On Semi-+-Open Sets in Topological Spaces

Carlos Granados
Corporación Universitaria Latinoamericana
carlosgranadosortiz@outlook.es

Abstract

In this paper, we used the notion of operator A^+ for defining a new class of set which will be called semi-+-open set, besides we define the concepts of generalized semi-+-closed sets and regular generalized semi-+-closed sets. Moreover, some of their properties are shown.

Keywords: Operator A^+, semi-+-open sets, generalized semi-+-closed sets, regular generalized semi-+-closed sets.

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1 Introduction

The concept of operator A^+ was introduced by Elez and Papaz [4], they defined an operator A^+ by $A^+ = Cl(A) - A$. In this paper, we show that A^+ does not induce a topology. Furthermore, the study of operator, closure operator or interior operator have been grown in several topics in general topology, see [2] and [5]. On the other hand, the idea of a semi-open set was introduced by Levine [7], a subset A of a topological space (X, τ) is called semi-open if $A \subseteq Cl(Int(A))$. Nevertheless, this type of set has been studied by many mathematicians, see [1], [3], [8], [6] and [9].

In this paper, motivated by the authors mentioned above, we used the operator A^+ for defining a new class of open set which will be called semi-+-open set. Besides, we prove some of its properties. Moreover, we define and study the notions of generalized semi-+-closed sets and regular generalized semi-+-closed sets.

Throughout this paper the terms (X, τ) is a topological spaces on which no separation axioms are assumed, unless otherwise be mentioned. Besides, we sometimes write X instead of (X, τ). On the other hand, for any subset A of X, $Int(A)$ and $Cl(A)$ represent the interior and closure of A respectively.

2 Semi-+-open sets

In this section, we show some properties on A^+ operator. Besides, we define and study the semi-open set on A^+ which will be called by semi-+-open sets.

Remark 2.1. A^+ does not induce a topological space, because $X^+ = Cl(X) - X = X - X = \emptyset$ and $\emptyset^+ = Cl(\emptyset) - \emptyset = \emptyset \cap X = \emptyset$. We can see that X will never be in the topology.

Remark 2.2. The operator $A^+ = Cl(A) - A = Cl(A) \cap A^c$, where A^c means the complement of the set A.

Definition 2.3. Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be semi-+-open if $A^+ \subseteq Cl(Int(A^+))$. The complement of a semi-+-open set is called semi-+-closed set.

Remark 2.4. The collection of all semi-+-open sets and semi-+-closed sets are denoted by $S{+}O(X,\tau)$ and $S{+}C(X,\tau)$ respectively.

The following example shows that every open need not be a semi-+-open set.

Example 2.5. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then, we can see that $(\{b\})^+ = Cl(\{b\}) - \{b\} = \{b,c\} - \{b\} = \{c\}$ and $Cl(Int((\{b\})^+)) = Cl(Int(\{c\})) = \emptyset$. Indeed, $\{c\} \nsubseteq \emptyset$, therefore $\{b\}$ is open, but it is not a semi-+-open.

The following example shows that the notion of semi-open set and semi-+-open set are independent.

Example 2.6. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then, $\{b\}$ is a semi-open set, but it is not a semi-+-open set. On the other hand, $\{c\}$ is a semi-+-open set, but it is not a semi-open set.

Theorem 2.7. *Every closed set is semi-+-open.*

Proof. Let A be a closed set of (X,τ). Since A is a closed set, then $Cl(A) = A$ and so $A^+ = Cl(A) - A = A \cap A^c = \emptyset$. Now, $\emptyset \subseteq Cl(Int(A^+)) = Cl(Int(\emptyset)) = \emptyset$. This shows that A is semi-+-open. □

The following example shows that the converse of the above Theorem need not be true.

Example 2.8. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then, $\{b,c,d\}$ is a semi-+-open set, but it is not a closed set.

Lemma 2.9. *Let $\{A_\delta : \delta \in \Delta\}$ be a collection of elements of (X,τ), then $\bigcup_{\delta\in\Delta} A_\delta^+ \subset (\bigcup_{\delta\in\Delta} A_\delta)^+$*

Proof. Let A_δ be a collection of elements of (X,τ), then

$$A_\delta^+ = Cl(A_\delta) \cap A_\delta^c$$

$$\bigcup_{\delta\in\Delta} A_\delta^+ = \bigcup_{\delta\in\Delta} (Cl(A_\delta) \cap A_\delta^c)$$

$$\subset (\bigcup_{\delta\in\Delta} Cl(A_\delta)) \cap (\bigcup_{\delta\in\Delta} A_\delta^c)$$

$$= Cl(\bigcup_{\delta\in\Delta} A_\delta)) \cap (\bigcup_{\delta\in\Delta} A_\delta^c)$$

$$= (\bigcup_{\delta\in\Delta} A_\delta)^+$$

□

Theorem 2.10. *The arbitrary union of semi-+-open sets is a semi-+-open set.*

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of semi-+-open sets, then

$$A_\delta^+ \subseteq Cl(Int(A_\delta^+))$$

$$\bigcup_{\delta\in\Delta} A_\delta^+ \subseteq \bigcup_{\delta\in\Delta} Cl(Int(A_\delta^+))$$

$$= Cl(\bigcup_{\delta\in\Delta} (Int(A_\delta^+)))$$

$$\subset Cl(Int(\bigcup_{\delta\in\Delta}(A_\delta^+)))$$

By the Lemma 2.9 we have that

$$\bigcup_{\delta\in\Delta} A_\delta^+ \subset Cl(Int((\bigcup_{\delta\in\Delta} A_\delta)^+))$$

Therefore, the arbitrary union of semi-+-open sets is semi-+-open. □

Lemma 2.11. *The arbitrary intersection of semi-+-closed sets is semi-+-closed set.*

Proof. The proof is followed by the Theorem 2.10. □

Definition 2.12. Let (X, τ) be a topological space and $A \subset X$. An element $x \in A$ is said to be semi-+-interior point of A if there exits a semi-+-open set U such that $x \in U \subseteq A$. The set of all semi-+-interior points of A is said to be semi+-interior of A an it is denoted by $Int_{s+}(A)$.

Theorem 2.13. *Let (X, τ) be a topological space and $A \subset X$. Then, A is semi-+-open if and only if $A = Int_{s+}(A)$.*

Proof. Let A be a semi-+-open set. Then, $A \subseteq A$ and this implies that $A \in \{U|U$ is semi-+-open and $U \subset A\}$. Since union of this collection is in A. Therefore, $A = Int_{s+}(A)$. Conversely, suppose that $A = Int_{s+}(A)$. Hence, A is semi-+-open. □

Definition 2.14. Let $A \subset X$. Then $x \in X$ is semi-+-adherent to A if $U \cap A \neq \emptyset$ for every semi-+-open set U containing x. The set of all semi-+-adherent points of A is said to be semi-+-closure of A and it is denoted by $Cl_{s+}(A)$.

Theorem 2.15. *Let (X, τ) be a topological space and $A, B \subset X$. Then, the following statements hold:*

1. *$A \subseteq Cl_{s+}(A)$.*
2. *$Cl_{s+}(A)$ is the smallest semi-+-closed set containing A, that is $Cl_{s+}(A) = \bigcap\{W|W$ is semi-+-closed and $A \subseteq W\}$.*
3. *A is semi-+-closed if and only if $A = Cl_{s+}(A)$.*
4. *If $A \subseteq B$, then $Cl_{s+}(A) \subseteq Cl_{s+}(B)$.*
5. *$Cl_{s+}(A) \cup Cl_{s+}(B) \subseteq Cl_{s+}(A \cup B)$.*
6. *$Cl_{s+}(A \cap B) \subseteq Cl_{s+}(A) \cap Cl_{s+}(B)$.*

Proof.
1. Let $x \in A$ and supposes that $x \notin Cl_{s+}(A)$. Then, there exits semi-+-open set V containing x such that $V \cap A = \emptyset$ and this is a contradiction. Therefore, $x \in Cl_{s+}(A)$.
2. Let $x \in Cl_{s+}(A)$. Then, $V \cap A \neq \emptyset$ for every semi-+-open set V containing x. Now, suppose the contrary, that

$$x \notin \bigcap\{W|W \text{ is semi-+-closed and } A \subseteq W\}$$

Then, $x \notin W$ for some semi-+-closed set W, so $x \in X-W$ for some semi-+-open set $X-W$. So, $(X-W) \cap A = \emptyset$ for some semi-+-open set $X-W$ containing x and this is a contradiction. Therefore, $x \notin \bigcap\{W|W$ is semi-+-closed and $A \subseteq W\}$. Conversely, let $y \in x \notin \bigcap\{W|W$ is semi-+-closed and $A \subseteq W\}$. Then, $y \in W$ for all semi-+-closed set W containing A. Now, suppose that $y \notin Cl_{s+}(A)$. Then, there exits semi-+-open set V containing y such that $V \cap A = \emptyset$. Therefore, $X-V$ is semi-+-closed set containing A and $y \notin X-V$ and this is a contradiction. Therefore, $y \in Cl_{s+}(A)$.

The proof of (3) and (4) are followed directly from the Definition 2.14. (5) and (6) are followed by applying part (4) of this Theorem. □

3 Generalized Semi-+-closed and regular generalized semi-+-closed sets

In this section, we use the notion of semi-+-closed sets for introducing the concepts of generalized semi-+-closed sets and regular generalized semi-+-closed sets. Furthermore, we show some of their properties.

Definition 3.1. Let (X, τ) be a topological space and $A \subset X$. Then, A is said to be a generalized semi-+-closed set or simply $gs+$-closed set if $Cl_{s+}(A) \subseteq U$, whenever $A \subseteq U$ and U is a open set. The complement of a $gs+$-closed set is called generalized semi-+-open set or simply $gs+$-open set.

Remark 3.2. The collection of all $gs+$-closed sets and $gs+$-open sets are denoted by $GS+(X, \tau)$ and $GS+(X, \tau)$, respectively.

Proposition 3.3. *Every semi-+-closed set is gs+-closed set.*

Proof. The proof is followed by the Definition 3.1. □

The following example shows that the converse of the above Proposition, it is not always true.

Example 3.4. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. Then, $\{a, b\}$ is a generalized semi-+-closed set, but it is not a semi-+-closed set.

Theorem 3.5. *Let A be a gs+-closed subset of X. Then, $Cl_{s+}(A) - A$ does not contain any non-empty closed sets.*

Proof. Let F be a closed set of X such that $F \subseteq Cl_{s+}(A) - A$. Since $X - F$ is a open set, then $A \subseteq X - F$ and A is $gs+$-closed, it follows $Cl_{s+}(A) \subseteq X - F$, in consequence $F \subseteq X - Cl_{s+}(A)$. This implies that $F \subseteq (X - Cl_{s+}(A)) \cap (Cl_{s+}(A) - A) = \emptyset$, therefore $F = \emptyset$. □

Corollary 3.6. *Let A be a gs+-closed set. Then, A is semi-+-closed if and only if $Cl_{s+}(A) - A$ is a closed set.*

Proof. Let A be $gs+$-closed set. If A is semi-+-closed, it has $Cl_{s+}(A) - A = \emptyset$ which is a closed set. Conversely, let $Cl_{s+}(A) - A$ be a closed set. Then, by the Theorem 3.5, $Cl_{s+}(A) - A$ does not contain any non-empty closed set and $Cl_{s+}(A)$ is a closed set of itself. Thus, $Cl_{s+}(A) - A = \emptyset$. Therefore, $A = Cl_{s+}(A)$, in consequence A is a semi-+-closed set. □

Corollary 3.7. *Let A be an open set and gs+-closed set. Then, $A \cap J$ is gs+-closed set whenever semi-+-closed set J of X.*

Proof. Since A is $gs+$-closed and open set, then $Cl_{s+}(A) \subseteq A$ and so A is a semi-+-closed. Therefore, $A \cap J$ is semi-+-closed set of X and this implies that $A \cap J$ is $gs+$-closed set of X. □

Theorem 3.8. *Let (X, τ) be a topological space and $A, B \subseteq X$. If A is a $gs+$-closed set and B is any set such that $A \subseteq B \subseteq Cl_{s+}(A)$, then B is a $gs+$-closed set of X.*

Proof. Let $B \subseteq V$ where V is an open set of X. Since A is a $gs+$-closed set and $A \subseteq V$, then $Cl_{s+}(A) \subseteq V$ and so $Cl_{s+}(A) = Cl_{s+}(B)$. Therefore, $Cl_{s+}(B) \subseteq V$ and hence B is a $gs+$-closed set of X. □

Theorem 3.9. *Let (X, τ) be a topological space and $A \subset X$. A is a $gs+$-open set if and only if $J \subseteq Int_{s+}(A)$ whenever J closed set and $J \subseteq A$.*

Proof. Let A be a $gs+$-open set and let $J \subseteq A$ where J is a closed set. Then, $X - A$ is a $gs+$-closed set contained in the open set $X - J$. Therefore, $Cl_{s+}(X - A) \subseteq X - J$ and $X - Int_{s+}(A) \subseteq X - J$. In consequence, $J \subseteq Int_{s+}(A)$.

Conversely, if A is a closed set with $J \subseteq Int_{s+}(A)$ and $J \subseteq A$, then $X - Int_{s+}(A) \subseteq X - J$. Therefore, $Cl_{s+}(X - A) \subseteq X - J$. Hence, $X - A$ is a $gs+$-closed set and A is a $gs+$-open set of X. □

Definition 3.10. Let (X, τ) be a topological space and $A \subseteq X$. Then, A is said to be regular generalized semi-+-closed or simply $rgs+$-closed set if $Cl_{s+}(A) \subseteq U$ whenever $A \subseteq U$ and U is a regular open set of X. The complement of a $rgs+$-closed set is called $rgs+$-open set.

Remark 3.11. The collection of all $rgs+$-closed sets and $rgs+$-open sets are denoted by $RGS+C(X, \tau)$ and $RGS+O(X, \tau)$, respectively.

Lemma 3.12. *Let A be a subset of a topological space (X, τ). Then, $Cl_{s+}(A) \subseteq Cl(A)$.*

Proof. The proof is clear. □

Theorem 3.13. *Every closed set is $rgs+$-closed set.*

Proof. Let B be any closed set of X such that $B \subseteq V$. where V is a regular open set. Since $Cl_{s+}(B) \subseteq Cl(B) = B$. Therefore, $Cl_{s+}(B) \subseteq V$. In consequence, B is a $rgs+$-closed set. □

The following example shows that the converse of the above Theorem need not be true.

Example 3.14. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. Then, $\{a, b, c\}$ is regular generalized semi-+-closed set, but it is not a closed set.

Theorem 3.15. *Every semi-+-closed set is $rgs+$-closed set.*

Proof. Let B be any semi-+-closed set of X such that V is any regular open set containing B. Since B is a semi-+-closed set, then $Cl_{s+}(B) = B$. Therefore, $Cl_{s+}(B) \subseteq V$. Hence, B is a $rgs+$-closed set. □

The following example shows that the converse of the above Theorem need not be true.

Example 3.16. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. Then, $\{a, b, c\}$ is regular generalized semi-+-closed set, but it is not a semi-+-closed set.

Open problems: In the Theorem 2.10, we showed that arbitrary union of semi-+-open sets is semi-+-open, but is the arbitrary intersection of semi-+-open sets a semi-+-open set? On the other hand, by consequence of the Theorem 2.10, we planted the Lemma 2.11 which says that arbitrary intersection of semi-+-closed sets is semi-+-closed, but is the arbitrary union of semi-+-closed sets a semi-+-closed set? We did not find any example, for that reason they are two open problems.

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