Fixed Point Theorems on Intuitionistic Fuzzy Bipolar Metric Spaces

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Abstract:

In this paper, we introduce the notion of intuitionistic fuzzy bipolar metric space, which can deal with the separation between purposes of two distinct sets. We characterize some fundamental definitions and expand the Banach contraction theorem for this new generalization. We give some non-trivial examples to support our result in intuitionistic fuzzy bipolar metric spaces.

Keywords: Fixed Point, Intuitionistic Fuzzy Metric Space, Complete, Bipolar Metric Spaces.

1 Introduction

Atanassov [1] presented and considered the idea of intuitionistic fuzzy sets. Park [12] utilizing the possibility of intuitionistic fuzzy sets characterized the thought of intuitionistic fuzzy metric spaces with the assistance of continuous t- norms and continuous t-conorms. George and Veeramani [5] indicated that each metric initiates fuzzy metric, each fuzzy metric instigates an intuitionistic fuzzy metric.

Various of generalizations of metric space have been presented by numerous famous mathematicians. As of late, Mutlu and Gurdal [10] generalized metric space which was considered bipolar metric spaces and give another idea of estimation of distance between the components of two unique sets. As in bipolar metric spaces a great deal of noteworthy work has been done by scientists (see, [5, 9, 10, 11]) and in fuzzy metric spaces, Grabeic [2] expand the notable fixed point theorem of Banach to fuzzy metric spaces in the feeling of Kramosil and Michalek [7].

The point of this paper is to present another speculation of the intuitionistic fuzzy bipolar metric space, which can deal with the separation between purposes of two distinct sets. We characterize some fundamental definitions and expand the Banach contraction theorem for this new generalization. Likewise, we additionally give examples to the legitimacy of our primary outcomes.

2 Preliminaries

Definition 2.1.[2]

Let χ and Y be two non-empty sets. A quadruple (χ, Y, M_b, $*$) is said to be fuzzy bipolar metric space (FB-space), where $*$ is continuous t-norm and M_b, is fuzzy set on $\chi \times Y \times (0,\infty)$, fulfilling the accompanying conditions: For all t, s, r > 0:

(i) $M_b(x, y, t) > 0$ for all $(x, y) \in \chi \times Y$;

(ii) $M_b(x, y, t) = 1$ if and only if $x = y$ for $x \in \chi$ and $y \in Y$;

(iii) $M_b(x, y, t) = M_b(y, x, t)$ for all $x, y \in \chi \cap Y$;

(iv) $M_b(x_1, y_2, t + s + r) \geq M_b(x_1, y_1, t) * M_b(x_2, y_1, s) * M_b(x_2, y_2, r)$, for all $x_1, x_2 \in \chi$ and $y_1, y_2 \in Y$;

(v) $M_b(x, y, .) : [0,\infty) \rightarrow [0, 1]$ is left continuous;

(vi) M_b (x, y, .) is non-decreasing for all x ∈ χ and y ∈ Y.

Definition 2.2.

Let χ and Y be two non-empty sets. A 6-tuple (χ, Y, M_b, N_b, ∗, ◊) is said to be intuitionistic fuzzy bipolar metric space (IFBM-space), where ∗ is continuous t-norm and M_b, N_b, are fuzzy set on χ × Y × (0,∞), fulfilling the accompanying conditions: For all t, s, r > 0:

(i) M_b (x, y, t) + N_b (x, y, t) ≤ 1,

(ii) M_b (x, y, t) > 0 for all (x, y) ∈ χ × Y;

(iii) M_b (x, y, t) = 1 if and only if x = y for x ∈ χ and y ∈ Y;

(iv) M_b (x, y, t) = M_b (y, x, t) for all x, y ∈ χ ∩ Y;

(v) M_b (x_1, y_2, t + s + r) ≥ M_b (x_1, y_1, t) ∗M_b (x_2, y_1, s) ∗M_b (x_2, y_2, r), for all x_1, x_2 ∈ χ and y_1, y_2 ∈ Y;

(vi) M_b (x, y, .) : [0,∞) → [0, 1] is left continuous;

(vii) M_b (x, y, .) is non-decreasing for all x ∈ χ and y ∈ Y.

(viii) N_b (x, y, t) < 1 for all (x, y) ∈ χ × Y;

(ix) N_b (x, y, t) = 0 if and only if x = y for x ∈ χ and y ∈ Y;

(x) N_b (x, y, t) = N_b (y, x, t) for all x, y ∈ χ ∩ Y;

(xi) N_b (x_1, y_2, t + s + r) ≤ N_b (x_1, y_1, t) ◊N_b (x_2, y_1, s) ◊N_b (x_2, y_2, r), for all x_1, x_2 ∈ χ and y_1, y_2 ∈ Y;

(xii) N_b (x, y, .) : [0,∞) → [0, 1] is right continuous;

(xiii) N_b (x, y, .) is non-increasing for all x ∈ χ and y ∈ Y.

Remark 2.3.

In IFBM-space (χ, Y, M_b, N_b, ∗, ◊), M_b (x, y, t) and N_b (x, y, t) can be thought of as the indicate the degree of nearness and degree of non-nearness among x and y concerning t, where (x, y) ∈ χ × Y.

Example 2.4.

Let (χ, Y, d) be an bipolar metric space. For all x ∈ χ, y ∈ Y, and t > 0, denote M_b (x, y, t) = $\frac{t}{t+d(x,y)}$ and N_b (x, y, t) = $\frac{d(x,y)}{t+d(x,y)}$. Then (χ, Y, M_b, N_b, ∗, ◊) is IFBM-space, where is a t-norm defined as $\mu * \nu = \min\{\mu, \nu\}$ and t-conorm defined as $\mu \lozenge \nu = \max\{\mu, \nu\}$.

Remark 2.5.

In a IFBM-space (χ, Y, M_b, N_b, ∗, ◊), in the event that χ = Y, at that point (χ, M_b, N_b, ∗, ◊) is intuitionistic fuzzy bipolar metric space.

Definition 2.6.

Let (χ, Y, M_b, N_b, ∗, ◊) be a IFBM-space. The points belong to χ, Y and χ ∩ Y are called as Left, Right and Central Points individually and sequences belong to χ, Y , χ ∩ Y and X × Y are named as Left sequences, Right sequences, Central sequences and bisequences separately.

Lemma 2.7.

Let (χ, Y, M_b, N_b, ∗, ◊) be a IFBM- space with the end goal that M_b (x, y, kt) ≥ M_b (x, y, t) and N_b (x, y, kt) ≤ N_b (x, y, t) for x ∈ χ, y ∈ Y and k ∈ (0, 1). Then x = y.

Proof.

We have M_b (x, y, kt) ≥ M_b (x, y, t) and N_b (x, y, kt) ≤ N_b (x, y, t) for t > 0. (2.7.1)

Since kt < t for all t > 0 and k ∈ (0, 1), by (vii) and (xiii) we have

M_b (x, y, kt) ≤ M_b (x, y, t) and N_b (x, y, kt) ≥ N_b (x, y, t). (2.7.2)

From (2.6.1) and (2.6.2) and definition (2.1) of IFBM-space, we get x = y.

Definition 2.8.

Let $(\chi, Y, M_b, N_b, *, \Diamond)$ be a IFBM-space. A left sequence (x_n) converges to a right point y if and only if for every $\epsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M_b(x_n, y, t) > 1 - \epsilon$ and $N_b(x_n, y, t) < \epsilon$ for all $n > n_0$, i.e., $M_b(x_n, y, t) \to 1$ and $N_b(x_n, y, t) \to 0$ as $n \to \infty$ for all $t > 0$.

Similarly, a right sequence (y_n) converges to a left point x if and only if for every $\epsilon > 0$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $M_b(x, y_n, t) > 1 - \epsilon$ and $N_b(x, y_n, t) < \epsilon$ for all $n \geq n_0$, i.e., $M_b(x, y_n, t) \to 1$ and $N_b(x, y_n, t) \to 0$ as $n \to \infty$ for all $t > 0$.

Remark 2.9.

If $\chi = Y$, then convergent of any sequences in IFBM-space is equivalent to its convergence in intuitionistic fuzzy bipolar metric space.

Definition 2.10.

Consider, IFBM-space

(i) If the sequences (x_n) and (y_n) are converge, then the bisequence $(x_n, y_n) \in \chi \times Y$ is said to be a convergent bisequence. If the sequences (x_n) and (y_n) are converge to same center point then the bisequence (x_n, y_n) is said to be a biconvergent bisequence.

(ii) A bisequence (x_n, y_n) is said to be a Cauchy bisequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ such that $M_b(x_n, y_m, t) > 1 - \epsilon$ and $N_b(x_n, y_m, t) < \epsilon$ for each $t > 0$, i.e., a bisequence (x_n, y_n) is said to be a Cauchy bisequence if $M_b(x_n, y_m, t) \to 1$ and $N_b(x_n, y_m, t) \to 0$ as $n, m \to \infty$ for all $t > 0$.

Definition 2.11.

The IFBM-space is said to be complete if every Cauchy bisequence in $\chi \times Y$ is convergent in it.

Proposition 2.12.

In a IFBM-space, every convergent Cauchy bisequence is biconvergent.

Proof.

Consider, IFBM-space and bisequence $(x_n, y_n) \in \chi \times Y$ such that $(x_n) \to y \in Y$ and $(y_n) \to x \in \chi$. Since (x_n, y_n) is convergent Cauchy bisequence, so for all $t > 0$, we have

$M_b(x_n, y_n, t) \to 1$ and $N_b(x_n, y_n, t) \to 0$ as $n \to \infty$, which implies that

$M_b(x, y, t) = 1$ and $N_b(x, y, t) = 0$, for all $t > 0$. By (iii) and (ix) of Definition 2.1, w get x = y.

Hence every convergent Cauchy bisequence (x_n, y_n) is biconvergent.

Proposition 2.13.

In a IFBM-space, every biconvergent bisequence is a Cauchy bisequence.

Proof.

Consider IFBM-space and bisequence (x_n, y_n), $\in \chi \times Y$ converges to a point

$x_0 \in \chi \cap Y$ for all $n, m \geq n_0 \in N$ and $t > 0$.

Now, $M_b(x_n, y_m, t) \geq M_b(x_n, x_0, \frac{t}{3}) * M_b(x_0, x_0, \frac{t}{3}) * M_b(x_0, y_m, \frac{t}{3})$ and

$N_b(x_n, y_m, t) \leq N_b(x_n, x_0, \frac{t}{3}) \Diamond N_b(x_0, x_0, \frac{t}{3}) \Diamond N_b(x_0, y_m, \frac{t}{3})$ as $n, m \to \infty$, we get

$M_b(x_n, y_m, t) > 1$ and $N_b(x_n, y_m, t) < 0$ for each $t > 0$.

Which implies that $M_b(x_n, y_m, t) \to 1$, $N_b(x_n, y_m, t) \to 0$ for each $t > 0$.

Hence, (x_n, y_n) is a Cauchy bisequence.

Lemma 2.14.

Consider, IFBM-space and $u \in \chi \cap Y$ is a limit of a sequence then it is a unique limit of the sequence.

Proof.

Let $(x_n) \in \chi$ be a sequence. Suppose that $(x_n) \to y \in Y$ and also $(x_n) \to u \in \chi \cap Y$, then for $t, s, r > 0$. We have $M_b(u, y, t + s + r) \geq M_b(u, u, t) * M_b(x_n, u, s) * M_b(x_n, y, r)$ and

$N_b(u, y, t + s + r) \leq N_b(u, u, t) \lozenge N_b(x_n, u, s) \lozenge N_b(x_n, y, r)$ as $n \to \infty$ we get,

$M_b(u, y, t + s + r) \geq 1$ and $N_b(u, y, t + s + r) \leq 0$,

which implies that $u = y$, i.e., sequence (x_n) have a unique limit.

3 Main results

Theorem 3.1

Let $(\chi, Y, M_b, N_b, *, \lozenge)$ be a complete IFBM-space such that

$$M_b(x, y, t) = 1 \text{ and } N_b(x, y, t) = 0 \text{ as } t \to \infty \text{ for all } x \in \chi,\ y \in Y. \tag{3.1.1}$$

Let $\zeta: \chi \cup Y \to \chi \cup Y$ be mapping satisfying

(i) $\zeta(\chi) \subseteq \chi$ and $\zeta(Y) \subseteq Y$;

(ii) $M_b(\zeta(x), \zeta(y), kt) \geq M_b(x, y, t)$ and $N_b(\zeta(x), \zeta(y), kt) \leq N_b(x, y, t)$ for all $x \in \chi,\ y \in Y$ and $t > 0$, where $k \in (0, 1)$.

Then ζ has a unique fixed point.

Proof.

Fix $x_0 \in \chi$ and $y_0 \in Y$ and assume that $\zeta(x_n) = x_{n+1}$ and $\zeta(y_n) = y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then we get (x_n, y_n) as a bisequence on IFBM-space $(\chi, Y, M_b, N_b, *, \lozenge)$.

Now, we have $M_b(x_1, y_1, t) = M_b(\zeta(x_0), \zeta(y_0), t) > M_b(x_0, y_0, \frac{t}{k})$ and

$N_b(x_1, y_1, t) = N_b(\zeta(x_0), \zeta(y_0), t) < N_b(x_0, y_0, \frac{t}{k})$ for every $t > 0$ and $n \in \mathbb{N}$.

By simple induction, we get

$M_b(x_n, y_n, t) = M_b(\zeta(x_{n-1}), \zeta(y_{n-1}), t) \geq M_b(x_0, y_0, \frac{t}{k^n})$ and

$$N_b(x_n, y_n, t) = N_b(\zeta(x_{n-1}), \zeta(y_{n-1}), t) \leq N_b(x_0, y_0, \frac{t}{k^n}) \tag{3.1.2}$$

Also, $M_b(x_{n+1}, y_n, t) = M_b(\zeta(x_n), \zeta(y_{n-1}), t) \geq M_b(x_1, y_0, \frac{t}{k^n})$ and

$$N_b(x_{n+1}, y_n, t) = N_b(\zeta(x_n), \zeta(y_{n-1}), t) \leq N_b(x_1, y_0, \frac{t}{k^n}) \tag{3.1.3}$$

for every $t > 0$ and $n \in \mathbb{N}$. Assume $n < m$, for $n, m \in \mathbb{N}$.

If, we take $g_i \in (0, 1)$ for $i = n, (n+1), \ldots, 2(m-n)+1$, fulfilling $g_n + g_{n+1} + \cdots + g_{2(m-n)+1} = 1$.

Then from the definition of IFBM-space , (3.1.2) and (3.1.3), we get

$$M_b(x_n, y_m, t) \geq M_b(x_n, y_n, g_n t) * M_b(x_{n+1}, y_n, g_{n+1} t) * \cdots * M_b(x_m, y_{m-1}, g_{2(m-n)} t) * M_b(x_m, y_m, g_{2(m-n)+1} t)$$

$$\geq M_b(x_0, y_0, \tfrac{g_n t}{k^n}) * M_b(x_1, y_0, \tfrac{g_{n+1} t}{k^n}) * \cdots * M_b(x_1, y_0, \tfrac{g_{2(m-n)} t}{k^n}) * M_b(x_0, y_0, \tfrac{g_{2(m-n)+1} t}{k^n}) \quad \text{and}$$

$$N_b(x_n, y_m, t) \leq N_b(x_n, y_n, g_n t) \lozenge N_b(x_{n+1}, y_n, g_{n+1} t) \lozenge \cdots \lozenge N_b(x_m, y_{m-1}, g_{2(m-n)} t) \lozenge N_b(x_m, y_m, g_{2(m-n)+1} t)$$

$$\leq N_b(x_0, y_0, \tfrac{g_n t}{k^n}) \lozenge N_b(x_1, y_0, \tfrac{g_{n+1} t}{k^n}) \lozenge \cdots \lozenge N_b(x_1, y_0, \tfrac{g_{2(m-n)} t}{k^n}) \lozenge N_b(x_0, y_0, \tfrac{g_{2(m-n)+1} t}{k^n})$$

From (3.1.1), as $n, m \to \infty$ we get $M_b(x_n, y_m, t) \geq 1$ and $N_b(x_n, y_m, t) \leq 0$

for every $t > 0$. Which implies that bisequence (x_n, y_n) is a Cauchy bisequence.

Now, IFBM-space (χ, Y, M_b, N_b, $*$, $\Diamond$) is a complete space. So, bisequence (x_n, y_n) is a convergent Cauchy bisequence. As indicated by the Proposition 2.12 the bisequence

(x_n, y_n) is biconvergent bisequence. As, bisequence (x_n, y_n) is biconvergent then there exist a point $u \in \chi \cap$ Y which is a limit of the both sequences (x_n) and (y_n). By Lemma 2.14, both sequences (x_n) and (y_n) has a unique limit. We claim that $u \in \chi \cap$ Y is a fixed point of ζ. From Definition 2.2 (v) and (xi), Consider

$M_b(\zeta(u), u, t) > M_b(\zeta(u), \zeta(y_n), \frac{t}{3}) * M_b(\zeta(x_n), \zeta(y_n), \frac{t}{3}) * M_b(\zeta(x_n), u, \frac{t}{3})$ and

$N_b(\zeta(u), u, t) < N_b(\zeta(u), \zeta(y_n), \frac{t}{3}) \Diamond N_b(\zeta(x_n), \zeta(y_n), \frac{t}{3}) \Diamond N_b(\zeta(x_n), u, \frac{t}{3})$

for every $n \in \mathbb{N}$ and $t > 0$ and as $n \to \infty$ we have

$M_b(\zeta(u), u, t) \to 1 * 1 * 1 = 1$ and $N_b(\zeta(u), u, t) \to 0 \Diamond 0 \Diamond 0 = 0$.

From Definition 2.2 (iii) and (ix), we get $\zeta(u) = u$.

For the uniqueness of the fixed point assume that a point $v \in \chi \cap$ Y is another fixed point of ζ. We have $M_b(u, v, t) = M_b(\zeta(u), \zeta(v), t) \geq M_b(u, v, \frac{t}{k})$ and $N_b(u, v, t) = N_b(\zeta(u), \zeta(v), t) \leq N_b(u, v, \frac{t}{k})$ for $k \in (0, 1)$ and for every $t > 0$. By Lemma 2.7, we get $u = v$.

Example 3.2

Let χ = [0, 1] and Y = {0} ∪ ℕ − {1}. Define $M_b = \frac{t}{t + |x - y|}$ and $N_b = \frac{|x - y|}{t + |x - y|}$ for all t > 0 and x ∈ χ and y ∈ Y. Clearly, (χ, Y, M_b, N_b, $*$, $\Diamond$) is a complete intuitionistic fuzzy bipolar metric space, where $*$ is a continuous t-norm and $\Diamond$ is a continuous t-conorm defined as $a * b = ab$ and $a \Diamond b = max\{a, b\}$.

Let $\zeta: \chi \cup$ Y → $\chi \cup$ Y be a mapping given by

$$\zeta(u) = \begin{cases} \frac{u}{2}, & u \in [0,1] \\ 0, & u \in \mathbb{N} - \{1\} \end{cases}$$

for all $u \in \chi \cup$ Y. ζ satisfies the condition (i) of Theorem (3.1).

Now, suppose that $k = \frac{1}{2}$ then for all t > 0, condition (ii) of Theorem 3.1 also satisfies by ζ. We can construct the bisequences $\zeta(x_n) = x_{n+1}$ and $\zeta(y_n) = y_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$.

By assuming $x_0 = 1$ and $y_0 = 2$, we obtain a non-trivial sequence as $(x_n, y_n) = \{(1, 2), (\frac{1}{2}, 0), (\frac{1}{2^2}, 0), \ldots\}$.

By Theorem 3.1, we get ζ have a unique fixed point, i.e., $x = 0$.

Corollary 3.3.

Let (χ, Y, M_b, N_b, $*$, $\Diamond$) be a complete intuitionistic fuzzy bipolar metric space such that

$M_b(x, y, t) = 1$ and $N_b(x, y, t) = 0$ as $t \to \infty$ for every $x \in \chi$, $y \in$ Y. (3.3.1)

Let $\zeta: \chi \cup$ Y → $\chi \cup$ Y be a mapping satisfying

(i) $\zeta(\chi) \subseteq$ Y and ζ(Y) $\subseteq \chi$;

(ii) $M_b(\zeta(y), \zeta(x), kt) > M_b(x, y, t)$ and $N_b(\zeta(y), \zeta(x), kt) < N_b(x, y, t)$ for every $\in \chi$, $y \in$ Y and $t > 0$, where $k \in (0, 1)$.

Then ζ has a unique fixed point.

Proof.

Fix $x_0 \in \chi$ and assume that $\zeta(x_n) = y_n$ and $\zeta(y_n) = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Then we get (x_n, y_n) as a bisequence on intuitionistic fuzzy bipolar metric space (χ, Y, M_b, N_b, $*$, $\Diamond$). We have

$M_b(x_1, y_0, t) = M_b(\zeta(y_0), \zeta(x_0), t) \geq M_b(x_0, y_0, \frac{t}{k})$ and

$N_b(x_1, y_0, t) = N_b(\zeta(y_0), \zeta(x_0), t) \leq N_b(x_0, y_0, \frac{t}{k})$ for every $t > 0$ and $n \in \mathbb{N}$.

By simple induction, we get

$M_b(x_n, y_n, t) = M_b(\zeta(y_{n-1}), \zeta(x_n), t) \geq M_b(x_0, y_0, \frac{t}{k^{2n}})$ and

$N_b(x_n, y_n, t) = N_b(\zeta(y_{n-1}), \zeta(x_n), t) \leq N_b(x_0, y_0, \frac{t}{k^{2n}})$ (3.3.2)

$M_b(x_{n+1}, y_n, t) = M_b(\zeta(y_n), \zeta(x_n), t) \geq M_b(x_0, y_0, \frac{t}{k^{2n+1}})$ and

$N_b(x_{n+1}, y_n, t) = N_b(\zeta(y_n), \zeta(x_n), t) \leq N_b(x_0, y_0, \frac{t}{k^{2n+1}})$ for every $t > 0$ and $n \in \mathbb{N}$. (3.3.3)

Letting $n < m$, for $n, m \in \mathbb{N}$; If we take $g_i \in (0,1)$ for $i = n, n+1 \ldots, 2(m-n)+1$, fulfilling $g_n + g_{n+1} + \cdots + g_{2(m-n)+1} = 1$, then from the definition of IFBM -space, (3.3.2) and (3.3.3), we get

$$M_b(x_n, y_m, t) \geq M_b(x_n, y_n, g_n t) * M_b(x_{n+1}, y_n, g_{n+1} t) * \cdots * M_b(x_m, y_{m-1}, g_{2(m-n)} t) * M_b(x_m, y_m, g_{2(m-n)+1} t)$$

$$\geq M_b\left(x_0, y_0, \frac{g_n t}{k^{2n}}\right) * M_b\left(x_0, y_0, \frac{g_{n+1} t}{k^{2n+1}}\right) * \cdots * M_b\left(x_0, y_0, \frac{g_{2(m-n)} t}{k^{2m+1}}\right) * M_b\left(x_0, y_0, \frac{g_{2(m-n)+1} t}{k^{2m}}\right) \quad \text{and}$$

$$N_b(x_n, y_m, t) \leq N_b(x_n, y_n, g_n t) \lozenge N_b(x_{n+1}, y_n, g_{n+1} t) \lozenge \cdots \lozenge N_b(x_m, y_{m-1}, g_{2(m-n)} t) \lozenge N_b(x_m, y_m, g_{2(m-n)+1} t)$$

$$\leq N_b\left(x_0, y_0, \frac{g_n t}{k^{2n}}\right) \lozenge N_b\left(x_0, y_0, \frac{g_{n+1} t}{k^{2n+1}}\right) \lozenge \cdots \lozenge N_b\left(x_0, y_0, \frac{g_{2(m-n)} t}{k^{2m+1}}\right) \lozenge N_b\left(x_0, y_0, \frac{g_{2(m-n)+1} t}{k^{2m}}\right).$$

From (3.3.1), as $n, m \to \infty$ we get $M_b(x_n, y_m, t) \geq 1$, $N_b(x_n, y_m, t) \leq 0$ for all $t > 0$.

Which implies that bisequence (x_n, y_n) is a Cauchy bisequence.

Now, IFBM-space (χ, Y, M_b, N_b, $*$, $\lozenge$) is a complete space. So, bisequence (x_n, y_n) is a convergent Cauchy bisequence. According to the Proposition 2.11 the bisequence (x_n, y_n) is biconvergent sequence.

As, bisequence (x_n, y_n) is biconvergent then there exist a point $u \in \chi \cap Y$ which is a limit of the both sequences (x_n) and (y_n). By, Lemma 2.13 sequences (x_n) and (y_n) have a unique limit.

Now, to prove $u \in \chi \cap Y$ as the fixed point of mapping ζ, Let we consider

$M_b(\zeta(u), u, t) \geq M_b(\zeta(u), \zeta(x_n), \frac{t}{3}) * M_b(\zeta(y_n), \zeta(x_n), \frac{t}{3}) * M_b(u, \zeta(x_n), \frac{t}{3})$,

$N_b(\zeta(u), u, t) \leq N_b(\zeta(u), \zeta(x_n), \frac{t}{3}) \lozenge N_b(\zeta(y_n), \zeta(x_n), \frac{t}{3}) \lozenge N_b(u, \zeta(x_n), \frac{t}{3})$

for all $n \in \mathbb{N}$ and $t > 0$ and as $n \to \infty$, we have $M_b(\zeta(u), u, t) \to 1 * 1 * 1 = 1$ and $N_b(\zeta(u), u, t) \to 0 \lozenge 0 \lozenge 0 = 0$ from Definition 2.1, (iii) and (ix), we get $\zeta(u) = u$.

For the uniqueness of the fixed point assume that a point $v \in \chi \cap Y$ is another fixed point of ζ.

We have, $M_b(u, v, t) = M_b(\zeta(v), \zeta(u), t) \geq M_b(u, v, \frac{t}{k})$, $N_b(u, v, t) = N_b(\zeta(v), \zeta(u), t) \leq N_b(u, v, \frac{t}{k})$ for $k \in (0,1)$ and for all $t > 0$.

By Lemma 2.6, we get $u = v$.

Theorem 3.4.

Let (χ, Y, M_b, N_b, $*$, $\lozenge$) be a complete intuitionistic fuzzy bipolar metric space and $\zeta : \chi \cup Y \to \chi \cup Y$ a mapping satisfying:

(i) $\zeta(\chi) \subseteq \chi$ and $\zeta(Y) \subseteq Y$;

(ii) $M_b(x, y, t) > 0 \Rightarrow M_b(\zeta(x), \zeta(y), t) \geq \Psi(M_b(x, y, t))$ and

$N_b(x, y, t) < 0 \Rightarrow N_b(\zeta(x), \zeta(y), t) \leq \Psi(N_b(x, y, t))$ for every $\in \chi$, $y \in Y$ and $t > 0$.

Then ζ has a fixed point.

Proof.

Let $x_0 \in \chi$ and $y_0 \in$ Y be such that $\zeta(x_n) = x_{n+1}$ and $\zeta(y_n) = y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then (x_n, y_n) be a bisequence on intuitionistic fuzzy bipolar metric space (χ, Y, M_b, N_b, $*$, $\lozenge$).

By the definition 2.2 for all $t > 0$ and condition (ii) and (iii) we have

$M_b(x_n, y_n, t) \geq \Psi^n(M_b(x_0, y_0, t))$ and $N_b(x_n, y_n, t) \leq \Psi^n(N_b(x_0, y_0, t))$ (3.4.1)

$M_b(x_{n+1}, y_n, t) \geq \Psi^n(M_b(x_1, y_0, t))$ and $N_b(x_{n+1}, y_n, t) \leq \Psi^n(N_b(x_1, y_0, t))$ (3.4.2)

Letting $n < m$, and defining as Theorem (3.1) for all i = n, . . . , 2(m − n) + 1, then from (3.4.1), (3.4.2),and definition of IFBM-space we have

$$M_b(x_n, y_m, t) \geq \Psi^n(M_b(x_0, y_0, g_n t)) * \Psi^n(M_b(x_1, y_0, g_{n+1} t)) * \cdots * \Psi^{m-1}(M_b(x_1, y_0, g_{2(m-n)} t)) * \Psi^m(M_b(x_0, y_0, g_{2(m-n)+1} t))$$

$$N_b(x_n, y_m, t) \leq \Psi^n(N_b(x_0, y_0, g_n t)) \lozenge \Psi^n(N_b(x_1, y_0, g_{n+1} t)) \lozenge \cdots \lozenge \Psi^{m-1}(N_b(x_1, y_0, g_{2(m-n)} t)) \lozenge \Psi^m(N_b(x_0, y_0, g_{2(m-n)+1} t)).$$

Now as n, m → ∞, we have $M_b(x_n, y_m, t) \to 1$ and $N_b(x_n, y_m, t) \to 0$ for all $t > 0$.

Apply same lines of the proof of Theorem 3.1 here. We have, if $u \in \chi \cap$ Y is a unique limit of sequences (x_n) and (y_n), then we have to show that u is a fixed point of ζ.

Since we have $M_b(x_n, u, t) \to 1$ and $N_b(x_n, u, t) \to 0$ for all $t > 0$. Also

$M_b(x_{n+1}, \zeta(u), t) = M_b(\zeta(x_n), \zeta(u), t) \geq \Psi(M_b(x_n, u, t)) \geq M_b(x_n, u, t)$ and

$N_b(x_{n+1}, \zeta(u), t) = N_b(\zeta(x_n), \zeta(u), t) \geq \Psi(N_b(x_n, u, t)) \geq N_b(x_n, u, t)$

and it follows that $x_{n+1} \to \zeta(u)$, which implies that $\zeta(u) = u$.

Corollary 3.5.

Let (χ, Y, M_b, N_b, $*$, $\lozenge$) be a complete intuitionistic fuzzy bipolar metric space and $\zeta : \chi \cup Y \to \chi \cup Y$ a mapping satisfying:

(i) $\zeta(\chi) \subseteq$ Y and ζ (Y) $\subseteq \chi$;

(ii) $M_b(x, y, t) > 0 \Rightarrow M_b(\zeta(y), \zeta(x), t) \geq \Psi(M_b(x, y, t))$ and

$N_b(x, y, t) < 1 \Rightarrow N_b(\zeta(y), \zeta(x), t) \leq \Psi(N_b(x, y, t))$ for $\in \chi$, $y \in$ Y and $t > 0$,

Then ζ has a fixed point.

Proof.

Proof of the Corollary follows on the lines of proof of the Theorem 3.4 and Corollary 3.3 .

4 Conclusion

In the current exploration, we need to research proposes of intuitionistic fuzzy bipolar metric spaces. We have introduced fixed point results by utilizing contractive conditions characterized on intuitionistic fuzzy bipolar metric spaces, appropriate models that underpins our principle results.

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