

## Solution Of Integral Equations Of Volterra Type Using The Adomian Decomposition Method (ADM)

Fadugba Sunday Emmanuel<sup>1</sup>

<sup>1</sup>Department of Mathematics, Faculty of Science, Ekiti State University, Ado Ekiti, Nigeria

sunday.fadugba@eksu.edu.ng

### Abstract:

This paper presents an alternative semi-analytical method for the solution of the Volterra integral equations. The solution of the Volterra integral equations has been decomposed into an infinite series of components via the Adomian decomposition method (ADM). Furthermore, four illustrative examples were solved to measure the performance of ADM in terms of suitability, convergence and accuracy. Moreover, the results obtained via ADM were compared with the exact solution. Hence, ADM is found to be a good tool for the solution of Volterra integral equations and also minimizes the volume of calculations.

**Keywords:** Adomian decomposition method, exact solution, integral equation, Volterra type

### Introduction

Many scientific and technological problems are modeled mathematically by both Differential Equations (DEs). For instance, in physics heat flow and wave propagation phenomenon are well defined by Partial Differential Equations (PDEs). Most of the DEs does not have exact solutions; therefore there is a need for numerical methods to obtain approximate solutions to these equations. In other words, numerical methods must be used to handle them. There are many developed methods for the solution of both linear and nonlinear DEs such as [7, 9, 10, 13, 14], just to mention a few.

Integral equations arise naturally in Physics, chemistry, biology and engineering applications modeled by Initial Value Problems (IVPs) for a finite interval [a, b]. A typical form of an integral equation is given by

$$u(x) = f(x) + \int_{\beta(x)}^{\alpha(x)} k(x,t)u(t)dt$$

where  $k(x,t)$  is the Kernel of the integral equation;  $\alpha(x)$  and  $\beta(x)$  are the limits of integration [15]. More details on the origin of the integral equations can be found in [6, 8, 11, 12].

Adomian Decomposition Method (ADM) was proposed by George Adomian in the 1980s. In the recent years, ADM has also become extremely popular in applied mathematics due to its effectiveness and suitability to solve DEs of different characteristics [1, 2, 3, 4, 5, 16, 17]. In this paper, an alternative semi-analytical method for the solution of the Volterra integral equations is presented. The rest of the paper is organized as follows. Section 2 presents the analysis of ADM for the Volterra integral equation. Numerical examples, results and discussion were presented in Section 3. Section 4 concludes the paper.

### Materials and Methods

This section presents the analysis of ADM for the solution of the Volterra integral equation of the second kind given by

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \quad (2.1)$$

Suppose that the solution of (2.1) is of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (2.2)$$

with

$$u_0(x) = f(x) \quad (2.3)$$

Using (2.1) and (2.2), one obtains

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t) \left( \sum_{n=0}^{\infty} u_n(t) \right) dt \quad (2.4)$$

Expanding (2.4), one gets

$$u_0(x) + u_1(x) + u_2(x) + \dots = f(x) + \lambda \int_0^x k(x,t) u_0(t) dt + \lambda \int_0^x k(x,t) u_1(t) dt + \lambda \int_0^x k(x,t) u_2(t) dt + \dots \quad (2.5)$$

The components  $u_j(x)$ ,  $j \geq 0$  of  $u(x)$  are completely determined as follows;

$$u_0(x) = f(x) \quad (2.6)$$

$$u_1(x) = \lambda \int_0^x k(x,t) u_0(t) dt \quad (2.7)$$

$$u_2(x) = \lambda \int_0^x k(x,t) u_1(t) dt \quad (2.8)$$

$$u_3(x) = \lambda \int_0^x k(x,t) u_2(t) dt \quad (2.9)$$

and so on. The ADM for the determination of the components  $u_j(x)$ ,  $j \geq 0$  of the solution of (2.1) can be written in a recurrence relation given by

$$u_0(x) = f(x) \quad (2.10)$$

$$u_{n+1}(x) = \lambda \int_0^x k(x,t) u_n(t) dt \quad (2.11)$$

## Results and Discussion

This section presents some numerical examples and discussion of results as follows.

**Example 1:** Consider the following Volterra integral equation of the form

$$u(x) = 1 + \int_0^x u(x) dt \quad (3.1)$$

It is evident here that  $f(x) = 1$ ,  $\lambda = 1$ ,  $k(x, t) = 1$

Let

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (3.2)$$

Substituting (3.2) into both sides of (3.1) yields

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} u_n(t) dt \quad (3.3)$$

Expanding (3.3) further yields

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + \dots = 1 + \int_0^x (u_0(t) + u_1(t) + u_2(t) + u_3(t) + \dots) dt \quad (3.4)$$

with

$$u_0(x) = 1 \quad (3.5)$$

Equating coefficients in (3.4), we have that

$$u_1(x) = \int_0^x u_0(t) dt \quad (3.6)$$

$$u_2(x) = \int_0^x u_1(t) dt \quad (3.7)$$

$$u_3(x) = \int_0^x u_2(t) dt \quad (3.8)$$

$$u_4(x) = \int_0^x u_3(t) dt \quad (3.9)$$

and so on. By means of (3.5), (3.6) yields

$$u_1(x) = \int_0^x dt = x \quad (3.10)$$

Similarly,

$$u_2(x) = \int_0^x u_1(t) dt = \int_0^x t dt = \frac{x^2}{2} \quad (3.11)$$

$$u_3(x) = \int_0^x u_2(t) dt = \int_0^x \frac{t^2}{2} dt = \frac{x^3}{3!} \quad (3.12)$$

$$u_4(x) = \int_0^x u_3(t) dt = \int_0^x \frac{t^3}{6} dt = \frac{x^4}{4!} \quad (3.13)$$

Proceeding in the same manner, other components can be obtained. Therefore,

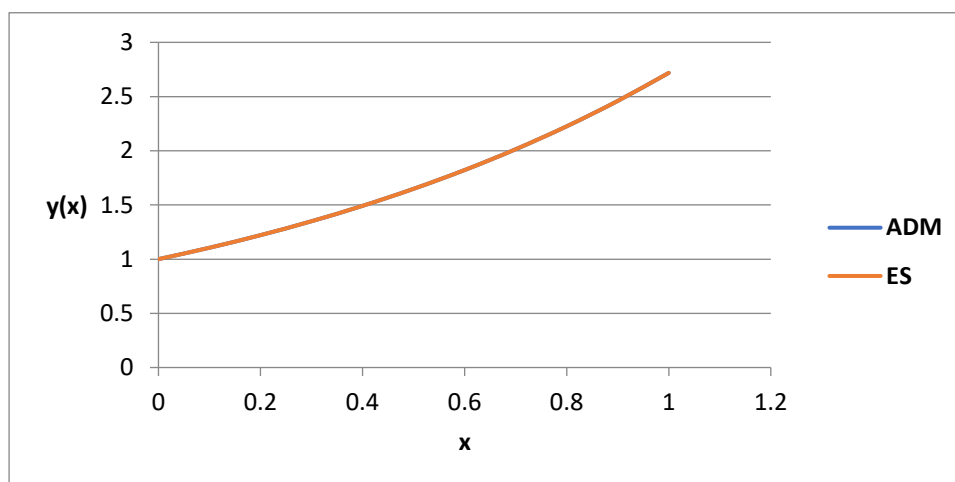
$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \dots \quad (3.14)$$

Using (3.5), (3.10)-(3.13), (3.14) becomes

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (3.15)$$

Equation (3.15) is the solution series obtained via ADM. The closed form of (3.15) is given by

$$u(x) = e^x \quad (3.16)$$



**Figure 1:** The graph of the solutions obtained via ADM against exact solution (ES) for different values of  $x$  for Problem

**Example 2:** Consider the Volterra integral equation of the form

$$u(x) = x + \int_0^x (t-x)u(t)dt \quad (3.17)$$

It is evident that

$$f(x) = x, \lambda = 1, k(x, t) = (t-x)$$

Using the decomposition series solution (2.10)

$$u_0(x) = f(x) \quad (3.18)$$

and the recursive scheme (2.11)

$$u_{n+1}(x) = \lambda \int_0^x k(x, t)u_n(t)dt, n \geq 0 \quad (3.19)$$

One obtains

$$u_0(x) = f(x) = x \quad (3.20)$$

and

$$u_{n+1}(x) = \int_0^x (t-x)u_n(t)dt, n \geq 0 \quad (3.21)$$

respectively.

For  $n = 0$ ,

$$\begin{aligned} u_1(x) &= \int_0^x (t-x)u_0(t)dt \\ &= \int_0^x (t-x)t dt \\ &= \int_0^x (t^2 - xt) dt \\ &= -\frac{x^3}{3!} \end{aligned} \quad (3.22)$$

For  $n = 1$ ,

$$\begin{aligned} u_2(x) &= \int_0^x (t-x)u_1(t)dt \\ &= \int_0^x (t-x)\left(-\frac{t^3}{6}\right)dt \\ &= \int_0^x \left(-\frac{t^4}{6} + \frac{xt^3}{6}\right)dt \end{aligned}$$

$$= \frac{-4x^5 + 5x^5}{120} = \frac{x^5}{120} = \frac{x^5}{5!} \tag{3.23}$$

For  $n = 2$ ,

$$u_3(x) = \int_0^x (t-x)u_2(t)dt$$

$$= \int_0^x (t-x)\left(\frac{t^5}{120}\right)dt = \int_0^x \left(\frac{t^6}{120} - \frac{xt^5}{120}\right)dt = \frac{x^7}{840} - \frac{x^7}{720} = \frac{-x^7}{7!} \tag{3.24}$$

and so on. Consequently, the solution of (3.17) in a series form is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{3.25}$$

and in a closed form

$$u(x) = \sin x \tag{3.26}$$



**Figure 2:** The graph of the solution obtained via ADM against exact solution (ES) for different values of  $x$  for Problem

**Example 3:** Consider the following Volterra integral equation

$$u(x) = 1 + \int_0^x (t-x)u(t)dt \tag{3.27}$$

Here, it is clearly seen that

$$f(x) = 0, \lambda = 1, k(x, t) = t - x$$

Using the decomposition series solution (2.10)

$$u_0(x) = f(x)$$

and the recursive scheme (2.11)

$$u_{n+1}(x) = \lambda \int_0^x k(x, t)u_n(t)dt, n \geq 0$$

To determine the components  $u_n(x), n \geq 0$ , we find

$$u_0(x) = f(x) = 1 \tag{3.28}$$

and

$$u_{n+1}(x) = \int_0^x (t-x)u_n(t) dt \tag{3.29}$$

Thus,

$$u_0(x) = 1 \tag{3.30}$$

$$u_1(x) = \int_0^x (t-x)u_0(t)dt$$

$$= \int_0^x (t-x)dt = -\frac{x^2}{2} \tag{3.31}$$

$$\begin{aligned}
 u_2(x) &= \int_0^x (t-x)u_1(t)dt \\
 &= \int_0^x -(t-x)\frac{x^2}{2}dt \\
 &= \int_0^x \left(\frac{t^3}{2} - \frac{xt^2}{2}\right)dt = -\frac{x^4}{8} + \frac{x^4}{6} = \frac{x^4}{4!}
 \end{aligned}
 \tag{3.32}$$

$$u_3(x) = \int_0^x (t-x)u_2(t)dt
 \tag{3.33}$$

$$\begin{aligned}
 &= \int_0^x (t-x)\frac{x^4}{4!}dt \\
 &= \int_0^x \left(\frac{t^5}{24} - \frac{xt^4}{24}\right)dt = \frac{x^6}{6(24)} - \frac{x^6}{120} = -\frac{x^6}{6!}
 \end{aligned}
 \tag{3.34}$$

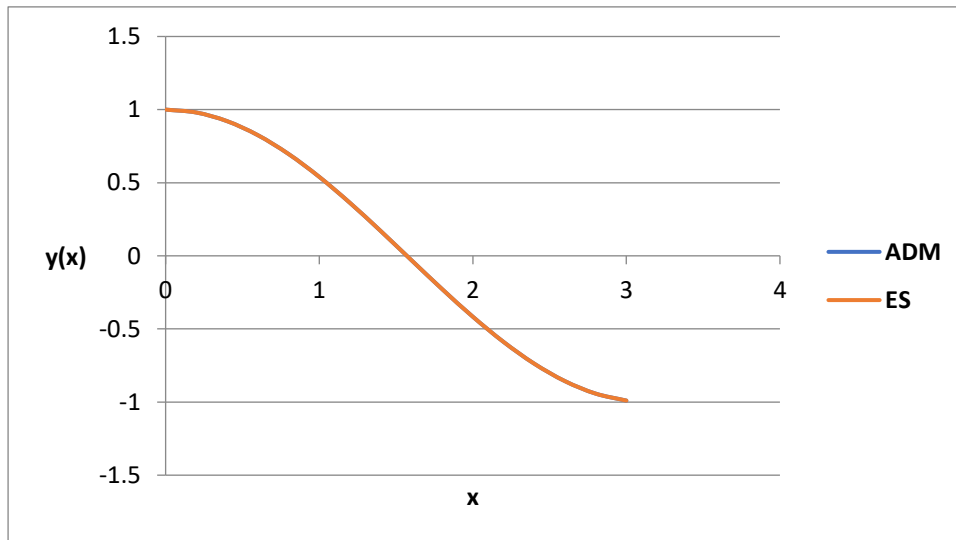
and so on.

Hence, the solution of (3.27) in a series form is obtained as

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots
 \tag{3.35}$$

and in a closed form

$$u(x) = \cos x
 \tag{3.36}$$



**Figure 3:** The graph of the solution obtained via ADM against exact solution (ES) for different values of x for Problem

**Example 4**

Consider the Volterra integral equation

$$u(x) = 2\cosh x - x\sinh x - 1 + \int_0^x tu(t)dt
 \tag{3.37}$$

Using the decomposition series solution (2.10)

$$u_0(x) = f(x)$$

and the recursive scheme (2.11)

$$u_{n+1}(x) = \lambda \int_0^x k(x,t)u_n(t)dt, n \geq 0$$

We have

$$u_0(x) = 2\cosh x - x\sinh x - 1
 \tag{3.38}$$

and

$$u_{n+1}(x) = \int_0^x t u_n(t) dt \tag{3.39}$$

By means of the Taylor series expansion, (3.38) can be evaluated as

$$u_0(x) = 2 \left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) - x \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) - 1 = 1. \tag{3.40}$$

Thus,

$$u_0(x) = 1 \tag{3.41}$$

$$u_1(x) = \int_0^x t u_0(t) dt = \int_0^x t dt = \frac{x^2}{2} \tag{3.42}$$

$$u_2(x) = \int_0^x t u_1(t) dt = \int_0^x \frac{t^3}{2} dt = \frac{x^4}{4!} \tag{3.43}$$

$$u_3(x) = \int_0^x t u_2(t) dt = \int_0^x \frac{t^5}{24} dt = \frac{x^6}{6!} \tag{3.44}$$

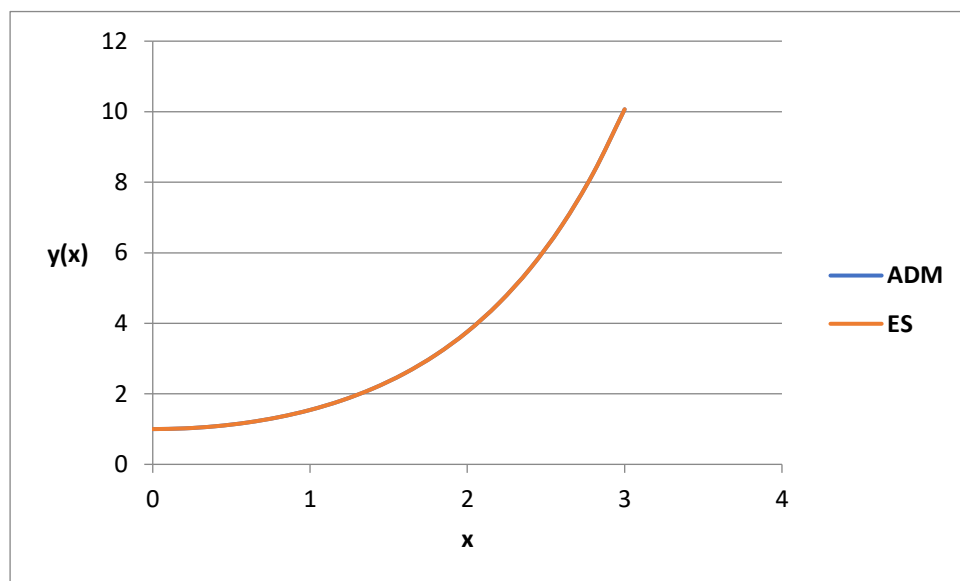
and so on.

The solution of (3.37) in a series form is given by

$$u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \tag{3.45}$$

and in a closed form

$$u(x) = \cosh x \tag{3.46}$$



**Figure 4:** The graph of the solution obtained via ADM against exact solution (ES) for different values of  $x$  for Problem

**Discussion of Results**

It is observed from Figs. 1-4 that the results obtained via ADM agreed and compared favourably with the exact solution. It is also observed that the ADM introduced promising improvements as an alternative approach to other existing techniques for the solution of the Volterra equations.

**Conclusions**

In this paper, ADM for the solution of Volterra integral equation has been implemented successfully. The solution of the Volterra equation was decomposed into an infinite series of components via the ADM. It is observed from the results obtained that ADM agreed with the exact solution. Hence, it can be concluded

that ADM is found to be computationally effective, accurate and a good tool for the solution of integral equations of Volterra type. The methodology can be applied to the solution of higher order non-linear partial differential equations emanated from real life situations with points of catastrophe.

### Conflicts of Interest

Author declares that the information provided in this paper has no conflict of interest.

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