On the Maximization of Investment Portfolios with Returns of Contributions

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Abstract

In this work, how investment portfolios of a pension scheme can be maximized in the presence of return clause of contributions is presented. This clause permits return of accumulated contributions together with predetermined interest from risk free asset to members' families whenever death occurs to their family members. Also considered herein are investments in cash, marketable security and loan to increase the total accumulated funds of the pension scheme left to be distributed among the surviving members such that the price models of marketable security and loan follows geometric Brownian motions. The game theoretic approach, separation of variable technique and mean variance utility are used to obtain closed form solutions of the optimal control plans for the assets and the efficient frontier. Next, the consequence of some parameters on the optimal control plans with time is numerically analysed. Furthermore, a theoretical comparison of our result with an existing result is given.

Keywords: Pension scheme, extended HJB equation, optimal control plan, return of contributions, game theoretic approach, interest rate.

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1. Introduction

Pension scheme was established to cater and managed the old age income of every retiree. There are two types of pension plan; these include the defined benefit plan (DB) and the defined contributions (DC) plan. In a DC plan, members remit some percentage of their income as ordered by the law into their retirement savings account (RSA) and these funds are managed by the pension fund administrators (PFA) with the aim of maximizing members expected returns. In investment, the PFAs need to know how investments in different assets are done since most of the assets in the market are highly volatile due to their random nature; these assets include cash, bond, stock, loan etc. This uncertainty in the market triggered by the random behaviours of these assets has led to the study of optimal control plan which gives a guide on how financial experts/institutions can combine different assets at a time to achieve optimal result with minimal risk. Based on this, a number of authors had engaged in studying optimal investment plan under diverse conditions; some of which include [7,17, 20], studied optimal investment problems under constant elasticity of variance model using either Legendary or power transformation method to solve the optimization problems. [6,14,19], studied optimal investment problems under affine interest rate.

From the objective function view point, different kind of utility functions used in solving optimization problem exist; they include utility functions which exhibit constant relative risk aversion (CRRA); example power and logarithmic utility functions were used by [5,19], the utility function which exhibit constant absolute risk aversion (CARA); example exponential utility were used by [13]. The mean-variance utility was first developed by Markowitz to investigate optimal portfolio selection problems [12] but the problem is that optimal investment plan under the mean-variance utility are not time consistent, since the mean-variance utility function does not have the iterated expectation condition, hence the Bellman's optimality condition does not hold. However, in many situations time consistency of strategies is a basic requirement for rational decision makers. [4] studied the general theory of Markovian time inconsistent stochastic control problems. The study of optimal investment plan with return of contributions has taken a centre stage since members can possibly die during the accumulations phase; these include [9] who investigated optimal control plan for a defined contribution pension scheme with the return of premiums clauses in a mean-variance utility function. [15] Studied the same problem above both for accumulation and distribution phase with the price of the risky asset modelled by Heston volatility model. Strategic optimal portfolio management for a DC pension scheme with return clause was studied by [2], in this work; they considered investment in one risk free and two risky assets which was an extension of the work of [9]. Investment strategies with return clause under inflation risk and volatility risk was investigated by [16]; they considered investment in a risk free asset, the inflation index bond and the stock whose price was modelled by Heston volatility.[10] Studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under (CEV) model; they considered investments in treasury, stock and bond. In all the literatures above, the authors assumed that the returned accumulations are without any form of interest. Most recently the studies of return of contributions with predetermined interest have been studied by some authors; in their work, they assumed that not only will the accumulated premium be returned to the next of kin but with interest from risk free asset investment which is predetermined at the beginning of the investment. They include [1], where they investigated optimal control plan in a DC plan with one risk free and one risky asset when the return contributions were with predetermined interest and [3] studied the optimal asset allocation strategy for a DC pension system with refund of contributions with predetermined interest under Heston's volatility model; in their work, they considered one risk free asset and a risky asset where the price of the risky asset follows the Heston's volatility model.Since the price process of the risk free asset is deterministic and the interest rate is predetermined, it is possible to determine the interest paid to each death member's family at each point in time during the accumulation phase. This form the basis of this paper, where we shall investigate how to manage a portfolio consisting of three assets whose return clause is with predetermined interest and the price process of the two risky assets are modelled by geometric Brownian motion.

2. The Control Problem

Assume the market is composed of cash, equity and loan and a complete probability space denoted as $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a real probability space, $\mathbb{P}$ a probability measure satisfying the condition $0 \leq t \leq T$. $\{\mathcal{B}_s(t), \mathcal{B}_l(t), \mathcal{B}_m(t) : t \geq 0\}$ are standard Brownian motions and $\mathcal{F}$ is the filtration which represents the available information provided by the Brownian motions. Assume the market is complete and frictionless and open continuously over a fix time $0 \leq t \leq T$, where T is the retirement age.

Let $\mathcal{C}(t)$ denote the price of the cash and the price process is driven by

$$d\mathcal{C}(t) = \Upsilon\, \mathcal{C}(t)\, dt, \tag{1}$$

$\mathcal{S}(t)$ and $\mathcal{L}(t)$ denote the prices of the equity and loan which are modelled by the GBM as follows

$$d\mathcal{S}(t) = \mathcal{S}(t)[(\Upsilon + \varkappa_1)dt + \mathfrak{h}_1 d\mathcal{B}_s(t)] \tag{2}$$

$$d\mathcal{L}(t) = \mathcal{L}(t)\left[(\Upsilon + \varkappa_2)\, dt + \mathfrak{h}_2 d\mathcal{B}_l(t) + \mathfrak{h}_3 d\mathcal{B}_m(t)\right] \tag{3}$$

where $r, \varkappa_1, \varkappa_2, \varkappa_3, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ are constant and Υ is the interest rate of cash which is predetermined, $(r + \varkappa_1)$ is the expected instantaneous rate of return of equity and $(\Upsilon+\varkappa_2)$ is the expected instantaneous rate of return of loan and $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$ are the instantaneous volatility of equity and loan and the Brownian motions $\mathcal{B}_s(t), \mathcal{B}_l(t), \mathcal{B}_m(t)$ are such that

$$\begin{aligned} d\mathcal{B}_s(t)\, d\mathcal{B}_l(t) &= d\mathcal{B}_s(t)\, d\mathcal{B}_m(t) = d\mathcal{B}_l(t)\, d\mathcal{B}_m(t) = 0, d\mathcal{B}_s(t)\, d\mathcal{B}_s(t) = d\mathcal{B}_l(t)\, d\mathcal{B}_l(t) \\ &= d\mathcal{B}_m(t)\, d\mathcal{B}_m(t) = dt \end{aligned}$$

Let $\mathcal{R}(t)$ represent the accumulated wealth of the pension fund at time t and considering the time interval $[t, t+\frac{1}{h}]$, we attempt to derive the differential form $d\mathcal{R}(t)$ associated with the pension wealth as follows

$$\mathcal{R}\left(t+\frac{1}{h}\right) = \frac{1}{1-\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}} \left(\begin{array}{c} \frac{1}{h}m - tmk\frac{1}{h}\mathfrak{M}_{\vartheta_0+t} - \mu_1 k\mathcal{R}(t)\frac{\mathcal{C}(t+\frac{1}{h})}{\mathcal{R}(t)}\frac{1}{h}\mathfrak{M}_{\vartheta_0+t} \\ +\mathcal{R}(t)\left(\mu_1\frac{\mathcal{C}(t+\frac{1}{h})}{\mathcal{C}(t)} + \mu_2\frac{\mathcal{S}(t+\frac{1}{h})}{\mathcal{S}(t)} + \mu_3\frac{\mathcal{L}(t+\frac{1}{h})}{\mathcal{L}(t)}\right) \end{array} \right) \tag{4}$$

$$\mathcal{R}\left(t+\frac{1}{h}\right) = \left(\frac{\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}}{1-\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}}+1\right) \left(\begin{array}{c} m\frac{1}{h} - tmk\frac{1}{h}\mathfrak{M}_{\vartheta_0+t} - (1-\mu_2-\mu_3)\, k\mathcal{R}(t)\frac{1}{h}\mathfrak{M}_{\vartheta_0+t} \\ +\mathcal{R}(t)\left(\begin{array}{c} 1+(1-\mu_2-\mu_3)\left(\frac{\mathcal{C}(t+\frac{1}{h})-\mathcal{C}(t)}{\mathcal{C}(t)}\right)\left(1-k\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}\right) \\ +\mu_2\left(\frac{\mathcal{S}(t+\frac{1}{h})-\mathcal{S}(t)}{\mathcal{S}(t)}\right)+\mu_3\left(\frac{\mathcal{L}(t+\frac{1}{h})-\mathcal{L}(t)}{\mathcal{L}(t)}\right) \end{array} \right) \end{array} \right) \tag{5}$$

where μ_1 , μ_2 , and μ_3 are the optimal control plans cash, equity and loan respectively such that $\mu_1 = 1-\mu_2-\mu_3$, m is the members' contributions received by the pension fund at any given time, ϑ_0 , the initial age of accumulation phase, T , the time frame of the accumulation period such that ϑ_0+T is the end age. Suppose $\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}$ is the mortality rate from time t to $t+\frac{1}{h}$, mt is the accumulated contributions at time t , $tmk\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}$ is the returned accumulated contributions of death members.

Since the price process of the risk free asset is deterministic and the interest rate is predetermined as seen in (1), it is possible to determine the interest paid to each death member's family at each point in time during the accumulation phase as $\mu_1 k\mathcal{R}(t)\frac{\mathcal{C}(t+\frac{1}{h})}{\mathcal{C}(t)}\frac{1}{h}M_{\vartheta_0+t}$ such that if $k=0$, there is no return of contributions and if $k=1$, there is return of contributions.

Following [9], we have the conditional death probability ${}_tb_y = 1 - {}_t a_y = 1 - e^{-\int_0^t \pi(\vartheta_0+t+s)ds}$, where $\pi(t)$ is the force function of the mortality at time t , and for $\frac{1}{h}\to 0$,

$$\frac{1}{h}\mathfrak{M}_{\vartheta_0+t} = 1-\exp\left\{-\int_0^{\frac{1}{h}}\pi(\vartheta_0+t+s)\, ds\right\} \approx \pi(\vartheta_0+t)\frac{1}{h}+O\left(\frac{1}{h}\right)$$

$$\frac{\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}}{1-\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}} = \frac{1-\exp\{-\int_0^{\frac{1}{h}}\pi(\vartheta_0+t+s)\, ds\}}{\exp\{-\int_0^{\frac{1}{h}}\pi(\vartheta_0+t+s)\, ds\}} = \exp\left\{\int_0^{\frac{1}{h}}\pi(\vartheta_0+t+s)\, ds\right\}-1 \approx \pi(\vartheta_0+t)\frac{1}{h}+O\left(\frac{1}{h}\right)$$

$$\begin{aligned} &\frac{1}{h}\to 0, \frac{\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}}{1-\frac{1}{h}\mathfrak{M}_{\vartheta_0+t}} = \pi(\vartheta_0+t)\, dt, \frac{1}{h}\mathfrak{M}_{\vartheta_0+t} = \pi(\vartheta_0+t)\, dt\ m\frac{1}{h}\to mdt, \frac{\mathcal{C}\left(t+\frac{1}{h}\right)-\mathcal{C}(t)}{\mathcal{C}(t)} \to \frac{d\mathcal{C}(t)}{\mathcal{C}(t)}, \\ &\left(\frac{\mathcal{S}(t+\frac{1}{h})-\mathcal{S}(t)}{\mathcal{S}(t)}\right) \to \frac{d\mathcal{S}(t)}{\mathcal{S}(t)}, \left(\frac{L\left(t+\frac{1}{h}\right)-L(t)}{L(t)}\right) \to \frac{dL(t)}{L(t)} \end{aligned} \tag{6}$$

substituting (6) into (5), we have

$$\mathcal{R}\left(t+\frac{1}{h}\right)=\left(1+\pi\left(\vartheta_0+t\right)dt\right)\left(\begin{array}{c} mdt-tmk\pi\left(\vartheta_0+t\right)dt-\left(1-\mu_2-\mu_3\right)k\mathcal{R}\left(t\right)\pi\left(\vartheta_0+t\right)dt \\ +\mathcal{R}\left(t\right)\left(\begin{array}{c} 1+\left(1-\mu_2-\mu_3\right)\frac{d\mathcal{C}(t)}{\mathcal{C}(t)}\left(1-k\pi\left(\vartheta_0+t\right)dt\right) \\ +\mu_2\frac{d\mathcal{S}(t)}{\mathcal{S}(t)}+\mu_3\frac{dL(t)}{L(t)} \end{array}\right) \end{array}\right) \tag{7}$$

Substituting (1), (2), (3) and (6) into (7), we have

$$d\mathcal{R}(t)=\left(\begin{array}{c} \left\{\begin{array}{c} \mathcal{R}\left(t\right)\left(\begin{array}{c} \mu_2((\varkappa_1-)+\frac{k}{\vartheta-\vartheta_0-t}) \\ +\mu_3\left((\varkappa_2-)+\frac{k}{\vartheta-\vartheta_0-t}\right)+ \\ (1-k)\frac{1}{\vartheta-\vartheta_0-t} \end{array}\right) \\ +m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right) \end{array}\right\}dt+ \\ \mathcal{R}(t)\left(\begin{array}{c} \mu_3\left(\mathfrak{H}_2dB_l(t)+_3d\mathcal{B}_m(t)\right) \\ +\mu_2\mathfrak{H}_1d\mathcal{B}_s\left(t\right) \end{array}\right) \end{array}\right)\mathcal{R}\left(0\right)=\Upsilon_0 \tag{8}$$

Where ϑ is the maximal age of the life table and $\pi\left(t\right)$ is the force function given by

$$\pi\left(t\right)=\frac{1}{\vartheta-t},\ 0\leq t<\vartheta$$

3. Mean-Variance Utility and the Optimization Problem

Consider a pension fund manager whose interest is to maximize his profit while penalising risk by using the mean-variance utility function given as

$$\mathcal{F}(t,r)=\sup_{\mu}\left\{E_{t,r}\mathcal{R}^{\mu}(T)-Var_{t,r}\mathcal{R}^{\mu}(T)\right\} \tag{9}$$

Applying the game theoretic method described in [4], the mean-variance control problem in (9) is similar to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{F}\left(t,r\right)$.

$$\left\{\begin{array}{c} \mathcal{G}\left(t,r,\mu\right)=E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)\right]-\frac{\gamma}{2}Var_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)\right] \\ G\left(t,r,\mu\right)=E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)\right]-\frac{\gamma}{2}\left(E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)^2\right]-\left(E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)\right]\right)^2\right) \\ \mathcal{F}\left(t,r\right)=\sup_{\mu}G\left(t,r,\mu\right) \end{array}\right\}$$

From [4], the optimal control plan μ^* satisfies:

$\mathcal{F}\left(t,r\right)=\sup_{\mu}\mathcal{G}\left(t,r,\mu^*\right)$

γ is a constant representing risk aversion coefficient of the member.

Let $p^{\mu}\left(t,r\right)=E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)\right]$, $q^{\mu}\left(t,r\right)=E_{t,r}\left[\mathcal{U}^{\mu}\left(T\right)^2\right]$ then

$\mathcal{F}\left(t,r\right)\quad=\sup_{\mu}u\left(t,r,p^{\mu}\left(t,r\right),q^{\mu}\left(t,r\right)\right)$

Where,

$$u\left(t,r,p,q\right)=p-\frac{\gamma}{2}\left(q-p^2\right) \tag{10}$$

Theorem 3.1 (verification theorem): If there exist three real functions $\mathcal{U},\mathcal{V},\mathcal{W}:[0,T]\times\mathcal{R}\to\mathcal{R}$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\sup_{\mu}\left\{\begin{array}{c}\mathcal{U}_t-u_t+\left[r\left(\begin{array}{c}\mu_2\left((\varkappa_1-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)\\+\mu_3\left((\varkappa_2-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)+r\\(1-k)\,\frac{1}{\vartheta-\vartheta_0-t}\end{array}\right)+m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)\right](\mathcal{U}_r-u_r)\\+\frac{1}{2}r^2\left[\mu_2^2\mathfrak{h}_1^2+\mu_3^2\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)\right](\mathcal{U}_{rr}-A_{rr})\\\mathcal{U}(T,r)=u\left(t,r,r,r^2\right)\end{array}\right\}=0, \tag{11}$$

where,

$$A_{rr}=u_{rr}+2u_{rp}p_r+2u_{rq}q_r+u_{pp}p_r^2+2u_{pq}p_rq_r+u_{qq}q_r^2=\gamma\mathcal{V}_r^2 \tag{12}$$

$$\left\{\begin{array}{c}\mathcal{V}_t+\left[r\left(\begin{array}{c}\mu_2\left((\varkappa_1-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)\\+\mu_3\left((\varkappa_2-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)+r\\(1-k)\,\frac{1}{\vartheta-\vartheta_0-t}\end{array}\right)+m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)\right]\mathcal{V}_r\\+\frac{1}{2}r^2\left[\mu_2^2\mathfrak{h}_1^2+\mu_3^2\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)\right]\mathcal{V}_{rr}\\\mathcal{V}(T,r)=r\end{array}\right\} \tag{13}$$

$$\left\{\begin{array}{c}\mathcal{W}_t+\left[r\left(\begin{array}{c}\mu_2\left((\varkappa_1-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)\\+\mu_3\left((\varkappa_2-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)+r\\(1-k)\,\frac{1}{\vartheta-\vartheta_0-t}\end{array}\right)+m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)\right]\mathcal{W}_r\\+\frac{1}{2}r^2\left[\mu_2^2\mathfrak{h}_1^2+\mu_3^2\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)\right]\mathcal{W}_{rr}\\\mathcal{W}(T,r)=r^2\end{array}\right\}=0 \tag{14}$$

Then $\mathcal{F}(T,r)=\mathcal{U}(T,r),p^{\mu^*}=\mathcal{V}(T,r),q^{\mu^*}=\mathcal{W}(T,r)$ for the optimal control plan μ^* .

Proof:

The details of the proof can be found in [8,11,18]

Next, we solve (11), (13) and (14) for the value function when $\mathcal{F}(T,r)=\mathcal{U}(T,r)$, and find the optimal control plan

Recall

$u(t,r,p,q)=p-\frac{\gamma}{2}\left(q-p^2\right)$

$$u_t=u_r=u_{rr}=u_{rp}=u_{rq}=u_{pq}=u_{qq}=0,u_p=1+\gamma p,u_{pp}=\gamma,u_q=-\frac{\gamma}{2} \tag{15}$$

Substituting (15) into (11) and differentiating it with respect to μ_2 and μ_3 and solving for μ_2 and μ_3 , we have

$$\mu_2^*=-\frac{\left((\varkappa_1-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)\mathcal{U}_r}{(\mathcal{U}_{rr}-A_{rr})\,r\mathfrak{h}_1^2} \tag{16}$$

$$\mu_3^*=-\frac{\left((\varkappa_2-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)\mathcal{U}_r}{(\mathcal{U}_{rr}-A_{rr})\,r\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)} \tag{17}$$

Substituting (12), (16) and (17) into (11), we have

$$\mathcal{U}_t+\mathcal{U}_r\left[\begin{array}{c}\left(r+\frac{1-k}{\vartheta-\vartheta_0-t}\right)r\\+m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)\end{array}\right]-\frac{\mathcal{U}_r^2}{2\left(\mathcal{U}_{rr}-\gamma\mathcal{V}_r^2\right)}\left(\frac{\left((\varkappa_1-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_1^2}+\frac{\left((\varkappa_2-r)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)=0 \tag{18}$$

$$\begin{aligned}\mathcal{V}_t+\mathcal{V}_r\left[\begin{array}{c}\left(\Upsilon+\frac{1-k}{\vartheta-\vartheta_0-t}\right)r\\+m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)\end{array}\right]-\frac{\mathcal{U}_r\mathcal{V}_r}{2\left(\mathcal{U}_{rr}-\gamma\mathcal{V}_r^2\right)}\left(\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_1^2}+\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)&\\+\frac{\mathcal{V}_{rr}}{2}\left[\frac{\mathcal{U}_r^2}{\left(\mathcal{U}_{rr}-\gamma\mathcal{V}_r^2\right)}\left(\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_1^2}+\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\right]&=0\end{aligned} \tag{19}$$

Next, we conjecture a solution for $\mathcal{U}(t,r)$ and $\mathcal{V}(t,r)$ as follows:

$$\left\{\begin{array}{c}\mathcal{U}(t,r)=r\mathcal{H}(t)+\mathcal{I}(t)\,\mathcal{H}(T)=1,\mathcal{I}(T)=0\\ \mathcal{V}(t,r)=r\mathcal{J}(t)+\mathcal{K}(t)\,J(T)=1,\mathcal{K}(T)=0\\ \mathcal{U}_t=r\mathcal{H}_t+\mathcal{I}_t,\mathcal{U}_r=\mathcal{H}(t),\mathcal{U}_{rr}=0,\mathcal{V}_t=r\mathcal{J}_t+\mathcal{K}_t,\mathcal{V}_r=\mathcal{J}(t),\mathcal{V}_{rr}=0\end{array}\right\} \tag{20}$$

Substituting (20) into (18) and (19), we have

$$\left\{\begin{array}{c}\mathcal{H}_t(t)+\left(\Upsilon+\frac{1-k}{\vartheta-\vartheta_0-t}\right)\mathcal{H}(t)=0\\ \mathcal{I}_t(t)+\mathcal{H}(t)\,m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)+\frac{\mathcal{H}^2(t)}{2\gamma\mathcal{J}^2(t)}\left(\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_1^2}+\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)=0\end{array}\right\} \tag{21}$$

$$\left\{\begin{array}{c}\mathcal{J}_t(t)+\left(\Upsilon+\frac{1-k}{\vartheta-\vartheta_0-t}\right)\mathcal{J}(t)=0\\ \mathcal{K}_t(t)+\mathcal{J}(t)\,m\left(\frac{\vartheta-\vartheta_0-(1+k)t}{\vartheta-\vartheta_0-t}\right)+\frac{\mathcal{H}(t)}{\gamma\mathcal{J}(t)}\left(\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_1^2}+\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)=0\end{array}\right\} \tag{22}$$

Solving (21) and (22), we have

$$\mathcal{H}(t)=\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)} \tag{23}$$

$$\mathcal{J}(t)=\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)} \tag{24}$$

$$\mathcal{I}(t)=m\left(\vartheta-\vartheta_0-T\right)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{\left(\vartheta-\vartheta_0-\tau\right)^k}\right)d\tau+\frac{1}{2\gamma}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{h}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{h}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{h}_1^2}+\frac{1}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right] \tag{25}$$

$$\mathcal{K}(t)=m\left(\vartheta-\vartheta_0-T\right)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{\left(\vartheta-\vartheta_0-\tau\right)^k}\right)d\tau+\frac{1}{\gamma}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{h}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{h}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{h}_1^2}+\frac{1}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right] \tag{26}$$

$$\mathcal{U}(t,r)=\left(\begin{array}{c}\frac{1}{2\gamma}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{H}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{H}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{H}_1^2}+\frac{1}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]\\ +r\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)}+m(\vartheta-\vartheta_0-T)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{(\vartheta-\vartheta_0-\tau)^k}\right)d\tau\end{array}\right) \tag{27}$$

$$\mathcal{V}(t,r)=\left(\begin{array}{c}\frac{1}{\gamma}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{H}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{H}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{H}_1^2}+\frac{1}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]\\ +r\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)}+m(\vartheta-\vartheta_0-T)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{(\vartheta-\vartheta_0-\tau)^k}\right)d\tau\end{array}\right) \tag{28}$$

From (20), we have

$$\mathcal{U}_r=\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)},\ \mathcal{U}_{rr}=0,\mathcal{V}_r=\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)},\ \mathcal{V}_{rr}=0 \tag{29}$$

Substituting (29) into (16) and (17), we have

$$\mu_2^*=\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\mathfrak{H}_1^2} \tag{30}$$

$$\mu_3^*=\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{H}_2^2+\mathfrak{H}_3^2\right)} \tag{31}$$

$$\mu_1^*=1-\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\mathfrak{H}_1^2}-\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{H}_2^2+\mathfrak{H}_3^2\right)} \tag{32}$$

Lemma 3.1

The optimal control plan with predetermine interest with and without return of contribution clause are given as

$$\begin{aligned}\mu_1^* &= 1-\frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\mathfrak{H}_1^2}-\frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{H}_2^2+\mathfrak{H}_3^2\right)}\\ \mu_2^* &= \frac{\left((\varkappa_1-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\mathfrak{H}_1^2}\\ \mu_3^* &= \frac{\left((\varkappa_2-\Upsilon)+\frac{k}{\vartheta-\vartheta_0-t}\right)\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^{k-1}e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{H}_2^2+\mathfrak{H}_3^2\right)}\end{aligned}$$

The corresponding optimal fund size $\mathcal{R}^{\mu^*}(t)$ is obtained by solving the SDE in (8) after substituting (30) and (31) in it.

Lemma 3.2

The efficient frontier of the pension plan is given

$$E_{t,r}\left[\mathcal{R}^{\mu^*}(t)\right]=\left(\begin{array}{c}\sqrt{\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{H}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{H}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{H}_1^2}+\frac{1}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]\mathcal{V}ar_{t,r}\left[\mathcal{R}^{\mu^*}(t)\right]}\\ +r\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)}+m\left(\vartheta-\vartheta_0-T\right)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{(\vartheta-\vartheta_0-\tau)^k}\right)d\tau\end{array}\right) \tag{33}$$

Proof:

Recall that

$Var_{t,r}[R^{\mu^*}(t)] = E_{t,r}[R^{\mu^*}(t)^2] - (E_{t,r}R^{\mu^*}(t)])^2$

$$Var_{t,r}\left[R^{\mu^*}(t)\right]=\frac{2}{\gamma}\left[\mathcal{V}(t,r)-\mathcal{U}(t,r)\right] \tag{34}$$

Substituting (27) and (28) into (34), we have

$$Var_{t,r}\left[R^{\mu^*}(t)\right]=\frac{1}{\gamma^2}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{H}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{H}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{H}_1^2}+\frac{1}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right] \tag{35}$$

$$\frac{1}{\gamma}=\sqrt{\frac{Var_{t,r}\left[R^{\mu^*}(t)\right]}{\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{H}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{H}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{H}_1^2}+\frac{1}{\mathfrak{H}_2^2+\mathfrak{H}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]}} \tag{36}$$

$$E_{t,r}\left[R^{\mu^*}(t)\right]=\mathcal{V}(t,r) \tag{37}$$

Substituting (28) into (37), we have

$$E_{t,r}\left[R^{\mu^*}(t)\ \right]=\left(\begin{array}{c}\frac{1}{\gamma}\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{h}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{h}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{h}_1^2}+\frac{1}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]\\ +r\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)}+m\left(\vartheta-\vartheta_0-T\right)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{(\vartheta-\vartheta_0-\tau)^k}\right)d\tau\end{array}\right) \tag{38}$$

$$E_{t,r}\left[R^{\mu^*}(t)\ \right]=\left(\begin{array}{c}\sqrt{\left[\begin{array}{c}\left(\frac{(\varkappa_1-\Upsilon)^2}{\mathfrak{h}_1^2}+\frac{(\varkappa_2-\Upsilon)^2}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)(T-t)+\\ \left(\frac{\varkappa_1-\Upsilon}{\mathfrak{h}_1^2}+\frac{\varkappa_2-\Upsilon}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\ln\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)^2+\\ \left(\frac{1}{\mathfrak{h}_1^2}+\frac{1}{\mathfrak{h}_2^2+\mathfrak{h}_3^2}\right)\left(\frac{T-t}{(\vartheta-\vartheta_0-t)(\vartheta-\vartheta_0-T)}\right)\end{array}\right]Var_{t,r}\left[R^{\mu^*}(t)\ \right]}\\ +r\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)^{k-1}e^{\Upsilon(T-t)}+m\left(\vartheta-\vartheta_0-T\right)^{k-1}e^{\Upsilon T}\int_t^T e^{\Upsilon\tau}\left(\frac{\vartheta-\vartheta_0-(1+k)\tau}{(\vartheta-\vartheta_0-\tau)^k}\right)d\tau\end{array}\right) \tag{39}$$

Remark 3.3

The optimal control plan with predetermined interest and return of contribution clause is obtained when $k=1$ and is given as

$$\mu_1^*=1-\frac{\left((\varkappa_1-\Upsilon)+\frac{1}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r\mathfrak{h}_1^2}-\frac{\left((\varkappa_2-\Upsilon)+\frac{1}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)} \tag{40}$$

$$\mu_2^*=\frac{\left((\varkappa_1-\Upsilon)+\frac{1}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r\mathfrak{h}_1^2} \tag{41}$$

$$\mu_3^*=\frac{\left((\varkappa_2-\Upsilon)+\frac{1}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)} \tag{42}$$

4. Theoretical Analysis

In this section, we present a proposition comparing our result with the result in [2]

Let μ_1^{1*} , μ_2^{1*} and μ_3^{1*} be the optimal control plan in [2] and are given as

$$\mu_1^{1*}=1-\frac{(\varkappa_1-\Upsilon)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\mathfrak{h}_1^2}-\frac{(\varkappa_2-\Upsilon)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)}$$

$$\mu_2^{1*}=\frac{(\varkappa_1-\Upsilon)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\mathfrak{h}_1^2}$$

$$\mu_3^{1*}=\frac{(\varkappa_2-\Upsilon)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)}$$

Proposition 4.1

Let $\varkappa_1 > 0, \varkappa_2 > 0, \vartheta > 0, \vartheta_0 > 0,\ \Upsilon > 0, \mathfrak{h}_1 > 0,\ \mathfrak{h}_2 > 0,\ \mathfrak{h}_3$>0, T>0, t>0, and $r(t) > 0,\ r^1 > 0$ for $t \in [0, T]$ such that $r^1(t) >\ r(t)$ and $(\vartheta - \vartheta_0 - t) \geq (\vartheta - \vartheta_0 - T)$, then

1. $\mu_2^* > \mu_2^{1*}$
2. $\mu_3^* > \mu_3^{1*}$
3. $\mu_1^* < \mu_1^{1*}$

Proof

(i) Recall that

$\mu_2^{1*} = \frac{(\varkappa_1 - \Upsilon)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r^1 \mathfrak{h}_1^2}$ and $\mu_2^* = \frac{\left((\varkappa_1 - \Upsilon) + \frac{1}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r \mathfrak{h}_1^2}$

Suppose $\mu_2^* > \mu_2^{1*}$, we want to show that $\mu_2^* - \mu_2^{1*} >\ 0$

$$\mu_2^* - \mu_2^{1*} = \frac{\left((\varkappa_1 - \Upsilon) + \frac{1}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r \mathfrak{h}_1^2} - \frac{(\varkappa_1 - \Upsilon)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r^1 \mathfrak{h}_1^2} = \frac{e^{\Upsilon(t-T)}}{\gamma r r^1 \mathfrak{h}_1^2}\left(\frac{r(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - T)}{\vartheta - \vartheta_0 - t}\right)$$

$$= \frac{e^{\Upsilon(t-T)}}{\gamma r r^1 \mathfrak{h}_1^2(\vartheta - \vartheta_0 - t)}\left(r(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - T)\right)$$

Since $r^1(t) >\ r(t), (\vartheta - \vartheta_0 - t) \geq (\vartheta - \vartheta_0 - T)$ and $r > 0$ then

$$\left(r(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - T)\right) > 0$$

Hence

$$\mu_2^* - \mu_2^{1*} = \frac{e^{\Upsilon(t-T)}}{\gamma r r^1 \mathfrak{h}_1^2(\vartheta - \vartheta_0 - t)}\left(r(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_1 - \Upsilon)(\vartheta - \vartheta_0 - T)\right) > 0$$

$\therefore \mu_2^* - \mu_2^{1*} > 0$

(ii) Recall that

$\mu_2^{1*} = \frac{(\varkappa_2 - \Upsilon)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)}$ and $\mu_2^* = \frac{\left((\varkappa_2 - \Upsilon) + \frac{1}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)}$

Suppose $\mu_3^* > \mu_3^{1*}$, we want to show that $\mu_3^* - \mu_3^{1*} > 0$

$$\mu_3^* - \mu_3^{1*} = \frac{\left((\varkappa_2 - \Upsilon) + \frac{1}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)} - \frac{(\varkappa_2 - \Upsilon)\left(\frac{\vartheta - \vartheta_0 - T}{\vartheta - \vartheta_0 - t}\right)e^{\Upsilon(t-T)}}{\gamma r^1\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)} = \frac{e^{\Upsilon(t-T)}}{\gamma r r^1\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)}\left(\frac{r(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - T)}{\vartheta - \vartheta_0 - t}\right)$$

$$= \frac{e^{\Upsilon(t-T)}}{\gamma r r^1\left(\mathfrak{h}_2^2 + \mathfrak{h}_3^2\right)(\vartheta - \vartheta_0 - t)}\left(r(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - T)\right)$$

Since $r^1(t) >\ r(t), (\vartheta - \vartheta_0 - t) \geq (\vartheta - \vartheta_0 - T)$ and $r > 0$ then

$$\left(r(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - t) + r - r^1(\varkappa_2 - \Upsilon)(\vartheta - \vartheta_0 - T)\right) > 0$$

Hence

$\mu_3^* - \mu_3^{1*} = \frac{e^{\Upsilon(t-T)}}{\gamma r r^1\left(\mathfrak{h}_2^2+\mathfrak{h}_3^2\right)(\vartheta-\vartheta_0-t)}\left(r\left(\varkappa_2-\curlyvee\right)(\vartheta-\vartheta_0-t)+r-r^1\left(\varkappa_2-\curlyvee\right)(\vartheta-\vartheta_0-T)\right) \quad > 0$

$\therefore \mu_3^* - \mu_3^{1*} > 0$

(iii) Recall that $\mu_1^{1*} = 1 - \mu_2^{1*} - \mu_3^{1*}$ and $\mu_1^* = 1 - \mu_2^* - \mu_3^*$

Suppose $\mu_1^* < \mu_1^{1*}$, we want to show that $\mu_1^* - \mu_1^{1*} < 0$

$$\mu_1^* - \mu_1^{1*} = 1 - \mu_2^* - \mu_3^* - \left(1 - \mu_2^{1*} - \mu_3^{1*}\right)$$

$$1 - 1 - \left(\mu_2^* - \mu_2^{1*}\right) - \left(\mu_3^* - \mu_3^{1*}\right)$$

$$= -\left[\left(\mu_2^* - \mu_2^{1*}\right) + \left(\mu_3^* - \mu_3^{1*}\right)\right]$$

Since $\mu_2^* - \mu_2^{1*} > 0$ and $\mu_3^* - \mu_3^{1*} > 0$, then

$$\left(\mu_2^* - \mu_2^{1*}\right) + \left(\mu_3^* - \mu_3^{1*}\right) > 0$$

Hence

$\mu_1^* - \mu_1^{1*} = -\left[\left(\mu_2^* - \mu_2^{1*}\right) + \left(\mu_3^* - \mu_3^{1*}\right)\right] < 0$

Therefore $\mu_1^* - \mu_1^{1*} < 0$

5. Numerical Simulations

In this section we present numerical simulations of the optimal investment policy with respect to time using the following parameters. $\vartheta = 100;\ \vartheta_0 = 20;\ \gamma = 0.05;\ \curlyvee = 0.02; \varkappa_1 = 0.055; \varkappa_2 = 0.065; \mathfrak{h}_1 = 0.85;\ \mathfrak{h}_2 = 1;\ \mathfrak{h}_3 = 0.60;\ r = R\left(t\right);\ r_0 = 1;\ T = 40;\ t = 0:5:20;$ Unless otherwise stated.

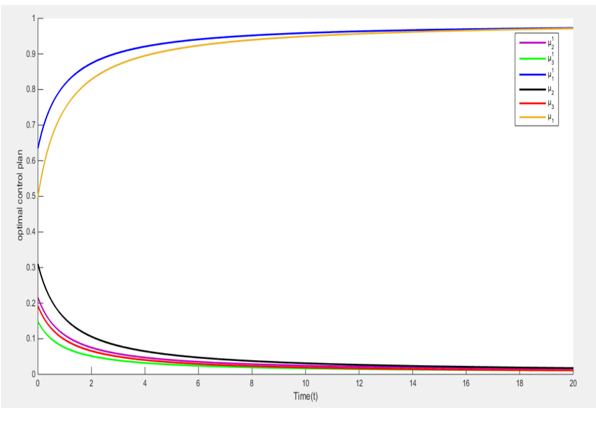

Figure 1: Time evolution of the optimal control plans with and without predetermined interest when $r = R^{\mu^*}\left(t\right)$

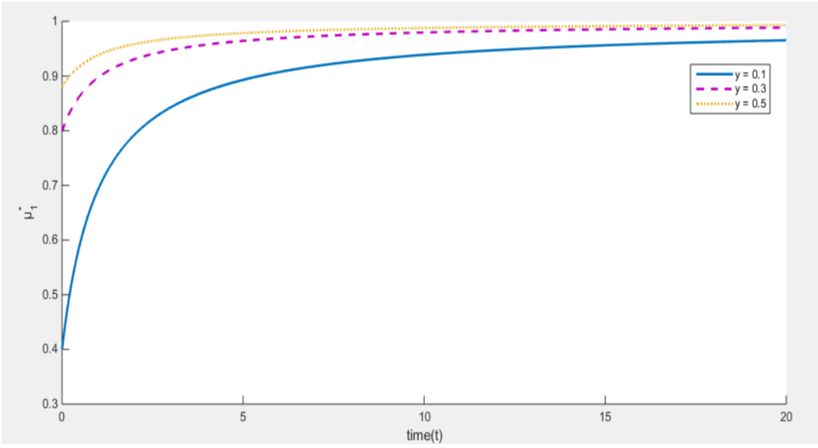

Figure 2: Time evolution of μ_1^* when $r = R^{\mu^*}(t)$ with different risk averse γ

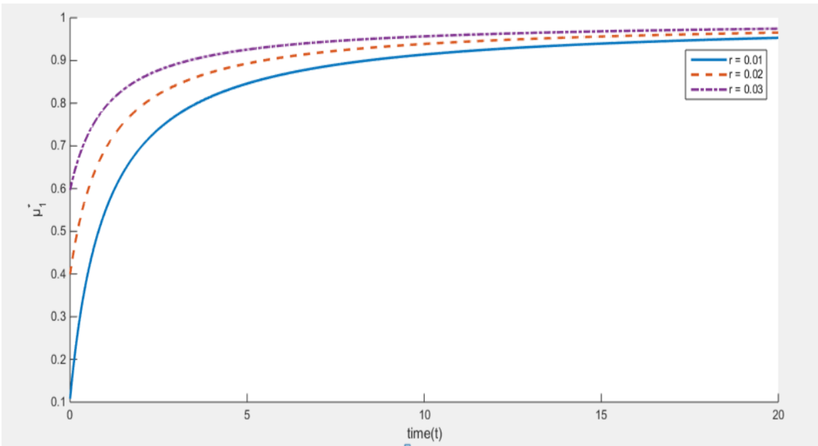

Figure 3: Time evolution of μ_1^* when $r = R^{\mu^*}(t)$ with different predetermined interest rate Υ

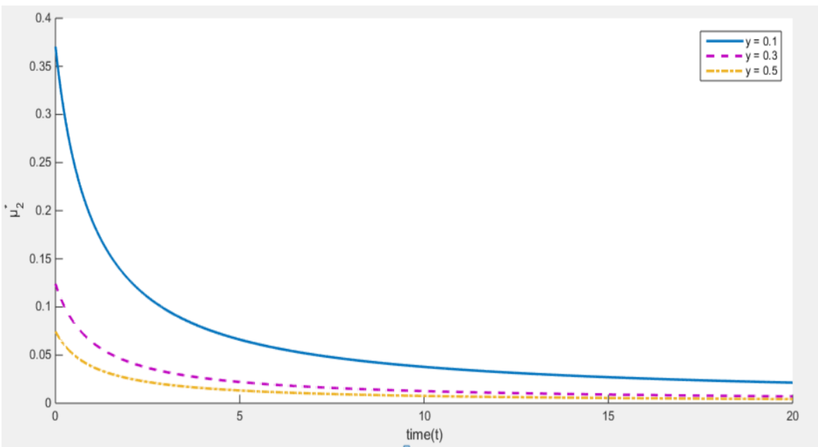

Figure 4: Time evolution of μ_2^* when $r = R^{\mu^*}(t)$ with different risk averse γ

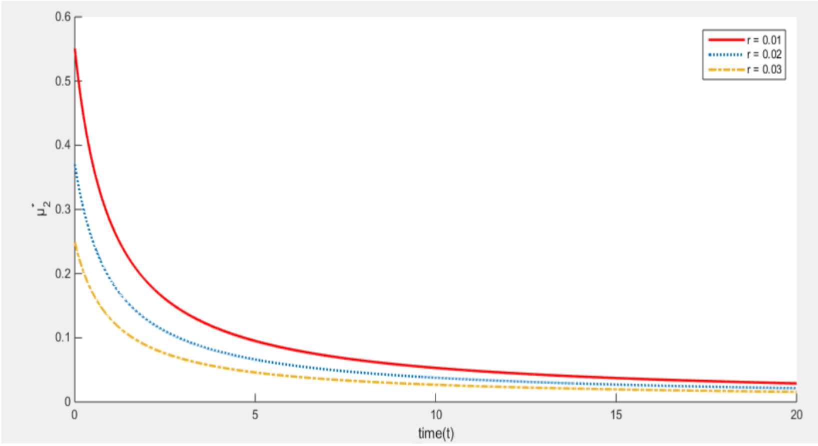

Figure 5: Time evolution of μ_2^* when $r = R^{\mu^*}(t)$ with different predetermined interest rate Υ

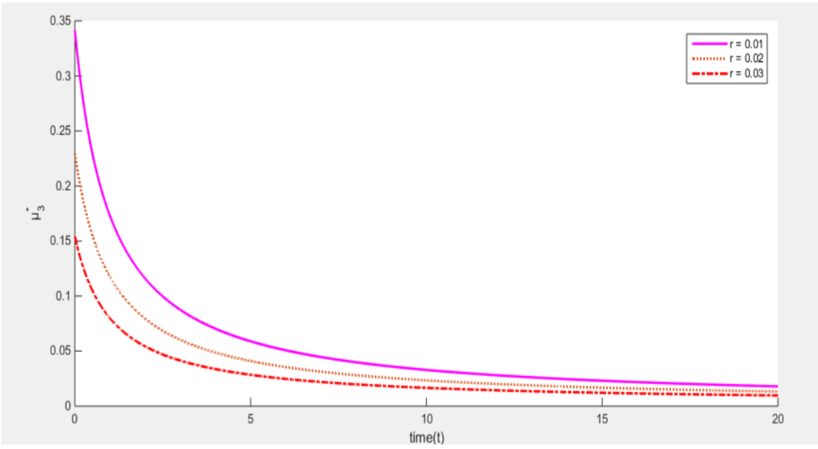

Figure 6: Time evolution of μ_3^* when $r = R^{\mu^*}(t)$ with different predetermined interest rate Υ

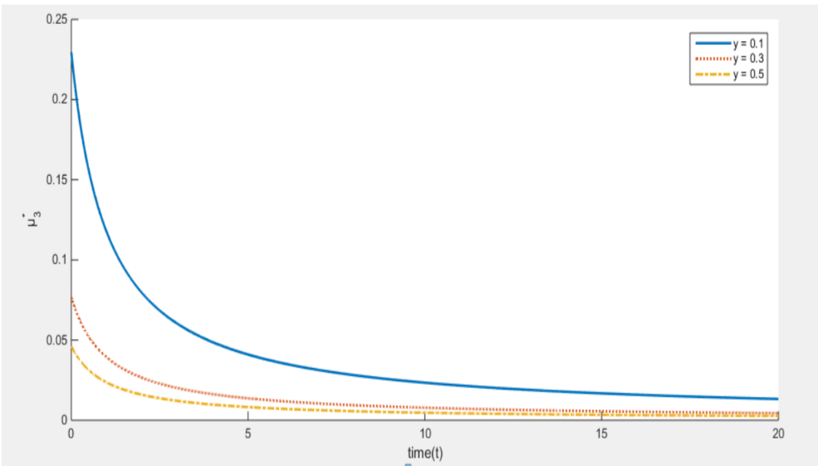

Figure 7: Time evolution of μ_3^* when $r = R^{\mu^*}(t)$ with different risk averse γ

6. Discussion

From proposition 4.1, we observed that the optimal control plan for the risky assets for the case where the return of clause is with predetermined interest is higher when compared to the case where the return clause is without predetermined interest. This is because when the returned contributions are with predetermined interest; the remaining fund in the pension system is less compared to when the return is without predetermined interest. Also, a careful observation of figure 1 shows that a decrease in wealth leads to an increase in the optimal control plan for the risky assets and a decrease in the optimal control plan of the risk free asset. This shows that both the numerical and theoretical analysis confirms each other. The implication of this is that pension fund managers in cases where the return contributions are with predetermined interest may decide to take more risk in investing in risky assets in order to increase the remaining accumulated funds such that the remaining members will be well provided for after retirement.

From figures 2 and 3, the optimal control plan of cash is directly proportional to both the risk aversion coefficient of the pension member and the predetermined interest rate of the risk free asset; this implies that when the risk free interest rate is high, the fund manager will increase his investment in risk free asset while reducing investment in risky assets and vice versa. Similarly, if the members have high risk aversion coefficient, they will invest more in risk free asset and vice versa. In Figures 4 and 7, we observed that the optimal control plan of the risky assets are inversely proportional to the risk averse level of the members; this implies that an investor with high risk averse level will invest less in equity and loan and vice versa as retirement age gets closer. Figure 5 and 6 shows that the optimal control plans of the risky assets are inversely proportional to predetermined interest; that is to say, as the predetermined interest rate increases, there is a decrease in investments in equity and loan.

7. Conclusion

In conclusion, the optimal control plan for a member in a DC plan with or without return clause of contributions with predetermined interest rate was studied. A continuous time mean-variance stochastic optimal control problem was developed using the actuarial symbol. Investments in cash, equity and loan were considered to increase the accumulated funds remaining for the surviving members. An optimization problem from the extended Hamilton Jacobi Bellman equations was obtained using game theoretic approach and mean variance utility function. Solving the optimization problem, some close form solutions of the optimal control plans for the three assets and the efficient frontier of the pension members were found. Furthermore, numerical simulation of the optimal control plans of the three assets with respect to time was given. Finally, theoretical comparison of the optimal control plans whose return of contributions are with and without interest and observed that optimal control plan of the risky assets is inversely proportional to risk averse level and predetermined interest rate.

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