\}$, $\{5, 7, 8\}$, $\{6, 7, 8\}$. It is clear that $\{5, 6, 8\}$ is g-closed set but it is not $\mathcal{I}_{\ddot{g}}$-closed. Also it is clear that $\{5\}$ is an $\mathcal{I}_{\ddot{g}}$-closed set but it is not g-closed.

3.19 Remark

We have the following implications for the subsets stated above.

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \ddot{g}\text{-closed} & \longrightarrow & \text{g-closed} \\ \downarrow & & \downarrow & & \downarrow \\ \star\text{-closed} & \longrightarrow & \mathcal{I}_{\ddot{g}}\text{-closed} & \longrightarrow & \mathcal{I}_{g}\text{-closed} \end{array}$$

Diagram

3.20 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_{\ddot{g}}$-closed if and only if $A = F - M$ where F is $\star$-closed and M contains no nonempty sg-closed set.

Proof If A is $\mathcal{I}_{\ddot{g}}$-closed, then by Theorem 3.4 (5), $M = A^* - A$ contains no nonempty sg-closed set. If $F = cl^*(A)$, then F is $\star$-closed such that $F - M = (A \cup A^*) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A \cup A^*) \cap ((A^*)^c \cup A\ ) = (A \cup A^*) \cap (A \cup (A^*)^c) = A \cup (A^* \cap (A^*)^c) = A$.

Conversely, suppose A = F − M where F is $\star$-closed and M contains no nonempty sg-closed set. Let U be a sg-open set such that A$\subseteq$U. Then, F − M $\subseteq$ U which implies that F $\cap$(X − U) $\subseteq$ M. Now, A $\subseteq$ F and $F^* \subseteq F$, then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq M$. By hypothesis, since $A^* \cap (X - U)$ is sg-closed, $A^* \cap (X - U) = \emptyset$ and so $A^* \subseteq U$. Hence A is $\mathcal{I}_{\ddot{g}}$-closed. □

3.21 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space and A $\subseteq$ X. If $A \subseteq B \subseteq A^*$, then $A^* = B^*$ and B is $\star$-dense in itself.

Proof Since A $\subseteq$ B, then $A^* \subseteq B^*$ and since $B \subseteq A^*$, then $B^* \subseteq (A^*)^* \subseteq A^*$. Therefore, $A^* = B^*$ and $B \subseteq A^* \subseteq B^*$. Hence proved. □

3.22 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$ and A is $\mathcal{I}_{\ddot{g}}$-closed, then B is $\mathcal{I}_{\ddot{g}}$-closed.

Proof Since A is $\mathcal{I}_{\ddot{g}}$-closed, then by Theorem 3.4 (5), $cl^*(A) - A$ contains no nonempty sg-closed set. Since $cl^*(B) - B \subseteq cl^*(A) - A$ and so $cl^*(B) - B$ contains no nonempty sg-closed set and so by Theorem 3.4 (4), B is $\mathcal{I}_{\ddot{g}}$-closed. □

3.23 Corollary

Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is $\mathcal{I}_{\ddot{g}}$-closed, then A and B are $\ddot{g}$-closed sets.

Proof Let A and B be subsets of X such that $A \subseteq B \subseteq A^*$ which implies that $A \subseteq B \subseteq A^* \subseteq cl^*(A)$ and A is $\mathcal{I}_{\ddot{g}}$-closed. By Theorem 3.22, B is $\mathcal{I}_{\ddot{g}}$-closed. Since $A \subseteq B \subseteq A^*$, then $A^* = B^*$ and so A and B are $\star$-dense in itself. By Theorem 3.14, A and B are $\ddot{g}$-closed. □

The following theorem gives a characterization of $\mathcal{I}_{\ddot{g}}$-open sets.

3.24 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space and A $\subseteq$ X. Then A is $\mathcal{I}_{\ddot{g}}$-open if and only if $F \subseteq int^*(A)$ whenever F is sg-closed and F $\subseteq$ A.

Proof Suppose A is $\mathcal{I}_{\ddot{g}}$-open. If F is sg-closed and F $\subseteq$ A, then $X - A \subseteq X - F$ and so $cl^*(X - A) \subseteq X - F$ by Theorem 3.4 (2). Therefore $F \subseteq X - cl^*(X - A) = int^*(A)$. Hence $F \subseteq int^*(A)$.

Conversely, suppose the condition holds. Let U be a sg-open set such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq int^*(A)$. Therefore $cl^*(X - A) \subseteq U$. By Theorem 3.4 (2), X − A is $\mathcal{I}_{\ddot{g}}$-closed. Hence A is $\mathcal{I}_{\ddot{g}}$-open. □

3.25 Corollary

Let (X, τ, $\mathcal{I}$) be an ideal space and A $\subseteq$ X. If A is $\mathcal{I}_{\ddot{g}}$-open, then $F \subseteq int^*(A)$ whenever F is closed and F $\subseteq$ A.

The following theorem gives a property of $\mathcal{I}_{\ddot{g}}$-closed.

3.26 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space and A $\subseteq$ X. If A is $\mathcal{I}_{\ddot{g}}$-open and int*(A) $\subseteq$ B $\subseteq$ A, then B is $\mathcal{I}_{\ddot{g}}$-open.

Proof Since A is $\mathcal{I}_{\ddot{g}}$-open, then X $-$ A is $\mathcal{I}_{\ddot{g}}$-closed. By Theorem 3.4 (4), cl*(X $-$ A) $-$ (X $-$ A) contains no nonempty sg-closed set. Since int*(A) $\subseteq$ int*(B) which implies that cl*(X $-$ B) $\subseteq$ cl*(X $-$ A) and so cl*(X $-$ B) $-$(X $-$ B) $\subseteq$ cl*(X $-$ A) $-$(X $-$ A). Hence B is $\mathcal{I}_{\ddot{g}}$-open. $\square$

The following theorem gives a characterization of $\mathcal{I}_{\ddot{g}}$-closed sets in terms of $\mathcal{I}_{\ddot{g}}$-open sets.

3.27 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space and A $\subseteq$ X. Then the following are equivalent.

1. A is $\mathcal{I}_{\ddot{g}}$-closed,
2. A $\cup$ (X $-$ A*) is $\mathcal{I}_{\ddot{g}}$-closed,
3. A* $-$ A is $\mathcal{I}_{\ddot{g}}$-open.

Proof (1) $\Rightarrow$ (2) Suppose A is $\mathcal{I}_{\ddot{g}}$-closed. If U is any sg-open set such that A $\cup$ (X $-$ A*) $\subseteq$ U, then X $-$ U $\subseteq$ X $-$ (A $\cup$ (X $-$ A*)) $=$ X $\cap$(A $\cup$ (A*)^c)^c $=$ A* $\cap$ A^c $=$ A* $-$ A. Since A is $\mathcal{I}_{\ddot{g}}$-closed, by Theorem 3.4 (5), it follows that X $-$ U $=$ $\emptyset$ and so X $=$ U. Therefore A $\cup$ (X $-$ A*) $\subseteq$ U which implies that A $\cup$ (X $-$ A*) $\subseteq$ X and so (A $\cup$ (X $-$ A*))* $\subseteq$ X* $\subseteq$ X $=$ U. Hence A $\cup$(X $-$A*) is $\mathcal{I}_{\ddot{g}}$-closed.

(2) $\Rightarrow$ (1) Suppose A $\cup$ (X $-$ A*) is $\mathcal{I}_{\ddot{g}}$-closed. If F is any sg-closed set such that F $\subseteq$ A* $-$ A, then F $\subseteq$ A* and F A which implies that X $-$ A* $\subseteq$ X $-$ F and A $\subseteq$ X $-$ F.
Therefore A $\cup$ (X $-$ A*) $\subseteq$ A $\cup$(X $-$ F) $=$ X $-$ F and X $-$ F is sg-open. Since (A $\cup$ (X $-$ A*))* $\subseteq$ X $-$ F which implies that A* $\cup$ (X $-$ A*)* $\subseteq$ X $-$ F and so A* $\subseteq$ X $-$ F which implies that F $\subseteq$ X $-$ A*. Since F $\subseteq$ A*, it follows that F $=$ $\emptyset$. Hence A is $\mathcal{I}_{\ddot{g}}$-closed.

(2) $\Leftrightarrow$ (3) Since X $-$ (A* $-$ A) $=$ X $\cap$ (A* $\cap$ A^c)^c $=$ X $\cap$((A*)^c $\cup$ A) $=$(X $\cap$ (A*)^c) $\cup$ (X $\cap$ A) $=$ A $\cup$ (X $-$ A*) is $\mathcal{I}_{\ddot{g}}$-closed. Hence, A* $-$ A is $\mathcal{I}_{\ddot{g}}$-open. $\square$

3.28 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}_{\ddot{g}}$-closed if and only if every sg-open set is $\star$-closed.

Proof Suppose every subset of X is $\mathcal{I}_{\ddot{g}}$-closed. If U$\subseteq$X is sg-open, then U is $\mathcal{I}_{\ddot{g}}$-closed and so U* $\subseteq$ U. Hence, U is $\star$-closed.

Conversely, suppose that every sg-open set is $\star$-closed. If U is a sg-open set such that A $\subseteq$ U $\subseteq$ X, then A* $\subseteq$ U* $\subseteq$ U and so A is $\mathcal{I}_{\ddot{g}}$-closed. $\square$

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\ddot{g}}$-open sets.

3.29 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. Then, the following are equivalent.

1. X is normal,
2. For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{\ddot{g}}$-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V,
3. For any closed set A and open set V containing A, there exists an $\mathcal{I}_{\ddot{g}}$-open set U such that A $\subseteq$ U $\subseteq$cl*(U) $\subseteq$ V.

Proof (1) $\Rightarrow$ (2) The proof follows from the fact that every open set is $\mathcal{I}_{\ddot{g}}$-open.

(2) $\Rightarrow$ (3) Suppose A is closed and V is an open set containing A. Since A and X $-$ V are disjoint closed sets, there exist disjoint $\mathcal{I}_{\ddot{g}}$-open sets U and W such that A $\subseteq$ U and X $-$ V $\subseteq$ W. Since X $-$ V is sg-closed and W is $\mathcal{I}_{\ddot{g}}$-open, X $-$ V $\subseteq$ int*(W) and so X $-$int*(W) $\subseteq$ V. Again, U $\cap$ W $= \emptyset$ which implies that U $\cap$ int*(W) $= \emptyset$ and so U $\subseteq$ X $-$ int*(W) which implies that cl*(U) $\subseteq$ X $-$ int*(W) $\subseteq$ V. U is the required $\mathcal{I}_{\ddot{g}}$-open set with A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.

(3) $\Rightarrow$ (1) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{\ddot{g}}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X $-$ B. Since U is $\mathcal{I}_{\ddot{g}}$-open, A $\subseteq$ int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 2.6, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X $-$ cl*(U) $\in \tau^\alpha$. Hence A $\subseteq$ int*(U) $\subseteq$ int(cl(int(int*(U)))) $=$ G and B $\subseteq$ X $-$ cl*(U) $\subseteq$ int(cl(int(X $-$ cl*(U)))) $=$ H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

3.30 Definition

A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be a $\ddot{g}_\alpha$-closed set [5] if cl_α(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg-open. The complement of $\ddot{g}_\alpha$-closed is said to be a $\ddot{g}_\alpha$-open set.

If $\mathcal{I} = \mathcal{N}$, then $\mathcal{I}_{\ddot{g}}$-closed sets coincide with $\ddot{g}_\alpha$-closed sets and so we have the following Corollary.

3.31 Corollary

Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I} = \mathcal{N}$. Then, the following are equivalent.

1. X is normal,
2. For any disjoint closed sets A and B, there exist disjoint $\ddot{g}_\alpha$-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V,
3. For any closed set A and open set V containing A, there exists an $\ddot{g}_\alpha$-open set U such that A $\subseteq$ U $\subseteq$ cl_α(U) $\subseteq$ V.

3.32 Definition

A subset A of an ideal space is said to be $\mathcal{I}$-compact [7] or compact modulo $\mathcal{I}$ [18] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that A $- \cup$ $U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space (X, τ, $\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.

3.33 Theorem

Let (X, τ, $\mathcal{I}$) be an ideal space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[17], Theorem 2.17].

3.34 Corollary

Let (X, τ, $\mathcal{I}$) be an ideal space. If A is an $\mathcal{I}_{\ddot{g}}$-closed subset of X, then A is $\mathcal{I}$-compact.

Proof The proof follows from the fact that every $\mathcal{I}_{\ddot{g}}$-closed is $\mathcal{I}_g$-closed. □

4 sg -$\mathcal{I}$-locally closed sets

4.1 Definition

A subset a of ideal topological space (X, τ, $\mathcal{I}$) is called a sg-$\mathcal{I}$-locally closed set(briefly sg-$\mathcal{I}$-LC) if A = M ∩ N where M is sg-open and N is $\star$-closed.

4.2 Proposition

Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following holds.

1. If A is sg-open , then A is sg-$\mathcal{I}$-LC set.
2. A is $\star$-closed, then A is sg-$\mathcal{I}$-LC set.
3. If A is a weakly $\mathcal{I}$-LC-set, then A is an sg-$\mathcal{I}$-LC set.

The converse of the above Proposition 4.2 need not be true as shown in the following examples.

4.3 Example

Let X, τ and $\mathcal{I}$ be as in the Example 3.6. Then sg-open sets are ϕ, X, {5}, {7}, {5, 7}, {5, 6, 7}, {5, 7, 8}, sg-$\mathcal{I}$-LC sets are ϕ, X, {5}, {6}, {7}, {8}, {5, 7}, {6, 8}, {5, 6, 7}, {5, 7, 8} and weakly $\mathcal{I}$-LC-set are {6}, {8}, {5, 7}, {6, 8}, {5, 6, 7}. (1) It is clear that {6, 8} is a sg-$\mathcal{I}$-LC set but it is not sg-open. (2) It is clear that {5, 7} is sg-$\mathcal{I}$-LC set but it is not $\star$-closed. In (3) it is also clear that {5, 7, 8} is a sg-$\mathcal{I}$-LC set but it is not weakly $\mathcal{I}$-LC-set.

4.4 Theorem

Let (X,τ,$\mathcal{I}$) be an ideal topological space. If A is a sg-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A ∩ B is a sg-$\mathcal{I}$-LC-set.

Proof Let B be $\star$-closed , then A ∩ B = (M ∩ N) ∩ B = M ∩ (N ∩ B), where N ∩ B is $\star$-closed. Hence A ∩ B is an sg-$\mathcal{I}$-LC-set. □

4.5 Theorem

A subset of an ideal topological space (X,τ,$\mathcal{I}$) is $\star$-closed if and only if it is

1. weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed [9].
2. sg-$\mathcal{I}$-LC-set and $\mathcal{I}_{\ddot{g}}$-closed.

Proof (2) Necessity is trivial. We prove only sufficiency. Let A be sg-$\mathcal{I}$-LC-set and $\mathcal{I}_{\ddot{g}}$-closed set. Since A is sg-$\mathcal{I}$-LC, A = M ∩ N , where M is sg-open and N is $\star$-closed. So we have A = M ∩ N ⊆ M. Since A is $\mathcal{I}_{\ddot{g}}$-closed, $A^* \subseteq$ M. Also since A = M ∩ N ⊆ N and N is $\star$-closed, we have $A^* \subseteq$ N. Consequently, $A^* \subseteq M \cap N = A$ and hence A is $\star$-closed. □

4.6 Remark

1. The notions of weakly $\mathcal{I}$-LC set and $\mathcal{I}_g$-closed set are independent [9].
2. The notions of sg-$\mathcal{I}$-LC-set and $\mathcal{I}_{\ddot{g}}$-closed set are independent.

4.7 Example

Let X, τ and $\mathcal{I}$ be as in the Example 4.3. It is clear that {5} is sg-$\mathcal{I}$-LC- set but it is not $\mathcal{I}_{\ddot{g}}$-closed. Also, is clear that {5, 6, 8} is an $\mathcal{I}_{\ddot{g}}$-closed but it is not sg-$\mathcal{I}$-LC set.

4.8 Definition

[4] Let A be a subset of a topological space (X, τ). then, sg-kernel of the set A, denoted by sg-ker(A), is the intersection of all sg-open supersets of A.

4.9 Definition

A subset A of a topological space (X, τ) is called $\wedge_{sg}$-set if A = sg-ker(A).

4.10 Definition

A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called ζ_{sg}-$\mathcal{I}$-closed if A = R ∩ S where R is a $\wedge_{sg}$-set and S is a $\star$-closed.

4.11 Lemma

1. Every $\star$-closed set is ζ_{sg}-$\mathcal{I}$-closed but not conversely.
2. Every $\wedge_{sg}$-set is ζ_{sg}-$\mathcal{I}$-closed but not conversely.

4.12 Example

Let X , τ and $\mathcal{I}$ be as in the Example 4.3. then, ζ_{sg}-$\mathcal{I}$-closed sets are ϕ, X, {5}, {6}, {7}, {8}, {5, 7}, {6, 8}, {5, 6, 7}, {5, 7, 8} and $\wedge_{sg}$-sets are ϕ, X, {5}, {7}, {5, 7}, {5, 6, 7}, {5, 7, 8}. It is clear that {7} is ζ_{sg}-$\mathcal{I}$-closed but it is not $\star$-closed. also, is clear that {6, 8} is ζ_{sg}-$\mathcal{I}$-closed but it is not $\wedge_{sg}$-set.

4.13 Remark

The concepts of $\star$-closed and $\wedge_{sg}$-set are independent.

4.14 Example

Let X , τ and $\mathcal{I}$ be as in the Example 4.12. It is clear that {7} is $\wedge_{sg}$-set but it is not $\star$-closed. also, it is clear that {8} is $\star$-closed set but it is not $\wedge_{sg}$-set.

4.15 Lemma

For a subset A of an ideal topological space (X,τ,$\mathcal{I}$) the following are equivalent.

1. A is ζ_{sg}-$\mathcal{I}$-closed.
2. A = R $\cap$ cl*(A) where R is a $\wedge_{sg}$-set .
3. A = sg-ker(A) $\cap$ cl*(A)

4.16 Lemma

A subset A $\subseteq$ (X,τ,$\mathcal{I}$) is $\mathcal{I}_{\ddot{g}}$-closed if and only if cl*(A) $\subseteq$ sg-ker(A).

Proof Suppose that A $\subseteq$ X is an $\mathcal{I}_{\ddot{g}}$-closed set. Suppose x $\notin$ sg-ker(A). then, there exists a sg-open set U containing A such that x $\notin$ U. Since A is an $\mathcal{I}_{\ddot{g}}$-closed set, A $\subseteq$ U and U is sg-open implies that cl*(A) $\subseteq$ U and so x $\notin$ cl*(A). therefore, cl*(A) $\subseteq$ sg-ker(A).
Conversely, suppose cl*(A) $\subseteq$ sg-ker(A). If A $\subseteq$ U and U is sg-open, then cl*(A) $\subseteq$ sg-ker(A) $\subseteq$ U . Therefore, A is $\mathcal{I}_{\ddot{g}}$-closed. □

4.17 Theorem

For a subset A of an ideal topological space (X,τ,$\mathcal{I}$) the following are equivalent.

1. A is $\star$-closed.
2. A is $\mathcal{I}_{\ddot{g}}$-closed and sg-$\mathcal{I}$-LC.
3. A is $\mathcal{I}_{\ddot{g}}$-closed and ζ_{sg}-$\mathcal{I}$-closed.

Proof (1) ⇒ (2) ⇒ (3) Obvious.
(3) ⇒ (1). Since A is $\mathcal{I}_{\ddot{g}}$-closed, by (2), Lemma 4.16, cl*(A) ⊆ sg-ker(A). Since A is ζ_{sg}-$\mathcal{I}$-closed, by Lemma 4.15, A = sg-ker(A) ∩ cl*(A) = cl*(A). Hence A is ⋆-closed. □

4.18 Remark

The concepts of $\mathcal{I}_{\ddot{g}}$-closedness and ζ_{sg}-$\mathcal{I}$-closedness are independent.

4.19 Example

Let X , τ and $\mathcal{I}$ be as in the Example 4.12. It is clear that {5, 7} is ζ_{sg}-$\mathcal{I}$-closed but it is not $\mathcal{I}_{\ddot{g}}$-closed. also, it is clear that {5, 6, 8} is $\mathcal{I}_{\ddot{g}}$-closed set but it is not ζ_{sg}-$\mathcal{I}$-closed.

5 $\mathcal{I}_{\ddot{g}}$-Continuous Function

5.1 Definition

[4] A function f: (X, τ, $\mathcal{I}$) → (Y, σ) is called $\mathcal{I}_{\ddot{g}}$-continuous if f^{-1}(V) is an $\mathcal{I}_{\ddot{g}}$-closed set of (X, τ, $\mathcal{I}$) for every closed set V of (Y, σ).

5.2 Proposition

Every ⋆-continuous is $\mathcal{I}_{\ddot{g}}$-continuous but not conversely.

Proof The proof follows from Theorem 3.5. □

5.3 Example

Let X, τ and $\mathcal{I}$ be defined as Example 3.6. Let Y = {5, 6, 7, 8} with σ = {ϕ, Y, {5}, {7}, {5, 7}}. Define f: (X, τ, $\mathcal{I}$) → (Y, σ) the identity function. then, is $\mathcal{I}_{\ddot{g}}$-continuous but not ⋆-continuous, since f^{-1}({5, 6, 8}) = {5, 6, 8} is not ⋆-closed in (X, τ, $\mathcal{I}$).

5.4 Proposition

Every $\mathcal{I}_{\ddot{g}}$-continuous is $\mathcal{I}_g$-continuous but not conversely.

Proof The proof follows from Theorem 3.2(2). □

5.5 Example

Let X, τ and $\mathcal{I}$ be defined as Example 3.3. Let Y = {5, 6, 7, 8} with σ = {$\emptyset$, Y, {7}, {5, 7}} and $\mathcal{J}$={$\emptyset$, {7}}. Define f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) the identity function. then, f is $\mathcal{I}_g$-continuous but not $\mathcal{I}_{\ddot{g}}$-continuous, since f^{-1}({6, 8})= {6, 8} is not $\mathcal{I}_{\ddot{g}}$-closed in (X, τ, $\mathcal{I}$).

5.6 Remark

The composition of two $\mathcal{I}_{\ddot{g}}$-continuous functions need not be $\mathcal{I}_{\ddot{g}}$-continuous and this is shown from the following example.

5.7 Example

Let X={5, 6, 7}, τ={ϕ, X, {5, 6}} and $\mathcal{I}$={$\emptyset$, {5}}. then, $\mathcal{I}_{\ddot{g}}$-closed sets are ϕ, X, {5}, {7}, {5, 7}, {6, 7}. Let Y = {5, 6, 7} with σ = {ϕ, Y, {5}} and $\mathcal{J}$ = {$\emptyset$, {5}}. then, $\mathcal{I}_{\ddot{g}}$-closed sets are ϕ, Y, {5}, {6, 7}. Let Z = {5, 6, 7} with γ = {ϕ, Z, {6}, {5, 7}} and $\mathcal{K}$ = {$\emptyset$}. Define f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) by f(5) = 6, f(6) = 5 and f(7) = 7. Define g : (Y, σ, $\mathcal{J}$) $\rightarrow$ (Z, γ, $\mathcal{K}$) by g(5) = 6, g(6) = 7 and g(7) = 5. Clearly f and g are n$\mathcal{I}_{\ddot{g}}$-continuous but their g $\circ$ f : (X, τ, $\mathcal{I}$) $\rightarrow$ (Z, γ, $\mathcal{K}$) is not $\mathcal{I}_{\ddot{g}}$-continuous, because V = {6} is closed in (Z, γ, $\mathcal{K}$) but (g $\circ$ f $^{-1}$({6})= $f^{-1}(g^{-1}$({6})) = f^{-1}({5}) = {6}, which is not $\mathcal{I}_{\ddot{g}}$-closed in (X, τ, $\mathcal{I}$).

5.8 Proposition

let f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) be $\mathcal{I}_{\ddot{g}}$-continuous if and only if f^{-1}(U) is $\mathcal{I}_{\ddot{g}}$-open in (X, τ, $\mathcal{I}$) for every open set U in (Y, σ).

Proof Let f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) be $\mathcal{I}_{\ddot{g}}$-continuous and U be an open set in (Y, σ). then, U^c is closed in (Y, σ) and since f is $\mathcal{I}_{\ddot{g}}$-continuous, $f^{-1}(U^c)$ is $\mathcal{I}_{\ddot{g}}$-closed in (X, τ, $\mathcal{I}$). But $f^{-1}(U^c) = f^{-1}((U))^c$ and so f^{-1}(U) is $\mathcal{I}_{\ddot{g}}$-open in (X, τ, $\mathcal{I}$).
Conversely, assume that f^{-1}(U) is $\mathcal{I}_{\ddot{g}}$-open in (X, τ, $\mathcal{I}$) for each open set U in (Y, σ). Let F be a closed set in (Y, σ). then, F^c is open in (Y, σ) and by assumption, $f^{-1}(F^c)$ is $\mathcal{I}_{\ddot{g}}$-open in (X, τ, $\mathcal{I}$). Since $f^{-1}(F^c) = f^{-1}((F))^c$, we have f^{-1}(F) is closed in (X, τ, $\mathcal{I}$) and so f is $\mathcal{I}_{\ddot{g}}$-continuous. □

We introduce the following definition

5.9 Definition

A function f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ, $\mathcal{J}$) is called $\mathcal{I}_{\ddot{g}}$-irresolute if f^{-1}(V) is an $\mathcal{I}_{\ddot{g}}$-closed set of (X, τ, $\mathcal{I}$) for every $\mathcal{I}_{\ddot{g}}$-closed set V of (Y, σ, $\mathcal{J}$).

5.10 Theorem

Every $\mathcal{I}_{\ddot{g}}$-irresolute function is $\mathcal{I}_{\ddot{g}}$-continuous but not conversely.

Proof Let f: (X, τ, $\mathcal{I}$) $\to$ (Y, σ, $\mathcal{J}$) be a $\mathcal{I}_{\ddot{g}}$-irresolute function. Let V be a closed set of (Y, σ). then, by the Theorems 3.2(1) and 3.5, V is $\mathcal{I}_{\ddot{g}}$-closed. Since f is $\mathcal{I}_{\ddot{g}}$-irresolute, then f^{-1}(V) is an $\mathcal{I}_{\ddot{g}}$-closed set of (X, τ, $\mathcal{I}$). therefore, f is $\mathcal{I}_{\ddot{g}}$-continuous. □

5.11 Example

Let X = {5, 6, 7}, τ = {ϕ, X, {7}, {5, 6}} and $\mathcal{I}$ = {$\emptyset$}. then, $\mathcal{I}_{\ddot{g}}$-closed sets are ϕ, X, {7}, {5, 6}. Let Y = {5, 6, 7}, σ = {ϕ, Y, {5, 6}} and $\mathcal{J}$ = {$\emptyset$, {5}}. then, $\mathcal{I}_{\ddot{g}}$-closed sets are ϕ, Y, {5}, {7}, {5, 7}, {6, 7}. Define f: (X, τ, $\mathcal{I}$) $\to$ (Y, σ, $\mathcal{J}$) by the identity function. (i) V = {7} is closed on (Y, σ, $\mathcal{J}$) it is clear that f^{-1}({7}) ={7} is $\mathcal{I}_{\ddot{g}}$-closed set of (X, τ, $\mathcal{I}$). (ii) It is clear that {6, 7} is an $\mathcal{I}_{\ddot{g}}$-closed set of (Y, σ, $\mathcal{J}$) but f^{-1}({6, 7}) ={6, 7} is not an $\mathcal{I}_{\ddot{g}}$-closed set of (X, τ, $\mathcal{I}$). thus, f is not $\mathcal{I}_{\ddot{g}}$-irresolute function. However, f is $\mathcal{I}_{\ddot{g}}$-continuous function.

5.12 Theorem

Let f: (X, τ, $\mathcal{I}$) $\to$ (Y, σ, $\mathcal{J}$) and g : (Y, σ, $\mathcal{J}$) $\to$ (Z, γ, $\mathcal{K}$) be any two functions. then,

1. g $\circ$ f is $\mathcal{I}_{\ddot{g}}$-continuous if g is $\star$-continuous and f is $\mathcal{I}_{\ddot{g}}$-continuous.
2. g $\circ$ f is $\mathcal{I}_{\ddot{g}}$-irresolute if both f and g are $\mathcal{I}_{\ddot{g}}$-irresolute.
3. g $\circ$ f is $\mathcal{I}_{\ddot{g}}$-continuous if g is $\mathcal{I}_{\ddot{g}}$-continuous and f is $\mathcal{I}_{\ddot{g}}$-irresolute.

Proof (1) Since g is a $\star$-continuous from (Y, σ, $\mathcal{J}$) $\to$ (Z, γ, $\mathcal{K}$), for any closed set z as a subset of Z, we get g^{-1}(z) = G is a closed set in (Y, σ, $\mathcal{J}$). As f is an $\mathcal{I}_{\ddot{g}}$-continuous function. We get $(g \circ f)^{-1}$(z)= $f^{-1}(g^{-1}(z))$ = f^{-1}(G) = S and S is an $\mathcal{I}_{\ddot{g}}$-closed set in (X, τ, $\mathcal{I}$). Hence (g $\circ$ f) is an $\mathcal{I}_{\ddot{g}}$-continuous function.
(2) Consider two $\mathcal{I}_{\ddot{g}}$-irresolute functions, f: (X, τ, $\mathcal{I}$) $\to$ (Y, σ, $\mathcal{J}$) and g : (Y, σ, $\mathcal{J}$) $\to$ (Z, γ, $\mathcal{K}$) is an $\mathcal{I}_{\ddot{g}}$-irresolute functions. As g is considered to be an $\mathcal{I}_{\ddot{g}}$-irresolute function, by Definition 5.9, for every $\mathcal{I}_{\ddot{g}}$-closed set z $\subseteq$ (Z, γ, $\mathcal{K}$), g^{-1}(z) = G is an $\mathcal{I}_{\ddot{g}}$-closed in (Y, σ, $\mathcal{J}$). Again since f is $\mathcal{I}_{\ddot{g}}$-irresolute, $(g \circ f)^{-1}$(z)= $f^{-1}(g^{-1}(z))$ = f^{-1}(G) = S and S is an $\mathcal{I}_{\ddot{g}}$-closed set in (X, τ, $\mathcal{I}$). Hence (g $\circ$ f) is an $\mathcal{I}_{\ddot{g}}$-irresolute function.
(3) Let g be an $\mathcal{I}_{\ddot{g}}$-continuous function from (Y, σ, $\mathcal{J}$) $\to$ (Z, γ, $\mathcal{K}$) and z subset of Z be a closed set. therefore, g^{-1}(z) is an $\mathcal{I}_{\ddot{g}}$-closed set in (Y, σ, $\mathcal{J}$), by Theorems 3.2(1) and 3.5, g^{-1}(z) = G is an $\mathcal{I}_{\ddot{g}}$-closed set in (Y, σ, $\mathcal{J}$). Also since f is $\mathcal{I}_{\ddot{g}}$-irresolute, we get $(g \circ f)^{-1}$(z)= $f^{-1}(g^{-1}(z))$ = f^{-1}(G) = S and S is an $\mathcal{I}_{\ddot{g}}$-closed set in (X, τ, $\mathcal{I}$). Hence (g $\circ$ f) is a $\mathcal{I}_{\ddot{g}}$-continuous function. □

6 Decompositions of $\star$-continuity

6.1 Definition

A function f: (X, τ, $\mathcal{I}$) $\to$ (Y, σ) is said to be sg-$\mathcal{I}$-LC-continuous (resp ζ_{sg}-$\mathcal{I}$-continuous) if f^{-1}(A) is sg-$\mathcal{I}$-LC-set (resp ζ_{sg}-$\mathcal{I}$-closed) in (X, τ, $\mathcal{I}$) for every closed set A of (Y, σ).

6.2 Theorem

A function f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) is $\star$-continuous if and only if it is

1. weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [9].
2. sg-$\mathcal{I}$-LC-continuous and $\mathcal{I}_{\ddot{g}}$-continuous.

Proof It is an immediate consequence of Theorem 4.5. $\square$

6.3 Theorem

A function f: (X, τ, $\mathcal{I}$) $\rightarrow$ (Y, σ) the following are equivalent.

1. f is $\star$-continuous.
2. f is $\mathcal{I}_{\ddot{g}}$–continuous and sg-$\mathcal{I}$-LC-continuous.
3. f is $\mathcal{I}_{\ddot{g}}$-continuous and ζ_{sg}-$\mathcal{I}$-continuous

Proof It is an immediate consequence of Theorem 4.17. $\square$

Conclusions

In this paper, characterizations and properties of $\mathcal{I}_{\ddot{g}}$-closed sets and $\mathcal{I}_{\ddot{g}}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\ddot{g}}$-open sets. Also, it is established that an $\mathcal{I}_{\ddot{g}}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact. We introduced the concepts of sg -$\mathcal{I}$-locally closed sets, $\wedge_{sg}$-sets and ζ_{sg}-$\mathcal{I}$-closed sets. We introduced $\mathcal{I}_{\ddot{g}}$-continuous, $\mathcal{I}_{\ddot{g}}$-irresolute, sg-$\mathcal{I}$-LC-continuous, ζ_{sg}-$\mathcal{I}$-continuous and to obtain decompositions of $\star$-continuity in ideal topological spaces. In future, we have extended this work in various ideal topological fields with some applications.

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