The Dynamics and Attractivity for a Rational Recursive Sequence of Order Three

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Abstract

This paper is concerned with the behavior of solution of the nonlinear difference equation
$x_{n+1} = ax_n + bx_{n-1} + cx_{n-2}d + ex_{n-1}x_{n-2}$, $n = 0, 1, ...,$where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d, e are positive constants.

Keywords: Stability, Periodicity, Boundedness, Recursive Sequence, Difference Equation.

Mathematics Subject Classification: 39A10

1 Introduction

In this paper we deal with the behavior of the solution of the following difference equation

$$x_{n+1} = ax_n + bx_{n-1} + cx_{n-2}d + ex_{n-1}x_{n-2},\quad n = 0, 1, ..., 1 \tag{1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and $a,\ b,\ c,\ d,\ e$ are positive constants.

During the last quarter of the twentieth century had been devloped the hypothesis of discrete dynamical systems and difference equations .In many years had grown by applications of difference equations also experienced enormous.Recently ,in the areas of bicology,economics,physics,resource management and others had been evidence by many applications of discrete dynamical sysrtems and difference equations.

Many researchers have investigated the behavior of the solution of difference equations for example: Elsayed and Abdul Khaliq [1] studied the dynamics of the following difference equation $x_{n+1} = ax_{n-l} + bx_{n-k} + cx_{n-s}d + ex_{n-t}$.Zayed [5] investigated the global stability and some properties of the nonnegative solutions of the following difference equations $x_{n+1} = Ax_n + Bx_{n-k} + px_n + x_{n-k}q + x_{n-k}$.In [8] Elabbasy et al. investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence $x_{n+1} = ax_n - bx_n cx_n - dx_{n-1}$.Elsayed et al. [6] investigated the global stability, boundedness, periodicity of solutions of the difference equation $x_{n+1} = ax_n + b + cx_{n-1}d + ex_{n-1}$.In [24] Ibrahim got the form of the solution of the rational difference equation $x_{n+1} = x_n x_{n-2} x_{n-1}(a + bx_n x_{n-2})$.Karatas et al. [25] gave that the solution of the difference equation
$x_{n+1} = x_{n-5}1 + x_{n-2}x_{n-5}$.
Yalçınkaya et al. [41] considered the dynamics of the difference equation $x_{n+1} = ax_{n-k}b + cx_n^p$.Elabbasy et al. [2] investigated the global asymptotic stability of the difference equation $x_{n+1} = ax_{n-2} + bx_{n-2}x_{n-3}cx_n + dx_{n-3}$.See also [28-38]. Other related results on rational difference equations can be found in refs. [30–44].

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $f: I^{k+1} \to I$,be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}),\quad n = 0, 1, ..., 2 \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [27].

Definition 1. (Equilibrium Point)

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$. That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$,we have $|x_n - \overline{x}| < \epsilon \quad for\ all \quad n \geq -k$.(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$,we have $\lim_{n\to\infty} \ x_n = \overline{x}$.(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} \ x_n = \overline{x}$.(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(1) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.3 \tag{3}$$

Theorem A [26]: Assume that $p, q \in \ R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$,is a sufficient condition for the asymptotic stability of the difference equation $x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$.

Remark. Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \ \ n = 0, 1, ..., 4 \tag{4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq.(4) is asymptotically stable provided that $\sum_{i=1}^{k} |p_i| < 1$.

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, ...x_{n-k}) \qquad n = 0, 1, 2...5 \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [27]: Let $[\alpha, \beta]$ be an interval of real numbers and assume that g:$[\alpha, \beta]^{k+1} \to [\alpha, \beta]$,is a continuous function satisfying the following properties :

(a) $g(x_1, x_2, ..., x_{k+1})$ is non-increasing in one component (for example x_σ) for each $x_r (r \neq \sigma)$ in $[\alpha, \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system M=g(m,m,...,m,M,m,...m,m) *and* $m = g(M, M, ..., M, m, M, ..., M, M)$,then m=M. Then Eq.(5) has a unique equilibrium $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(5) converges to $\overline{x}$.

Definition 3. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2 Local Stability of Eq.(1)

In this section we investigate the local stability character of the solutions of Eq.(1). Eq.(1) has a unique equilibrium point and is given by

$$\begin{aligned} \overline{x} &= a\overline{x} + b\overline{x} + c\overline{x}d + e\overline{x}^2, \\ (1-a)\overline{x} &= \frac{(b+c)\overline{x}}{d + e\overline{x}^2}, \end{aligned}$$

$$\begin{aligned} (1-a)d\overline{x} + (1-a)e\overline{x}^3 - (b+c)\overline{x} &= 0, \\ [(1-a)d - (1-a)e\overline{x}^2 - (b+c)]\overline{x} &= 0, \end{aligned}$$

$\overline{x}=0,\ or\ \ \overline{x}=\sqrt{\frac{(b+c)-(1-a)d}{e(1-a)}}\ if a\neq 1$.There are two positive equilibrium points, if $a\neq 1$.

Let $f:(0,\infty)^3\longrightarrow(0,\infty)$ be a function defined by

$$f(u,v,w)=au+\frac{bv+cw}{d+evw}.6 \tag{6}$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w) &= a,\ 7 && (7)\\ f_v(u,v,w) &= \frac{bd-ecw^2}{(d+evw)^2}, && (8)\\ f_w(u,v,w) &= \frac{cd-ebv^2}{(d+evw)^2}. && (9)\end{aligned}$$

Therefore at $\overline{x}=\sqrt{(b+c)-(1-a)de(1-a)}$

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x}) &= a=-c_0,\\ f_v(\overline{x},\overline{x},\overline{x}) &= \frac{bd-ec\overline{x}^2}{(d+e\overline{x}^2)^2}=\frac{bd-ec\left((b+c)-(1-a)de(1-a)\right)}{\left[d+e\left((b+c)-(1-a)de(1-a)\right)\right]^2}\\ &= \frac{(bd(1-a)-c(b+c)+cd(1-a)(1-a))}{(d(1-a)+(b+c)-(1-a)d(1-a))^2}=\frac{bd-abd-cb-c^2+cd-acd(1-a)}{(b+c1-a)^2}\\ &= d(c+b)(1-a)-c(b+c)(b+c)(1-a)^2=\frac{(d(1-a)-c)(b+c)}{(b+c)(1-a)^2}\\ &= \frac{(d(1-a)-c)(1-a)}{b+c}=-c_1,\\ f_w(\overline{x},\overline{x},\overline{x}) &= \frac{cd-eb\overline{x}^2}{(d+e\overline{x})^2}=\frac{bd-eb\left((b+c)-(1-a)de(1-a)\right)}{\left[d+e\left((b+c)-(1-a)de(1-a)\right)\right]^2}\\ &= \frac{(bd(1-a)-b(b+c)+bd(1-a)(1-a))}{(d(1-a)+(b+c)-(1-a)d(1-a))^2}\\ &= bd-abd-cb-b^2+bd-abd(1-a)(b+c1-a)^2\\ &= \frac{d(c+b)(1-a)-b(b+c)}{(b+c)(1-a)^2}=\frac{(d(1-a)-b)(b+c)}{(b+c)(1-a)^2}\\ &= \frac{(d(1-a)-b)(1-a)}{b+c}=-c_2.\end{aligned}$$

Also, we see that at $\overline{x}=0$,

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x}) &= a=-c_0,\\ f_v(\overline{x},\overline{x},\overline{x}) &= \frac{b}{d}=-c_1,\\ f_w(\overline{x},\overline{x},\overline{x}) &= \frac{c}{d}=-c_2.\end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1}+c_0y_n+c_1y_{n-1}+c_2y_{n-1}=0.8 \tag{10}$$

Theorem 2.1. Assume that d(1-a)¿c+b. Then the equilibrium point $\overline{x}=0$ of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(7) is asymptotically stable if

$$\begin{aligned} |c_2|+|c_1|+|c_0| &< 1, \\ \left|\frac{c}{d}\right|+\left|\frac{b}{d}\right|+|a| &< 1, \end{aligned}$$

or,

$$\begin{aligned} c+b+ad &< d, \\ c+b &< d(1-a). \end{aligned}$$

The proof is complete.

Theorem 2.2. Assume that $|d(1-a)-c|+|d(1-a)-b|<(b+c)$.Then the positive equilibrium point $\overline{x}=\sqrt{(b+c)-(1-a)de(1-a)}$ of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(7) is asymptotically stable if

$$\begin{aligned} \left|\frac{(d(1-a)-b)(1-a)}{b+c}\right|+\left|\frac{(d(1-a)-c)(1-a)}{b+c}\right|+|a| &< 1, \\ \left|\frac{(d(1-a)-b)(1-a)}{b+c}\right|+\left|\frac{(d(1-a)-c)(1-a)}{b+c}\right| &< 1-a, \end{aligned}$$

$$\begin{aligned} \left|\frac{(d(1-a)-b)}{b+c}\right|+\left|\frac{(d(1-a)-c)}{b+c}\right| &< 1, \\ |(d(1-a)-b)|+|(d(1-a)-c)| &< b+c. \end{aligned}$$

The proof is complete.

3 Global Attractor of the Equilibrium Point of Eq.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

Let α,β are a real numbers and assume that $f:[\alpha,\beta]^3\rightarrow[\alpha,\beta]$ be a function defined by

f(u,v,w)=au+bv+cw(d+evw).

Then

$$\begin{aligned} \frac{\partial f(u,v,w)}{\partial u} &= a, \\ \frac{\partial f(u,v,w)}{\partial v} &= \frac{bd-ecw^2}{(d+evw)^2}, \\ \frac{\partial f(u,v,w)}{\partial w} &= \frac{cd-ebv^2}{(d+evw)^2}. \end{aligned}$$

We can easily see that the function $f(u,v,w)$ is always non-decreasing in u and unknown in the other elements so, we have four cases (in all cases we suppose that $a<1$.):

Theorem 3.1. If $f(u,v,w)$ is non-decreasing in v and w. Then the equilibrium point $\overline{x}$ of Eq.(1) is global attractor if $d>ad+b+c$.

Proof: Suppose that (m,M) is a solution of the system $M=f(M,M,M)$ and $m=f(m,m,m)$. Thus from Eq.(1), we see that

M=aM+bM+cM(d+eM^2), $m = am + bm + cm(d + em^2)$,or
M(1-a)=(b+c)M(d+eM^2), $m(1 - a) = (b + c)m(d + em^2)$,then d(1-a)M+e(1-a)M^3 $= (b + c)M$, $d(1 - a)m + e(1 - a)m^3 = (b + c)m$.Subtracting we obtain (M-m)$\{d(1 - a) + e(1 - a)(M^2 + Mm + m^2) - (b + c)\} = 0$.under the conditions $a < 1$, and $d > ad + b + c$, we see that M=m. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1).

Theorem 3.2. If $f\ (u, v, w)$ is non-decreasing in v and non-increasing in w. Then the equilibrium point $\overline{x}$ of Eq.(1) is global attractor if $d > ad + b - c$.

Proof: Suppose that (m, M) is a solution of the system $M = f(M, M, m)$ and $m = f(m, m, M)$. Then from Eq.(1), we see that

M=aM+bM+cm(d+eMm), m=am+bm+cM(d+emM), or M(1-a)=bM+cm(d+eMm), m(1-a)=bm+cM(d+emM), then d(1-a)M+e(1-a)M^2$m - bM - cm = 0$, $d(1 - a)m + e(1 - a)Mm^2 - bm - cM = 0$.Subtracting we obtain (M-m)$\{d(1 - a) + e(1 - a)mM - b + c\} = 0$.Under the conditions $a < 1$, and $d > ad + b - c$, we see that M=m. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1).

Theorem 3.3. If $f(u, v, w)$ is non-increasing in v and nondecreasing w. Then the equilibrium point $\overline{x}$ of Eq.(1) is global attractor if $d > ad - b + c$.

Proof: Suppose that (m, M) is a solution of the system $M = f(M, m, M)$ and $m = f(m, M, m)$. Thus from Eq.(1), we see that

M=aM+bm+cM(d+emM), m=am+bM+cm(d+eMm), or M(1-a)=bm+cM(d+emM), m(1-a)=bM+cm(d+eMm), then d(1-a)M+e(1-a)mM^2 $= bm + cM$, $d(1 - a)m + e(1 - a)m^2M = bM + cm$.Subtracting we obtain
(M-m)$\{d(1 - a) + e(1 - a)mM + b - c\} = 0$.under the conditions $a < 1$, and $d > ad - b + c$, we see that M=m. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1).

Theorem 3.4. If $f(u, v, w)$ is non-increasing in v and w. Then the equilibrium point $\overline{x}$ of Eq.(1) is global attractor if $d < ad - b - c$.

Proof: Suppose that (m, M) is a solution of the system $M = f(M, m, m)$ and $m = f(m, M, M)$. Then from Eq.(1), we see that

M=aM+bm+cm(d+em^2), $m = am + bM + cM(d + eM^2)$,or
M(1-a)=(b+c)m(d+em^2), $m(1 - a) = (b + c)M(d + eM^2)$,then d(1-a)M+e(1-a)Mm^2 $= (b + c)m$, $d(1 - a)m + e(1 - a)M^2m = (b + c)M$.Subtracting we obtain (m-M)$\{-d(1 - a) + e(1 - a)Mm - (b + c)\} = 0$.Under the conditions $a < 1$, and $d < ad - b - c$, we see that M=m. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

4 Existence of Bounded and Unbounded Solutions of Eq.(1)

In this section we study the boundedness of solutions of Eq.(1).

Theorem 4.1. Every solution of Eq.(1) is bounded if $a + bd + cd < 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that $x_{n+1} = ax_n + bx_{n-1} + cx_{n-2}d + ex_{n-1}x_{n-2} \leq ax_n + bx_{n-1} + cx_{n-2}d = ax_n + bdx_{n-1} + cdx_{n-2}$.Then $x_{n+1} \leq ax_n + bdx_{n-1} + cdx_{n-2}\ foralln \geq 1$.

By using a comparison, the right hand side can be written as follows

$y_{n+1} = ay_n + bdy_{n-1} + cdy_{n-2}$.

This equation is locally asymptotically stable if

a+bd+cd¡1.

Hence, the solution is bounded.

Theorem 4.2. Every solution of Eq.(1) is unbounded if $a > 1$

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we find that

$x_{n+1} = ax_n + bx_{n-1} + cx_{n-2}d + ex_{n-1}x_{n-2} > ax_n \quad \forall n \geq 1$.

Then, the right hand side can be written as follows

$y_{n+1} = ay_n \Rightarrow y_n = a^n y_0$.

and this equation is unstable because $a > 1$, and $n \longrightarrow \infty \lim y_n = \infty$. Then using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

5 Existence of Periodic Solutions of Eq.(1)

In this section we study the existence of periodic solutions with period tow of Eq.(1)

Theorem 5.1. Eq.(1) has period two solutions if and only if

(i) b¡c+d(1+a).

Proof: First suppose that there exists a prime period two solutions ...,p,q,p,q,...,

of Eq.(1). We will show that condition (i) holds. From Eq.(1), we get

$$p = aq + \frac{bp + cq}{d + epq},$$
$$q = ap + \frac{bq + cp}{d + epq}.$$

Therefore,

$$pd + ep^2q = aqd + eapq^2 + bp + cq14 \tag{11}$$

and

$$qd + epq^2 = apd + eap^2q + bq + cp15 \tag{12}$$

Hence, by addition (14) from (15), we get

(p+q){d+epq-ad-eapq-b-c}=0.

Then

$$p + q = 0.16 \tag{13}$$

Again, subtracting (14) and (15) yields

d(p-q)+epq(p-q)+ad(p-q)+aepq(p-q)=b(p-q)-c(p-q), (p-q){d+epq+ad+eapq-b+c}=0, since $p \neq q$, it follows that

d+pq(e+ae)+ad-b+c=0.

Then,

$$pq = \frac{b - c - d(1 + a)}{e(1 + a)}.17 \tag{14}$$

Now it is obvious from Eqs. (16) and (17) that p and q are two distinct roots of the quadratic equation

$$t^2 + \left(\frac{b - c - d(1 + a)}{e(1 + a)}\right) = 0, 18 \tag{15}$$

and so

t=$\pm\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}$,

thus,

c-b+d(1+a)¿0,

or b¡c+d(1+a).

Therefore inequality (i) holds.

Conversely, suppose that inequality (i) is true. We will prove that Eq.(1) has a prime period two solution.

Suppose that

p=$\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}$ $\quad and \quad q=-\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}$,

We see from the inequality (i) that b¡c+d(1+a)

Therefore$p$ and q are distinct real numbers.

Set

x$_{-2}=x_0=q \; and \;\; x_{-1}=p.$

We would like to show that

x$_1=x_{-1}=p \; and \;\; x_2=x_0=q.$

It follows from Eq.(1) that

$$\begin{aligned} x_1 &= aq+\frac{bp+cq}{d+epq} \\ &= -a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(b-c)\sqrt{c-b+d(1+a)e(1+a)}}{d-e\left(c-b+d(1+a)e(1+a)\right)} \end{aligned}$$

=-a$\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(1+a)(b-c)\sqrt{c-b+d(1+a)e(1+a)}}{d(1+a)+b-c-d(1+a)}$

$$\begin{aligned} &= -a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(1+a)(b-c)\sqrt{c-b+d(1+a)e(1+a)}}{(b-c)} \\ &= -a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+(1+a)\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right) \\ &= \sqrt{\frac{c-b+d(1+a)}{e(1+a)}}. \end{aligned}$$

We obtain x$_1=p.$

Similarly as before,it is easy to show that

$$\begin{aligned} x_2 &= ap+\frac{bq+cb}{d+epq} \\ &= a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(c-b)\sqrt{c-b+d(1+a)e(1+a)}}{d-e\left(c-b+d(1+a)e(1+a)\right)} \end{aligned}$$

=a$\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(1+a)(c-b)\sqrt{c-b+d(1+a)e(1+a)}}{d(1+a)+b-c-d(1+a)}$

$$\begin{aligned} &= a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)+\frac{(1+a)(c-b)\sqrt{c-b+d(1+a)e(1+a)}}{-(c-b)} \\ &= a\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right)-(1+a)\left(\sqrt{\frac{c-b+d(1+a)}{e(1+a)}}\right). \end{aligned}$$

We obtain $x_2 = -\sqrt{\frac{c-b+d(1+a)}{e(1+a)}} = q.$

Then by induction we get

$x_{2n} = q \quad and \quad x_{2n+1} = p \qquad forall \;\; n \geq -2.$

Thus Eq.(1) has the prim period two solution

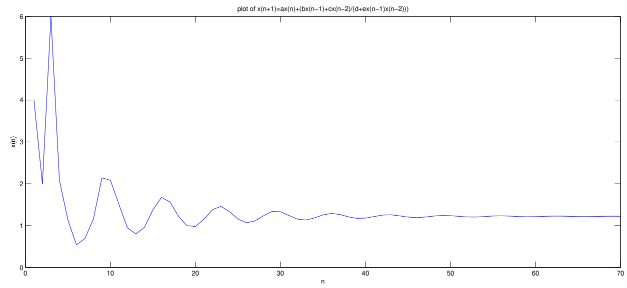
...,p,q,p,q,... .

Where p and q are the distinct roots of the quadratic equation (18) and the proof is complete.

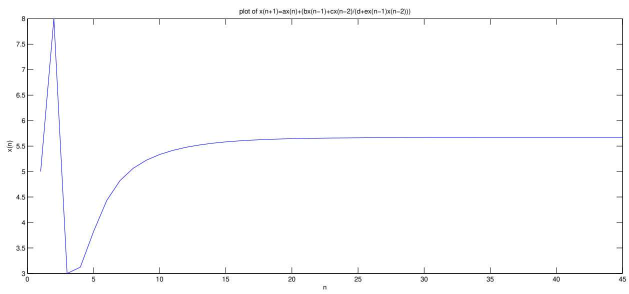
6 Numerical Examples

To confirm the results of this paper,we consider numerical examples which represent different types of solutions to Eq.(1).

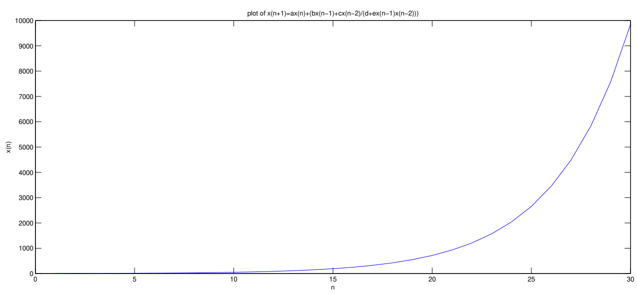
Example 6.1. See figure 1, since $x_{-2} = 4,\ x_{-1} = 2,\ x_0 = 6,\ a = 0.3,\ b = 3,\ c = 0.2,\ d = 0.1,\ e = 3.$



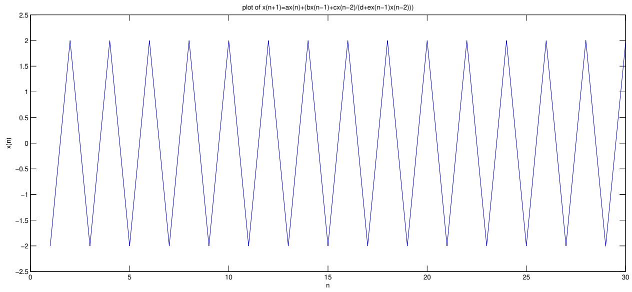
Example 6.2. We assume that $x_{-2} = 5,\ x_{-1} = 8, x_0 = 3,\ a = 0.9,\ b = 0.8,\ c = 9,\ d = 1.6,\ e = 3.$



Example 6.3. Figure 3 shows the solutions when $x_{-2} = 7,\ x_{-1} = 11,\ x_0 = 8,\ a = 1.3,\ b = 2,\ c = 5,\ d = 3,\ e = 7.$



Example 6.4. We consider $a = 0.02$, $b = 1$, $c = 2$, $d = 11$, $e = 3$, $x_{-2} = x_0 = p$, $x_{-1} = q$. $\left(Sincep, q = \pm\sqrt{c - b + d(1 + a)e(1 + a)}\right)$.



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